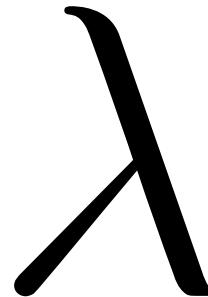

COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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Exercises from last time

- Download and install Isabelle from <http://mirror.cse.unsw.edu.au/pub/isabelle/>
- Step through the demo files from the lecture web page
- Write your own theory file, look at some theorems in the library, try 'find_theorems'

- How many theorems can help you if you need to prove something like “ $\text{Suc}(\text{Suc } x)$ ”?
- What is the name of the theorem for associativity of addition of natural numbers in the library?

Content

- Intro & motivation, getting started [1]

- Foundations & Principles
 - Lambda Calculus, natural deduction [1,2]
 - Higher Order Logic [3^a]
 - Term rewriting [4]

- Proof & Specification Techniques
 - Isar [5]
 - Inductively defined sets, rule induction [6^b]
 - Datatypes, recursion, induction [7^c, 8]
 - Calculational reasoning, code generation [9]
 - Hoare logic, proofs about programs [10^d,11,12]

^a a1 due; ^b a2 due; ^c session break; ^d a3 due

λ -calculus

Alonzo Church

- lived 1903–1995
- supervised people like Alan Turing, Stephen Kleene
- famous for Church-Turing thesis, lambda calculus, first undecidability results
- invented λ calculus in 1930's



λ -calculus

- originally meant as foundation of mathematics
- important applications in theoretical computer science
- foundation of computability and functional programming

untyped λ -calculus

- turing complete model of computation
- a simple way of writing down functions

Basic intuition:

instead of $f(x) = x + 5$
write $f = \lambda x. x + 5$

$\lambda x. x + 5$

- a term
- a nameless function
- that adds 5 to its parameter

Function Application

For applying arguments to functions

instead of $f(a)$

write $f a$

Example: $(\lambda x. x + 5) a$

Evaluating: in $(\lambda x. t) a$ replace x by a in t
(computation!)

Example: $(\lambda x. x + 5) (a + b)$ evaluates to $(a + b) + 5$

THAT'S IT!

NOW FORMAL

Syntax

Terms: $t ::= v \mid c \mid (t t) \mid (\lambda x. t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- v, x variables
- c constants
- $(t t)$ application
- $(\lambda x. t)$ abstraction

Conventions

- leave out parentheses where possible
- list variables instead of multiple λ

Example: instead of $(\lambda y. (\lambda x. (x y)))$ write $\lambda y x. x y$

Rules:

- list variables: $\lambda x. (\lambda y. t) = \lambda x y. t$
- application binds to the left: $x y z = (x y) z \neq x (y z)$
- abstraction binds to the right: $\lambda x. x y = \lambda x. (x y) \neq (\lambda x. x) y$
- leave out outermost parentheses

Getting used to the Syntax

Example:

$$\lambda x y z. x z (y z) =$$

$$\lambda x y z. (x z) (y z) =$$

$$\lambda x y z. ((x z) (y z)) =$$

$$\lambda x. \lambda y. \lambda z. ((x z) (y z)) =$$

$$(\lambda x. (\lambda y. (\lambda z. ((x z) (y z))))))$$

Computation

Intuition: replace parameter by argument
this is called β -reduction

Example

$$(\lambda x y. f (y x)) 5 (\lambda x. x) \longrightarrow_{\beta}$$

$$(\lambda y. f (y 5)) (\lambda x. x) \longrightarrow_{\beta}$$

$$f ((\lambda x. x) 5) \longrightarrow_{\beta}$$

$$f 5$$

Defining Computation

β reduction:

$$\begin{array}{l}
 (\lambda x. s) t \longrightarrow_{\beta} s[x \leftarrow t] \\
 s \longrightarrow_{\beta} s' \implies (s t) \longrightarrow_{\beta} (s' t) \\
 t \longrightarrow_{\beta} t' \implies (s t) \longrightarrow_{\beta} (s t') \\
 s \longrightarrow_{\beta} s' \implies (\lambda x. s) \longrightarrow_{\beta} (\lambda x. s')
 \end{array}$$

Still to do: define $s[x \leftarrow t]$

Defining Substitution

Easy concept. Small problem: variable capture.

Example: $(\lambda x. x z)[z \leftarrow x]$

We do **not** want: $(\lambda x. x x)$ as result.

What do we want?

In $(\lambda y. y z)[z \leftarrow x] = (\lambda y. y x)$ there would be no problem.

So, solution is: rename bound variables.

Free Variables

Bound variables: in $(\lambda x. t)$, x is a bound variable.

Free variables FV of a term:

$$FV(x) = \{x\}$$

$$FV(c) = \{\}$$

$$FV(st) = FV(s) \cup FV(t)$$

$$FV(\lambda x. t) = FV(t) \setminus \{x\}$$

Example: $FV(\lambda x. (\lambda y. (\lambda x. x) y) y x) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

Our problematic substitution example, $(\lambda x. x z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term “ x ”.

Substitution

$$x [x \leftarrow t] = t$$

$$y [x \leftarrow t] = y \quad \text{if } x \neq y$$

$$c [x \leftarrow t] = c$$

$$(s_1 s_2) [x \leftarrow t] = (s_1[x \leftarrow t] s_2[x \leftarrow t])$$

$$(\lambda x. s) [x \leftarrow t] = (\lambda x. s)$$

$$(\lambda y. s) [x \leftarrow t] = (\lambda y. s[x \leftarrow t]) \quad \text{if } x \neq y \text{ and } y \notin FV(t)$$

$$(\lambda y. s) [x \leftarrow t] = (\lambda z. s[y \leftarrow z][x \leftarrow t]) \quad \text{if } x \neq y \\ \text{and } z \notin FV(t) \cup FV(s)$$

Substitution Example

$$\begin{aligned} & (x (\lambda x. x) (\lambda y. z x))[x \leftarrow y] \\ = & (x[x \leftarrow y]) ((\lambda x. x)[x \leftarrow y]) ((\lambda y. z x)[x \leftarrow y]) \\ = & y (\lambda x. x) (\lambda y'. z y) \end{aligned}$$

α Conversion

Bound names are irrelevant:

$\lambda x. x$ and $\lambda y. y$ denote the same function.

α conversion:

$s =_{\alpha} t$ means $s = t$ up to renaming of bound variables.

Formally:

$$\begin{array}{l}
 (\lambda x. t) \longrightarrow_{\alpha} (\lambda y. t[x \leftarrow y]) \text{ if } y \notin FV(t) \\
 s \longrightarrow_{\alpha} s' \implies (s \ t) \longrightarrow_{\alpha} (s' \ t) \\
 t \longrightarrow_{\alpha} t' \implies (s \ t) \longrightarrow_{\alpha} (s \ t') \\
 s \longrightarrow_{\alpha} s' \implies (\lambda x. s) \longrightarrow_{\alpha} (\lambda x. s')
 \end{array}$$

$$s =_{\alpha} t \text{ iff } s \longrightarrow_{\alpha}^* t$$

($\longrightarrow_{\alpha}^*$ = transitive, reflexive closure of \longrightarrow_{α} = multiple steps)

α Conversion

Equality in Isabelle is equality modulo α conversion:

if $s =_{\alpha} t$ then s and t are syntactically equal.

Examples:

$$\begin{aligned} & x (\lambda x y. x y) \\ =_{\alpha} & x (\lambda y x. y x) \\ =_{\alpha} & x (\lambda z y. z y) \\ \neq_{\alpha} & z (\lambda z y. z y) \\ \neq_{\alpha} & x (\lambda x x. x x) \end{aligned}$$

Back to β

We have defined β reduction: \longrightarrow_{β}

Some notation and concepts:

- β **conversion**: $s =_{\beta} t$ iff $\exists n. s \longrightarrow_{\beta}^* n \wedge t \longrightarrow_{\beta}^* n$
- t is **reducible** if there is an s such that $t \longrightarrow_{\beta} s$
- $(\lambda x. s) t$ is called a **redex** (reducible expression)
- t is reducible iff it contains a redex
- if it is not reducible, t is in **normal form**

Does every λ term have a normal form?

No!

Example:

$$(\lambda x. x x) (\lambda x. x x) \longrightarrow_{\beta}$$

$$(\lambda x. x x) (\lambda x. x x) \longrightarrow_{\beta}$$

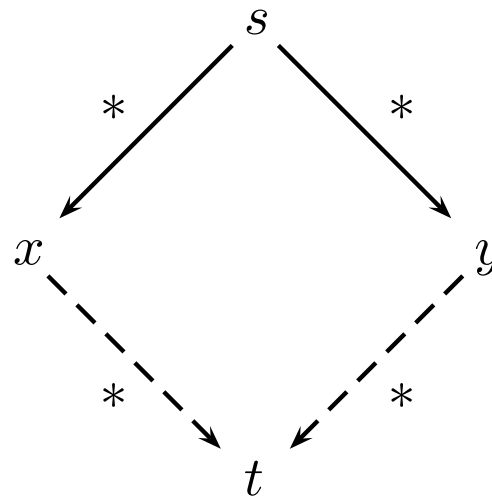
$$(\lambda x. x x) (\lambda x. x x) \longrightarrow_{\beta} \dots$$

$$\text{(but: } (\lambda x y. y) ((\lambda x. x x) (\lambda x. x x)) \longrightarrow_{\beta} \lambda y. y)$$

λ calculus is not terminating

β reduction is confluent

Confluence: $s \longrightarrow_{\beta}^* x \wedge s \longrightarrow_{\beta}^* y \implies \exists t. x \longrightarrow_{\beta}^* t \wedge y \longrightarrow_{\beta}^* t$



Order of reduction does not matter for result
Normal forms in λ calculus are unique

β reduction is confluent



Example:

$$(\lambda x y. y) ((\lambda x. x x) a) \longrightarrow_{\beta} (\lambda x y. y) (a a) \longrightarrow_{\beta} \lambda y. y$$

$$(\lambda x y. y) ((\lambda x. x x) a) \longrightarrow_{\beta} \lambda y. y$$

η Conversion

Another case of trivially equal functions: $t = (\lambda x. t x)$

$$\text{Definition: } \begin{array}{l} s \longrightarrow_{\eta} s' \implies (\lambda x. t x) \longrightarrow_{\eta} t \quad \text{if } x \notin FV(t) \\ t \longrightarrow_{\eta} t' \implies (s t) \longrightarrow_{\eta} (s' t) \\ s \longrightarrow_{\eta} s' \implies (\lambda x. s) \longrightarrow_{\eta} (\lambda x. s') \end{array}$$

$$s =_{\eta} t \quad \text{iff} \quad \exists n. s \longrightarrow_{\eta}^* n \wedge t \longrightarrow_{\eta}^* n$$

Example: $(\lambda x. f x) (\lambda y. g y) \longrightarrow_{\eta} (\lambda x. f x) g \longrightarrow_{\eta} f g$

- η reduction is confluent and terminating.
- $\longrightarrow_{\beta\eta}$ is confluent.
- $\longrightarrow_{\beta\eta}$ means \longrightarrow_{β} and \longrightarrow_{η} steps are both allowed.
- **Equality in Isabelle is also modulo η conversion.**

In fact ...

Equality in Isabelle is modulo α , β , and η conversion.

We will see later why that is possible.

So, what can you do with λ calculus?

λ calculus is very expressive, you can encode:

- logic, set theory
- turing machines, functional programs, etc.

Examples:

$\text{true} \equiv \lambda x y. x$ $\text{if true } x y \longrightarrow_{\beta}^* x$

$\text{false} \equiv \lambda x y. y$ $\text{if false } x y \longrightarrow_{\beta}^* y$

$\text{if} \equiv \lambda z x y. z x y$

Now, not, and, or, etc is easy:

$\text{not} \equiv \lambda x. \text{if } x \text{ false true}$

$\text{and} \equiv \lambda x y. \text{if } x y \text{ false}$

$\text{or} \equiv \lambda x y. \text{if } x \text{ true } y$

More Examples

Encoding natural numbers (Church Numerals)

$$0 \equiv \lambda f x. x$$

$$1 \equiv \lambda f x. f x$$

$$2 \equiv \lambda f x. f (f x)$$

$$3 \equiv \lambda f x. f (f (f x))$$

...

Numeral n takes arguments f and x , applies f n -times to x .

$$\text{iszero} \equiv \lambda n. n (\lambda x. \text{false}) \text{true}$$

$$\text{succ} \equiv \lambda n f x. f (n f x)$$

$$\text{add} \equiv \lambda m n. \lambda f x. m f (n f x)$$

Fix Points

$$(\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t \longrightarrow_{\beta}$$

$$(\lambda f. f ((\lambda x f. f (x x f)) (\lambda x f. f (x x f)) f)) t \longrightarrow_{\beta}$$

$$t ((\lambda x f. f (x x f)) (\lambda x f. f (x x f)) t)$$

$$\mu = (\lambda x f. f (x x f)) (\lambda x f. f (x x f))$$

$$\mu t \longrightarrow_{\beta} t (\mu t) \longrightarrow_{\beta} t (t (\mu t)) \longrightarrow_{\beta} t (t (t (\mu t))) \longrightarrow_{\beta} \dots$$

$(\lambda x f. f (x x f)) (\lambda x f. f (x x f))$ is Turing's fix point operator

Nice, but ...

As a mathematical foundation, λ does not work. **It is inconsistent.**

- **Frege** (Predicate Logic, \sim 1879):
allows arbitrary quantification over predicates
- **Russell** (1901): Paradox $R \equiv \{X \mid X \notin X\}$
- **Whitehead & Russell** (Principia Mathematica, 1910-1913):
Fix the problem
- **Church** (1930): λ calculus as logic, true, false, \wedge , ... as λ terms

with $\{x \mid P x\} \equiv \lambda x. P x$ $x \in M \equiv M x$

Problem: you can write $R \equiv \lambda x. \text{not } (x x)$
and get $(R R) =_{\beta} \text{not } (R R)$

ISABELLE DEMO

We have learned so far...

- λ calculus syntax
- free variables, substitution
- β reduction
- α and η conversion
- β reduction is confluent
- λ calculus is very expressive (turing complete)
- λ calculus is inconsistent