COMP 4161

NICTA Advanced Course

Advanced Topics in Software Verification

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NICTA

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^{a}a1 due; ^{b}a2 due; ^{c}session break; ^{d}a3 due

Last Time

- ➜ Sets
- ➜ Type Definitions
- ➜ Inductive Definitions

How Inductive Definitions Work

The Nat Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

➜ N is the set of natural numbers $\mathbb{N}$

➜ But why not the set of real numbers? $0 \in \mathbb{R}$, $n \in \mathbb{R} \Longrightarrow n + 1 \in \mathbb{R}$

➜ $\mathbb{N}$ is the **smallest** set that is **consistent** with the rules.

Why the smallest set?

➜ Objective: **no junk**. Only what must be in X shall be in X.

➜ Gives rise to a nice proof principle (rule induction)

Formally

Rules $\dfrac{a_1 \in X \quad \ldots \quad a_n \in X}{a \in X}$ with $a_1, \ldots, a_n, a \in A$

define set $X \subseteq A$

Formally: set of rules $R \subseteq A$ set $\times$ A (R, X possibly infinite)

Applying rules R to a set B: $\hat{R}\ B \equiv \{x.\ \exists H.\ (H, x) \in R \wedge H \subseteq B\}$

Example:

$$\begin{array}{lcl} R & \equiv & \{(\{\}, 0)\} \cup \{(\{n\}, n+1).\ n \in \mathbb{R}\} \\ \hat{R}\ \{3, 6, 10\} & = & \{0, 4, 7, 11\} \end{array}$$

The Set

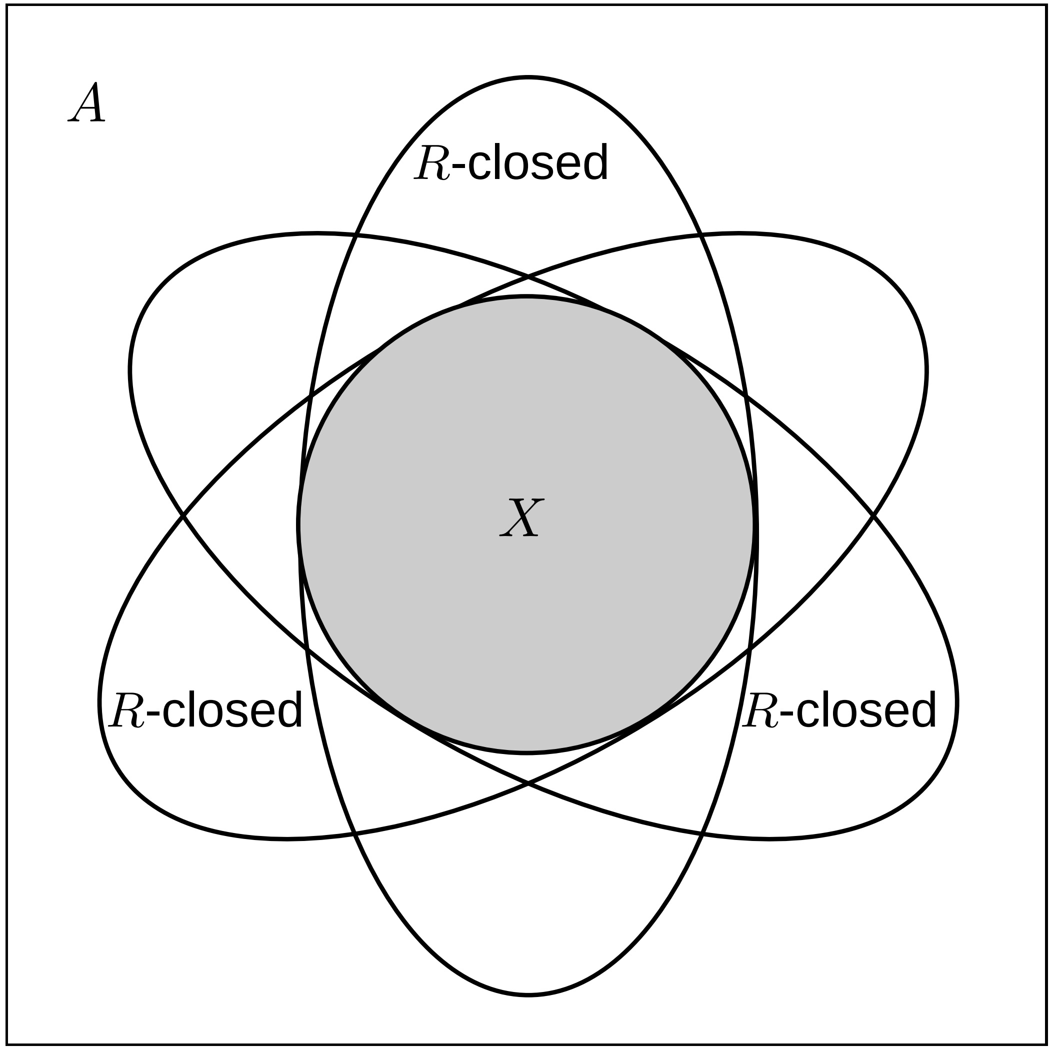
Definition: B is R-closed iff $\hat{R}\ B \subseteq B$

Definition: X is the least R-closed subset of A

This does always exist:

Fact: $X = \bigcap\{B \subseteq A.\ B\ R\text{--closed}\}$

Generation from Above



Rule Induction

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

induces induction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n+1)]\!] \Longrightarrow \forall x \in X.\ P\ x$$

In general:

$$\frac{\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

Why does this work?

$$\frac{\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

$$\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a$$

says

$\{x.\ P\ x\}$ is R-closed

but:	X is the least R-closed set
hence:	$X \subseteq \{x.\ P\ x\}$
which means:	$\forall x \in X.\ P\ x$

qed

Rules with side conditions

$$\frac{a_1 \in X \quad \ldots \quad a_n \in X \qquad C_1 \quad \ldots \quad C_m}{a \in X}$$

induction scheme:

$$(\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \wedge$$
$$C_1 \wedge \ldots \wedge C_m \wedge$$
$$\{a_1, \ldots, a_n\} \subseteq X \Longrightarrow P\ a)$$

$$\Longrightarrow$$

$$\forall x \in X.\ P\ x$$

X as Fixpoint

How to compute X?

$X = \bigcap\{B \subseteq A.\ B\ R-\text{closed}\}$ hard to work with.

Instead: view X as least fixpoint, X least set with $\hat{R}\ X = X$.

Fixpoints can be approximated by iteration:

$$X_0 = \hat{R}^0\ \{\} = \{\}$$

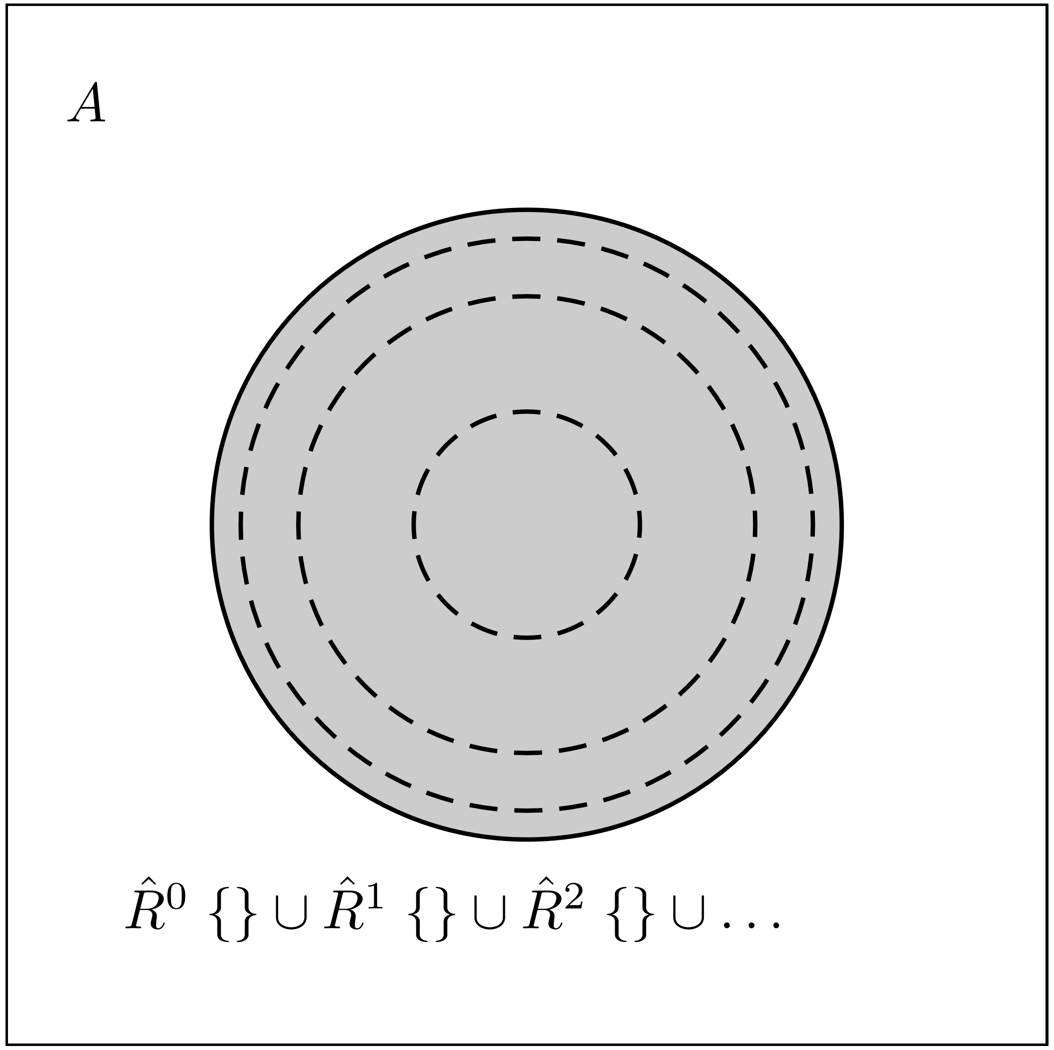
$$X_1 = \hat{R}^1\ \{\} = \text{rules without hypotheses}$$

$$\vdots$$

$$X_n = \hat{R}^n\ \{\}$$

$$X_\omega = \bigcup_{n \in \mathbb{N}} (R^n\ \{\}) = X$$

Generation from Below


A
$\hat{R}^0\ \{\} \cup \hat{R}^1\ \{\} \cup \hat{R}^2\ \{\} \cup \ldots$

Does this always work?

Knaster-Tarski Fixpoint Theorem:

Let $(A, \leq)$ be a complete lattice, and $f :: A \Rightarrow A$ a monotone function.
Then the fixpoints of f again form a complete lattice.

Lattice:

Finite subsets have a greatest lower bound (meet) and least upper bound (join).

Complete Lattice:

All subsets have a greatest lower bound and least upper bound.

Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

Exercise

Formalize the this lecture in Isabelle:

➜ Define **closed** $f\ A :: (\alpha \text{ set} \Rightarrow \alpha \text{ set}) \Rightarrow \alpha \text{ set} \Rightarrow \text{bool}$

➜ Show closed $f\ A \wedge$ closed $f\ B \Longrightarrow$ closed $f\ (A \cap B)$ if f is monotone (**mono** is predefined)

➜ Define **lfpt** f as the intersection of all f-closed sets

➜ Show that lfpt f is a fixpoint of f if f is monotone

➜ Show that lfpt f is the least fixpoint of f

➜ Declare a constant $R :: (\alpha \text{ set} \times \alpha) \text{ set}$

➜ Define $\hat{R} :: \alpha \text{ set} \Rightarrow \alpha \text{ set}$ in terms of R

➜ Show soundness of rule induction using R and lfpt $\hat{R}$

Rule Induction in Isar

Inductive definition in Isabelle

inductive $X :: \alpha \Rightarrow$ bool

where

rule$_1$: "$[\![X\ s; A]\!] \Longrightarrow X\ s'$"

$\vdots$

$|$ rule$_n$: $\ldots$

Rule induction

```
**show** "$X\ x \Longrightarrow P\ x$"
**proof** (induct rule: X.induct)
    **fix** $s$ and $s'$ **assume** "$X\ s$" and "$A$" and "$P\ s$"
    . . .
    **show** "$P\ s'$"
**next**
⋮
**qed**
```

Abbreviations

```
show "$X\ x \Longrightarrow P\ x$"
proof (induct rule: X.induct)
   case rule$_1$
   . . .
   show ?case
next
⋮
next
   case rule$_n$
   . . .
   show ?case
qed
```

Implicit selection of induction rule

assume A: "$X\ x$"

⋮

show "$P\ x$"

using A **proof** induct

⋮

qed

lemma assumes A: "$X\ x$" **shows** "$P\ x$"

using A **proof** induct

⋮

qed

Renaming free variables in rule

case (rule$_i$ $x_1 \dots x_k$)

Renames first k variables in rule$_i$ to $x_1 \dots x_k$.

A remark on style

➜ **case** (rule$_i$ x y) . . . **show** ?case
is easy to write and maintain

➜ **fix** x y **assume** $formula$ . . . **show** $formula'$
is easier to read:

- all information is shown locally
- no contextual references (e.g. ?case)

Demo: Rule Induction in Isar

We have learned today ...

- ➜ Formal background of inductive definitions
- ➜ Definition by intersection
- ➜ Computation by iteration
- ➜ Formalisation in Isabelle
- ➜ Rule Induction in Isar