COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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Slide 1

Exercises from last time

- ➜ Download and install Isabelle from `http://mirror.cse.unsw.edu.au/pub/isabelle/`
- ➜ Step through the demo files from the lecture web page
- ➜ Write your own theory file, look at some theorems in the library, try 'find_theorems'

- ➜ How many theorems can help you if you need to prove something like "Suc(Suc x))"?
- ➜ What is the name of the theorem for associativity of addition of natural numbers in the library?

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Content

[a] a1 due; [b] a2 due; [c] session break; [d] a3 due

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λ-calculus

Alonzo Church

- ➜ lived 1903–1995
- ➜ supervised people like Alan Turing, Stephen Kleene
- ➜ famous for Church-Turing thesis, lambda calculus, first undecidability results
- ➜ invented λ calculus in 1930's

λ-calculus

- ➜ originally meant as foundation of mathematics
- ➜ important applications in theoretical computer science
- ➜ foundation of computability and functional programming

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untyped λ-calculus

- ➜ turing complete model of computation
- ➜ a simple way of writing down functions

Basic intuition:

instead of $f(x) = x + 5$

write $f = \lambda x.\ x + 5$

$\lambda x.\ x + 5$

- ➜ a term
- ➜ a nameless function
- ➜ that adds 5 to its parameter

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Function Application

For applying arguments to functions

instead of $f(a)$

write $f\ a$

Example: $(\lambda x.\ x + 5)\ a$

Evaluating: in $(\lambda x.\ t)\ a$ replace x by a in t

(computation!)

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

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THAT'S IT!

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NOW FORMAL

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Syntax

Terms: $t \;::=\; v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- ➜ v, x variables
- ➜ c constants
- ➜ $(t\ t)$ application
- ➜ $(\lambda x.\ t)$ abstraction

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Getting used to the Syntax

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

$\lambda x\ y\ z.\ (x\ z)\ (y\ z) =$

$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$

$\lambda x.\ \lambda y.\ \lambda z.\ ((x\ z)\ (y\ z)) =$

$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

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Conventions

- ➜ leave out parentheses where possible
- ➜ list variables instead of multiple λ

Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

Rules:

- ➜ list variables: $\lambda x.\ (\lambda y.\ t) \ = \ \lambda x\ y.\ t$
- ➜ application binds to the left: $x\ y\ z \ = \ (x\ y)\ z \neq x\ (y\ z)$
- ➜ abstraction binds to the right: $\lambda x.\ x\ y \ = \ \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$
- ➜ leave out outermost parentheses

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Computation

Intuition: replace parameter by argument
this is called β-reduction

Example

$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$

$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$

$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$

$f\ 5$

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Defining Computation

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β **reduction:**

$$
\begin{array}{ccccccc}
 & & & & (\lambda x.\ s)\ t & \longrightarrow_\beta & s[x \leftarrow t] \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\
t & \longrightarrow_\beta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\beta & (\lambda x.\ s')
\end{array}
$$

Still to do: define $s[x \leftarrow t]$

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Defining Substitution

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Easy concept. Small problem: variable capture.

Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

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Free Variables

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Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

Free variables FV of a term:

$$
\begin{array}{ll}
FV\ (x) & = \{x\} \\
FV\ (c) & = \{\} \\
FV\ (s\ t) & = FV(s) \cup FV(t) \\
FV\ (\lambda x.\ t) & = FV(t) \setminus \{x\}
\end{array}
$$

Example: $FV(\quad \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x \quad) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

Our problematic substitution example, $(\lambda x.\ x\ z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term "x".

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Substitution

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$$
\begin{array}{lll}
x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
c\ [x \leftarrow t] & = c &
\end{array}
$$

$$(s_1\ s_2)\ [x \leftarrow t] = (s_1[x \leftarrow t]\ s_2[x \leftarrow t])$$

$$
\begin{array}{lll}
(\lambda x.\ s)\ [x \leftarrow t] & = (\lambda x.\ s) & \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda y.\ s[x \leftarrow t]) & \text{if } x \neq y \text{ and } y \notin FV(t) \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda z.\ s[y \leftarrow z][x \leftarrow t]) & \text{if } x \neq y \\
 & & \text{and } z \notin FV(t) \cup FV(s)
\end{array}
$$

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Substitution Example

$$
\begin{aligned}
& (x\ \ (\lambda x.\ x)\ \ (\lambda y.\ z\ x))[x \leftarrow y] \\
= \ & (x[x \leftarrow y])\ \ ((\lambda x.\ x)[x \leftarrow y])\ \ ((\lambda y.\ z\ x)[x \leftarrow y]) \\
= \ & y\ (\lambda x.\ x)\ (\lambda y'.\ z\ y)
\end{aligned}
$$

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α Conversion

Bound names are irrelevant:

$\lambda x.\ x$ and $\lambda y.\ y$ denote the same function.

α conversion:

$s =_\alpha t$ means $s = t$ up to renaming of bound variables.

Formally:

$$
\begin{array}{llcccl}
 & & & (\lambda x.\ t) & \longrightarrow_\alpha & (\lambda y.\ t[x \leftarrow y]) \text{ if } y \notin FV(t) \\
s \longrightarrow_\alpha s' & \Longrightarrow & & (s\ t) & \longrightarrow_\alpha & (s'\ t) \\
t \longrightarrow_\alpha t' & \Longrightarrow & & (s\ t) & \longrightarrow_\alpha & (s\ t') \\
s \longrightarrow_\alpha s' & \Longrightarrow & & (\lambda x.\ s) & \longrightarrow_\alpha & (\lambda x.\ s')
\end{array}
$$

$s =_\alpha t$ iff $s \longrightarrow_\alpha^* t$

($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

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α Conversion

Equality in Isabelle is equality modulo α conversion:

if $s =_\alpha t$ then s and t are syntactically equal.

Examples:

$$
\begin{array}{ll}
 & x\ (\lambda x\ y.\ x\ y) \\
=_\alpha & x\ (\lambda y\ x.\ y\ x) \\
=_\alpha & x\ (\lambda z\ y.\ z\ y) \\
\neq_\alpha & z\ (\lambda z\ y.\ z\ y) \\
\neq_\alpha & x\ (\lambda x\ x.\ x\ x)
\end{array}
$$

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Back to β

We have defined β reduction: $\longrightarrow_\beta$

Some notation and concepts:

- ➜ β **conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- ➜ t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- ➜ $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- ➜ t is reducible iff it contains a redex
- ➜ if it is not reducible, t is in **normal form**

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Does every λ term have a normal form?

No!

Example:

$$\begin{array}{ll}(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) & \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) & \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) & \longrightarrow_\beta \ldots\end{array}$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x))\ \longrightarrow_\beta\ \lambda y.\ y$)

λ calculus is not terminating

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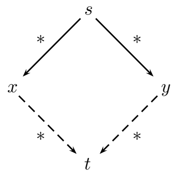
β reduction is confluent

Confluence: $s \longrightarrow^*_\beta x \wedge s \longrightarrow^*_\beta y \Longrightarrow \exists t.\ x \longrightarrow^*_\beta t \wedge y \longrightarrow^*_\beta t$



Order of reduction does not matter for result
Normal forms in λ calculus are unique

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β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta \lambda y.\ y$

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η Conversion

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

Definition:
$$\begin{array}{llllllll} & & & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t & \text{if } x \notin FV(t) \\ s & \longrightarrow_\eta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s'\ t) & \\ t & \longrightarrow_\eta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s\ t') & \\ s & \longrightarrow_\eta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\eta & (\lambda x.\ s') & \end{array}$$

$$s =_\eta t \ \text{ iff } \ \exists n.\ s \longrightarrow^*_\eta n \wedge t \longrightarrow^*_\eta n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta (\lambda x.\ f\ x)\ g \longrightarrow_\eta f\ g$

- ➜ η reduction is confluent and terminating.
- ➜ $\longrightarrow_{\beta\eta}$ is confluent.
  $\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.
- ➜ **Equality in Isabelle is also modulo η conversion.**

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In fact ...

Equality in Isabelle is modulo α, β, and η conversion.

We will see later why that is possible.

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So, what can you do with λ calculus?

λ calculus is very expressive, you can encode:

- ➜ logic, set theory
- ➜ turing machines, functional programs, etc.

Examples:

$\texttt{true} \equiv \lambda x\ y.\ x$ $\qquad$ $\texttt{if true}\ x\ y \longrightarrow^*_\beta x$

$\texttt{false} \equiv \lambda x\ y.\ y$ $\qquad$ $\texttt{if false}\ x\ y \longrightarrow^*_\beta y$

$\texttt{if} \equiv \lambda z\ x\ y.\ z\ x\ y$

Now, `not`, `and`, `or`, etc is easy:

$\texttt{not} \equiv \lambda x.\ \texttt{if}\ x\ \texttt{false true}$
$\texttt{and} \equiv \lambda x\ y.\ \texttt{if}\ x\ y\ \texttt{false}$
$\texttt{or} \equiv \lambda x\ y.\ \texttt{if}\ x\ \texttt{true}\ y$

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More Examples

Encoding natural numbers (Church Numerals)

$0 \equiv \lambda f\ x.\ x$
$1 \equiv \lambda f\ x.\ f\ x$
$2 \equiv \lambda f\ x.\ f\ (f\ x)$
$3 \equiv \lambda f\ x.\ f\ (f\ (f\ x))$
$\dots$

Numeral n takes arguments f and x, applies f n-times to x.

$\texttt{iszero} \equiv \lambda n.\ n\ (\lambda x.\ \texttt{false})\ \texttt{true}$
$\texttt{succ} \equiv \lambda n\ f\ x.\ f\ (n\ f\ x)$
$\texttt{add} \equiv \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)$

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Fix Points

$(\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ \ t \longrightarrow_\beta$
$(\lambda f.\ f\ ((\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ f))\ \ t \longrightarrow_\beta$
$t\ \ ((\lambda x\ f.\ f\ (x\ x\ f))\ (\lambda x\ f.\ f\ (x\ x\ f))\ t)$

$\mu = (\lambda x f.\ f\ (x\ x\ f))\ (\lambda x f.\ f\ (x\ x\ f))$
$\mu\ t \longrightarrow_\beta t\ (\mu\ t) \longrightarrow_\beta t\ (t\ (\mu\ t)) \longrightarrow_\beta t\ (t\ (t\ (\mu\ t))) \longrightarrow_\beta \dots$

$(\lambda x f.\ f\ (x\ x\ f))\ (\lambda x f.\ f\ (x\ x\ f))$ is Turing's fix point operator

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Nice, but ...

As a mathematical foundation, λ does not work. **It is inconsistent.**

- **Frege** (Predicate Logic, $\sim$ 1879):
  allows arbitrary quantification over predicates
- **Russell** (1901): Paradox $R \equiv \{X | X \notin X\}$
- **Whitehead & Russell** (Principia Mathematica, 1910-1913):
  Fix the problem
- **Church** (1930): λ calculus as logic, `true`, `false`, $\wedge$, ... as λ terms

	with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x$	$x \in M \equiv M\ x$
Problem:	you can write	$R \equiv \lambda x.\ \texttt{not}\ (x\ x)$	
	and get	$(R\ R) =_\beta \texttt{not}\ (R\ R)$	

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Isabelle Demo

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We have learned so far...

- λ calculus syntax
- free variables, substitution
- β reduction
- α and η conversion
- β reduction is confluent
- λ calculus is very expressive (turing complete)
- λ calculus is inconsistent

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