COMP 4161

NICTA Advanced Course

Advanced Topics in Software Verification

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$$\lambda$$

Exercises from last time

- Download and install Isabelle from
  `http://mirror.cse.unsw.edu.au/pub/isabelle/`
- Step through the demo files from the lecture web page
- Write your own theory file, look at some theorems in the library, try 'find_theorems'

- How many theorems can help you if you need to prove something like "Suc(Suc x))"?
- What is the name of the theorem for associativity of addition of natural numbers in the library?

Content

➜ Intro & motivation, getting started [1]

➜ Foundations & Principles

- Lambda Calculus, natural deduction [1,2]
- Higher Order Logic [3[a]]
- Term rewriting [4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [5]
- Datatypes, recursion, induction [6[b], 7]
- Code generation, type classes [7]
- Hoare logic, proofs about programs, refinement [8,9[c],10[d]]
- Isar, locales [11,12]

[a] a1 due; [b] a2 due; [c] session break; [d] a3 due

Alonzo Church

- ➜ lived 1903–1995
- ➜ supervised people like Alan Turing, Stephen Kleene
- ➜ famous for Church-Turing thesis, lambda calculus, first undecidability results
- ➜ invented λ calculus in 1930's

λ-calculus

- ➜ originally meant as foundation of mathematics
- ➜ important applications in theoretical computer science
- ➜ foundation of computability and functional programming

untyped λ-calculus

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- ➜ turing complete model of computation
- ➜ a simple way of writing down functions

Basic intuition:

instead of	$f(x) = x + 5$
write	$f = \lambda x.\ x + 5$

$\lambda x.\ x + 5$

- ➜ a term
- ➜ a nameless function
- ➜ that adds 5 to its parameter

Function Application

For applying arguments to functions

instead of	$f(a)$
write	$f\ a$

Example: $(\lambda x.\ x + 5)\ a$

Evaluating: in $(\lambda x.\ t)\ a$ replace x by a in t

(computation!)

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

THAT'S IT!

Now Formal

Syntax

Terms: $t \quad ::= \quad v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- ➜ v, x variables
- ➜ c constants
- ➜ $(t\ t)$ application
- ➜ $(\lambda x.\ t)$ abstraction

Conventions

➜ leave out parentheses where possible

➜ list variables instead of multiple λ

Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

Rules:

➜ list variables: $\lambda x.\ (\lambda y.\ t) \ =\ \lambda x\ y.\ t$

➜ application binds to the left: $x\ y\ z \ =\ (x\ y)\ z \neq x\ (y\ z)$

➜ abstraction binds to the right: $\lambda x.\ x\ y \ =\ \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$

➜ leave out outermost parentheses

Getting used to the Syntax

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

$\lambda x\ y\ z.\ (x\ z)\ (y\ z) =$

$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$

$\lambda x.\ \lambda y.\ \lambda z.\ ((x\ z)\ (y\ z)) =$

$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

Computation

Intuition: replace parameter by argument

this is called β-reduction

Example

$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$

$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$

$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$

$f\ 5$

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β reduction:

$$
\begin{array}{lllllll}
 & & & & (\lambda x.\ s)\ t & \longrightarrow_\beta & s[x \leftarrow t] \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\
t & \longrightarrow_\beta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\beta & (\lambda x.\ s')
\end{array}
$$

Still to do: define $s[x \leftarrow t]$

Defining Substitution

Easy concept. Small problem: variable capture.

Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

Free Variables

Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

Free variables FV of a term:

$$
\begin{aligned}
FV\ (x) &= \{x\} \\
FV\ (c) &= \{\} \\
FV\ (s\ t) &= FV(s) \cup FV(t) \\
FV\ (\lambda x.\ t) &= FV(t) \setminus \{x\}
\end{aligned}
$$

Example: $FV(\ \ \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x\ \ ) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

Our problematic substitution example, $(\lambda x.\ x\ z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term "x".

Substitution

$$
\begin{array}{lll}
x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
c\ [x \leftarrow t] & = c & \\
 & & \\
(s_1\ s_2)\ [x \leftarrow t] & = (s_1[x \leftarrow t]\ s_2[x \leftarrow t]) & \\
 & & \\
(\lambda x.\ s)\ [x \leftarrow t] & = (\lambda x.\ s) & \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda y.\ s[x \leftarrow t]) & \text{if } x \neq y \text{ and } y \notin FV(t) \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda z.\ s[y \leftarrow z][x \leftarrow t]) & \text{if } x \neq y \\
 & & \text{and } z \notin FV(t) \cup FV(s)
\end{array}
$$

Substitution Example

$$
\begin{array}{ll}
 & (x \;\; (\lambda x.\, x) \;\; (\lambda y.\, z\ x))[x \leftarrow y] \\
= & (x[x \leftarrow y]) \;\; ((\lambda x.\, x)[x \leftarrow y]) \;\; ((\lambda y.\, z\ x)[x \leftarrow y]) \\
= & y\ (\lambda x.\, x)\ (\lambda y'.\, z\ y)
\end{array}
$$

α Conversion

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Bound names are irrelevant:

$\lambda x.\ x$ and $\lambda y.\ y$ denote the same function.

α conversion:

$s =_\alpha t$ means $s = t$ up to renaming of bound variables.

Formally:

$$
\begin{array}{rcl}
 & & (\lambda x.\ t) \longrightarrow_\alpha (\lambda y.\ t[x \leftarrow y]) \text{ if } y \notin FV(t) \\
s \longrightarrow_\alpha s' & \Longrightarrow & (s\ t) \longrightarrow_\alpha (s'\ t) \\
t \longrightarrow_\alpha t' & \Longrightarrow & (s\ t) \longrightarrow_\alpha (s\ t') \\
s \longrightarrow_\alpha s' & \Longrightarrow & (\lambda x.\ s) \longrightarrow_\alpha (\lambda x.\ s')
\end{array}
$$

$$s =_\alpha t \quad \text{iff} \quad s \longrightarrow_\alpha^* t$$

($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

α Conversion

Equality in Isabelle is equality modulo α conversion:

if $s =_\alpha t$ then s and t are syntactically equal.

Examples:

$$
\begin{array}{ll}
 & x\ (\lambda x\ y.\ x\ y) \\
=_\alpha & x\ (\lambda y\ x.\ y\ x) \\
=_\alpha & x\ (\lambda z\ y.\ z\ y) \\
\neq_\alpha & z\ (\lambda z\ y.\ z\ y) \\
\neq_\alpha & x\ (\lambda x\ x.\ x\ x)
\end{array}
$$

Back to β

We have defined β reduction: $\longrightarrow_\beta$

Some notation and concepts:

- β **conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- t is reducible iff it contains a redex
- if it is not reducible, t is in **normal form**

Does every λ term have a normal form?

No!

Example:

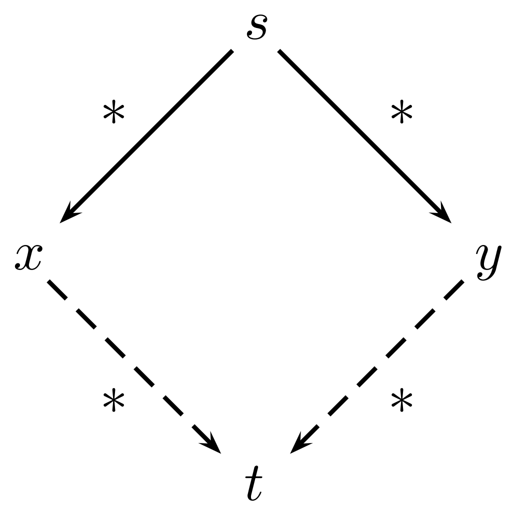
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta$$
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta$$
$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \ \longrightarrow_\beta \ \ldots$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \ \longrightarrow_\beta \ \lambda y.\ y$)

λ calculus is not terminating

β reduction is confluent

Confluence: $s \longrightarrow_\beta^* x \wedge s \longrightarrow_\beta^* y \Longrightarrow \exists t.\ x \longrightarrow_\beta^* t \wedge y \longrightarrow_\beta^* t$



Order of reduction does not matter for result

Normal forms in λ calculus are unique

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta \lambda y.\ y$

η Conversion

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

Definition:

$$
\begin{array}{lllllll}
 & & & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t \quad \text{if } x \notin FV(t) \\
s & \longrightarrow_\eta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s'\ t) \\
t & \longrightarrow_\eta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s\ t') \\
s & \longrightarrow_\eta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\eta & (\lambda x.\ s')
\end{array}
$$

$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta (\lambda x.\ f\ x)\ g \longrightarrow_\eta f\ g$

➜ η reduction is confluent and terminating.

➜ $\longrightarrow_{\beta\eta}$ is confluent.

$\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.

➜ **Equality in Isabelle is also modulo η conversion.**

In fact ...

Equality in Isabelle is modulo α, β, and η conversion.

We will see later why that is possible.

So, what can you do with λ calculus?

λ calculus is very expressive, you can encode:

- ➜ logic, set theory
- ➜ turing machines, functional programs, etc.

Examples:

$$\begin{aligned} \texttt{true} &\equiv \lambda x\ y.\ x & \qquad \texttt{if true}\ x\ y \longrightarrow_\beta^* x \\ \texttt{false} &\equiv \lambda x\ y.\ y & \qquad \texttt{if false}\ x\ y \longrightarrow_\beta^* y \\ \texttt{if} &\equiv \lambda z\ x\ y.\ z\ x\ y \end{aligned}$$

Now, `not`, `and`, `or`, etc is easy:

$$\begin{aligned} \texttt{not} &\equiv \lambda x.\ \texttt{if}\ x\ \texttt{false true} \\ \texttt{and} &\equiv \lambda x\ y.\ \texttt{if}\ x\ y\ \texttt{false} \\ \texttt{or} &\equiv \lambda x\ y.\ \texttt{if}\ x\ \texttt{true}\ y \end{aligned}$$

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Encoding natural numbers (Church Numerals)

$$0 \equiv \lambda f\ x.\ x$$
$$1 \equiv \lambda f\ x.\ f\ x$$
$$2 \equiv \lambda f\ x.\ f\ (f\ x)$$
$$3 \equiv \lambda f\ x.\ f\ (f\ (f\ x))$$
$$\ldots$$

Numeral n takes arguments f and x, applies f n-times to x.

$$\texttt{iszero} \equiv \lambda n.\ n\ (\lambda x.\ \texttt{false})\ \texttt{true}$$
$$\texttt{succ} \equiv \lambda n\ f\ x.\ f\ (n\ f\ x)$$
$$\texttt{add} \equiv \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)$$

Fix Points

$(\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ \ t \longrightarrow_\beta$

$(\lambda f.\ f\ ((\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ f))\ \ t \longrightarrow_\beta$

$t\ \ ((\lambda x\ f.\ f\ (x\ x\ f))\ (\lambda x\ f.\ f\ (x\ x\ f))\ t)$

$\mu = (\lambda x f.\ f\ (x\ x\ f))\ (\lambda x f.\ f\ (x\ x\ f))$

$\mu\ t \longrightarrow_\beta t\ (\mu\ t) \longrightarrow_\beta t\ (t\ (\mu\ t)) \longrightarrow_\beta t\ (t\ (t\ (\mu\ t))) \longrightarrow_\beta \ldots$

$(\lambda x f.\ f\ (x\ x\ f))\ (\lambda x f.\ f\ (x\ x\ f))$ is Turing's fix point operator

Nice, but ...

As a mathematical foundation, λ does not work. **It is inconsistent.**

- ➜ **Frege** (Predicate Logic, $\sim$ 1879):
  allows arbitrary quantification over predicates
- ➜ **Russell** (1901): Paradox $R \equiv \{X | X \notin X\}$
- ➜ **Whitehead & Russell** (Principia Mathematica, 1910-1913):
  Fix the problem
- ➜ **Church** (1930): λ calculus as logic, $\texttt{true}$, $\texttt{false}$, $\wedge$, ... as λ terms

	with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x$	$x \in M \equiv M\ x$
Problem:	you can write	$R \equiv \lambda x.\ \texttt{not}\ (x\ x)$	
	and get	$(R\ R) =_\beta \texttt{not}\ (R\ R)$	

ISABELLE DEMO

We have learned so far...

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- ➜ λ calculus syntax
- ➜ free variables, substitution
- ➜ β reduction
- ➜ α and η conversion
- ➜ β reduction is confluent
- ➜ λ calculus is very expressive (turing complete)
- ➜ λ calculus is inconsistent