COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

Gerwin Klein, June Andronick, Toby Murray, Rafal Kolanski

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Content

[a] a1 due; [b] a2 due; [c] session break; [d] a3 due

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Last Time

- ➜ Sets
- ➜ Type Definitions
- ➜ Inductive Definitions

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HOW INDUCTIVE DEFINITIONS WORK

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The Nat Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

- ➜ N is the set of natural numbers $\mathbb{N}$
- ➜ But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n+1 \in \mathbb{R}$
- ➜ $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

- ➜ Objective: **no junk**. Only what must be in X shall be in X.
- ➜ Gives rise to a nice proof principle (rule induction)

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Formally

Rules $\dfrac{a_1 \in X \quad \dots \quad a_n \in X}{a \in X}$ with $a_1, \dots, a_n, a \in A$

define set $X \subseteq A$

Formally: set of rules $R \subseteq A$ set $\times A$ $\quad (R, X$ possibly infinite)

Applying rules R to a set B: $\quad \hat{R}\ B \equiv \{x.\ \exists H.\ (H, x) \in R \wedge H \subseteq B\}$

Example:

$$\begin{array}{lcl} R & \equiv & \{(\{\}, 0)\} \cup \{(\{n\}, n+1).\ n \in \mathbb{R}\} \\ \hat{R}\ \{3, 6, 10\} & = & \{0, 4, 7, 11\} \end{array}$$

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The Set

Definition: B is R-closed iff $\hat{R}\ B \subseteq B$

Definition: X is the least R-closed subset of A

This does always exist:

Fact: $X = \bigcap\{B \subseteq A.\ B\ R\text{–closed}\}$

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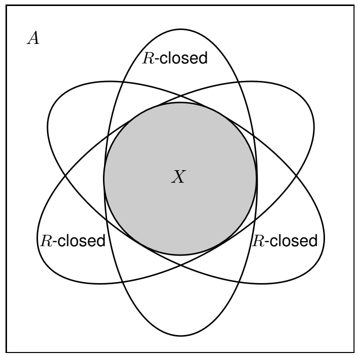
Generation from Above



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Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n+1 \in N}$$

induces induction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n+1)]\!] \Longrightarrow \forall x \in X.\ P\ x$$

In general:

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

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Rules with side conditions

$$\frac{a_1 \in X \quad \ldots \quad a_n \in X \qquad C_1 \quad \ldots \quad C_m}{a \in X}$$

induction scheme:

$$(\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \wedge$$
$$C_1 \wedge \ldots \wedge C_m \wedge$$
$$\{a_1, \ldots, a_n\} \subseteq X \Longrightarrow P\ a)$$
$$\Longrightarrow$$
$$\forall x \in X.\ P\ x$$

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Why does this work?

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

$\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a$

says

$\{x.\ P\ x\}$ is R-closed

but: X is the least R-closed set

hence: $X \subseteq \{x.\ P\ x\}$

which means: $\forall x \in X.\ P\ x$

qed

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X as Fixpoint

How to compute X?

$X = \bigcap \{B \subseteq A.\ B\ R - \text{closed}\}$ hard to work with.

Instead: view X as least fixpoint, X least set with $\hat{R}\ X = X$.

Fixpoints can be approximated by iteration:

$X_0 = \hat{R}^0\ \{\} = \{\}$

$X_1 = \hat{R}^1\ \{\} =$ rules without hypotheses

$\vdots$

$X_n = \hat{R}^n\ \{\}$

$X_\omega = \bigcup_{n \in \mathbb{N}} (R^n\ \{\}) = X$

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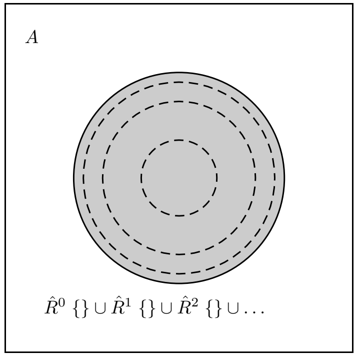
Generation from Below



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Does this always work?

Knaster-Tarski Fixpoint Theorem:
Let $(A, \leq)$ be a complete lattice, and $f :: A \Rightarrow A$ a monotone function.
Then the fixpoints of f again form a complete lattice.

Lattice:
Finite subsets have a greatest lower bound (meet) and least upper bound (join).

Complete Lattice:
All subsets have a greatest lower bound and least upper bound.

Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

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Exercise

Formalize the this lecture in Isabelle:

- ➜ Define **closed** $f\ A :: (\alpha$ set $\Rightarrow \alpha$ set$) \Rightarrow \alpha$ set $\Rightarrow$ bool
- ➜ Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)
- ➜ Define **lfpt** f as the intersection of all f-closed sets
- ➜ Show that lfpt f is a fixpoint of f if f is monotone
- ➜ Show that lfpt f is the least fixpoint of f
- ➜ Declare a constant $R :: (\alpha$ set $\times\ \alpha)$ set
- ➜ Define $\hat{R} :: \alpha$ set $\Rightarrow \alpha$ set in terms of R
- ➜ Show soundness of rule induction using R and lfpt $\hat{R}$

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We have learned today ...

- ➜ Formal background of inductive definitions
- ➜ Definition by intersection
- ➜ Computation by iteration
- ➜ Formalisation in Isabelle

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