COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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Slide 1

ENOUGH THEORY!
GETTING STARTED WITH ISABELLE

Slide 2

System Architecture

Proof General – user interface

HOL, ZF – object-logics

Isabelle – generic, interactive theorem prover

Standard ML – logic implemented as ADT

User can access all layers!

Slide 3

System Requirements

- ➜ **Linux**, **FreeBSD**, **MacOS X** or **Solaris**
- ➜ **Standard ML**
  (PolyML fastest, SML/NJ supports more platforms)
- ➜ **XEmacs** or **Emacs**
  (for ProofGeneral)

If you have only Windows, try IsaMorph
`http://www.brucker.ch/projects/isamorph/` or
install Cygwin.

Slide 4

Documentation

Available from `http://isabelle.in.tum.de`

➜ Learning Isabelle
- Tutorial on Isabelle/HOL (LNCS 2283)
- Tutorial on Isar
- Tutorial on Locales

➜ Reference Manuals
- Isabelle/Isar Reference Manual
- Isabelle Reference Manual
- Isabelle System Manual

➜ Reference Manuals for Object-Logics

Slide 5

ProofGeneral

➜ User interface for Isabelle
➜ Runs under XEmacs or Emacs
➜ Isabelle process in background

Interaction via

➜ Basic editing in XEmacs (with highlighting etc)
➜ Buttons (tool bar)
➜ Key bindings
➜ ProofGeneral Menu (lots of options, try them)

Slide 6

X-Symbol Cheat Sheet

Input of funny symbols in ProofGeneral

➜ via menu (“X-Symbol”)
➜ via ASCII encoding (similar to LaTeX): \<and>, \<or>, . . .
➜ via abbreviation: /\, \/, -->, . . .
➜ via *rotate*: `l C-.` = λ (cycles through variations of letter)

	∀	∃	λ	¬	∧	∨	⟶	⇒
①	\<forall>	\<exists>	\<lambda>	\<not>	/\	\/	-->	=>
②	ALL	EX	%	~	&	|		

① converted to X-Symbol
② stays ASCII

Slide 7

DEMO

Slide 8

Content

- ➜ Intro & motivation, getting started with Isabelle
- ➜ **Foundations & Principles**
  - **Lambda Calculus**
  - Higher Order Logic, natural deduction
  - Term rewriting
- ➜ Proof & Specification Techniques
  - Datatypes, recursion, induction
  - Inductively defined sets, rule induction
  - Calculational reasoning, mathematics style proofs
  - Hoare logic, proofs about programs

λ-calculus

Alonzo Church

- ➜ lived 1903–1995
- ➜ supervised people like Alan Turing, Stephen Kleene
- ➜ famous for Church-Turing thesis, lambda calculus, first undecidability results
- ➜ invented λ calculus in 1930's

λ-calculus

- ➜ originally meant as foundation of mathematics
- ➜ important applications in theoretical computer science
- ➜ foundation of computability and functional programming

untyped λ-calculus

- ➜ turing complete model of computation
- ➜ a simple way of writing down functions

Basic intuition:

instead of $f(x) = x + 5$
write $f = \lambda x.\ x + 5$

$\lambda x.\ x + 5$

- ➜ a term
- ➜ a nameless function
- ➜ that adds 5 to its parameter

Function Application

For applying arguments to functions

instead of $f(x)$
write $f\ x$

Example: $(\lambda x.\ x + 5)\ a$

Evaluating: in $(\lambda x.\ t)\ a$ replace x by a in t
(computation!)

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

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THAT'S IT!

Slide 13

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NOW FORMAL

Slide 14

Syntax

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Terms: $t \ ::= \ v \ | \ c \ | \ (t\ t) \ | \ (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- ➜ v, x variables
- ➜ c constants
- ➜ $(t\ t)$ application
- ➜ $(\lambda x.\ t)$ abstraction

Slide 15

Conventions

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- ➜ leave out parentheses where possible
- ➜ list variables instead of multiple λ

Example: instead of $(\lambda y.\ (\lambda x.\ (x\ y)))$ write $\lambda y\ x.\ x\ y$

Rules:

- ➜ list variables: $\lambda x.\ (\lambda y.\ t) \ = \ \lambda x\ y.\ t$
- ➜ application binds to the left: $x\ y\ z \ = \ (x\ y)\ z \neq x\ (y\ z)$
- ➜ abstraction binds to the right: $\lambda x.\ x\ y \ = \ \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$
- ➜ leave out outermost parentheses

Slide 16

Getting used to the Syntax

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

$\lambda x\ y\ z.\ (x\ z)\ (y\ z) =$

$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$

$\lambda x.\ \lambda y.\ \lambda z.\ ((x\ z)\ (y\ z)) =$

$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

Slide 17

Computation

Intuition: replace parameter by argument
this is called β-reduction

Example

$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$
$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$
$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$
$f\ 5$

Slide 18

Defining Computation

β **reduction:**

$$\begin{array}{lllclcl} & & & & (\lambda x.\ s)\ t & \longrightarrow_\beta & s[x \leftarrow t] \\ s & \longrightarrow_\beta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\ t & \longrightarrow_\beta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\ s & \longrightarrow_\beta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\beta & (\lambda x.\ s') \end{array}$$

Still to do: define $s[x \leftarrow t]$

Slide 19

Defining Substitution

Easy concept. Small problem: variable capture.

Example: $(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

Slide 20

Free Variables

Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

Free variables FV of a term:

$$\begin{aligned}
FV\ (x) &= \{x\} \\
FV\ (c) &= \{\} \\
FV\ (s\ t) &= FV(s) \cup FV(t) \\
FV\ (\lambda x.\ t) &= FV(t) \setminus \{x\}
\end{aligned}$$

Example: $FV(\quad \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x \quad) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

Our problematic substitution example, $(\lambda x.\ x\ z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term "x".

Slide 21

Substitution

$x\ [x \leftarrow t]$	$= t$	
$y\ [x \leftarrow t]$	$= y$	if $x \neq y$
$c\ [x \leftarrow t]$	$= c$	

$(s_1\ s_2)\ [x \leftarrow t] = (s_1[x \leftarrow t]\ s_2[x \leftarrow t])$

$(\lambda x.\ s)\ [x \leftarrow t]$	$= (\lambda x.\ s)$	
$(\lambda y.\ s)\ [x \leftarrow t]$	$= (\lambda y.\ s[x \leftarrow t])$	if $x \neq y$ and $y \notin FV(t)$
$(\lambda y.\ s)\ [x \leftarrow t]$	$= (\lambda z.\ s[y \leftarrow z][x \leftarrow t])$	if $x \neq y$ and $z \notin FV(t) \cup FV(s)$

Slide 22

Substitution Example

$$\begin{aligned}
& (x\ \ (\lambda x.\ x)\ \ (\lambda y.\ z\ x))[x \leftarrow y] \\
= \ & (x[x \leftarrow y])\ \ ((\lambda x.\ x)[x \leftarrow y])\ \ ((\lambda y.\ z\ x)[x \leftarrow y]) \\
= \ & y\ (\lambda x.\ x)\ (\lambda y'.\ z\ y)
\end{aligned}$$

Slide 23

α Conversion

Bound names are irrelevant:
$\lambda x.\ x$ and $\lambda y.\ y$ denote the same function.

α conversion:
$s =_\alpha t$ means $s = t$ up to renaming of bound variables.

Formally:

$$\begin{array}{ccccccl}
 & & & (\lambda x.\ t) & \longrightarrow_\alpha & (\lambda y.\ t[x \leftarrow y]) & \text{if } y \notin FV(t) \\
s \longrightarrow_\alpha s' & \Longrightarrow & & (s\ t) & \longrightarrow_\alpha & (s'\ t) & \\
t \longrightarrow_\alpha t' & \Longrightarrow & & (s\ t) & \longrightarrow_\alpha & (s\ t') & \\
s \longrightarrow_\alpha s' & \Longrightarrow & & (\lambda x.\ s) & \longrightarrow_\alpha & (\lambda x.\ s') &
\end{array}$$

$s =_\alpha t$ iff $s \longrightarrow_\alpha^* t$

($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

Slide 24

α Conversion

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Equality in Isabelle is equality modulo α conversion:

if $s =_\alpha t$ then s and t are syntactically equal.

Examples:

$$\begin{array}{ll} & x\ (\lambda x\ y.\ x\ y) \\ =_\alpha & x\ (\lambda y\ x.\ y\ x) \\ =_\alpha & x\ (\lambda z\ y.\ z\ y) \\ \neq_\alpha & z\ (\lambda z\ y.\ z\ y) \\ \neq_\alpha & x\ (\lambda x\ x.\ x\ x) \end{array}$$

Slide 25

Back to β

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We have defined β reduction: $\longrightarrow_\beta$

Some notation and concepts:

- ➜ β **conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- ➜ t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- ➜ $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- ➜ t is reducible iff it contains a redex
- ➜ if it is not reducible, t is in **normal form**

Slide 26

Does every λ term have a normal form?

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No!

Example:

$$\begin{array}{l} (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \\ (\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \ldots \end{array}$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \longrightarrow_\beta \lambda y.\ y$)

λ calculus is not terminating

Slide 27

β reduction is confluent

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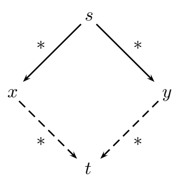
Confluence: $s \longrightarrow_\beta^* x \wedge s \longrightarrow_\beta^* y \Longrightarrow \exists t.\ x \longrightarrow_\beta^* t \wedge y \longrightarrow_\beta^* t$



Order of reduction does not matter for result
Normal forms in λ calculus are unique

Slide 28

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta (\lambda x\ y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a) \longrightarrow_\beta \lambda y.\ y$

Slide 29

η Conversion

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

Definition:

$$\begin{array}{ccccrcll} & & & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t & \text{if } x \notin FV(t) \\ s & \longrightarrow_\eta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s'\ t) & \\ t & \longrightarrow_\eta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s\ t') & \\ s & \longrightarrow_\eta & s' & \Longrightarrow & (\lambda x.\ s) & \longrightarrow_\eta & (\lambda x.\ s') & \end{array}$$

$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow^*_\eta n \wedge t \longrightarrow^*_\eta n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta (\lambda x.\ f\ x)\ g \longrightarrow_\eta f\ g$

- ➜ η reduction is confluent and terminating.
- ➜ $\longrightarrow_{\beta\eta}$ is confluent.
  $\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.
- ➜ **Equality in Isabelle is also modulo η conversion.**

Slide 30

In fact ...

Equality in Isabelle is modulo α, β, and η conversion.

We will see later why that is possible.

Slide 31

Exercises

- ➜ Download and install Isabelle from `http://mirror.cse.unsw.edu.au/pub/isabelle/`
- ➜ Switch on X-Symbol in ProofGeneral
- ➜ Step through the demo files from the lecture web page
- ➜ Write your own theory file, look at some theorems in the library, try 'find theorem'

Slide 32