Advanced Algorithms
COMP4121

Aleks Ignjatović

School of Computer Science and Engineering
University of New South Wales

Introduction to signal processing and MP3

Infinitely dimensional vector spaces

- So far our vectors were elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ where n is an integer.
- Such spaces are of dimension n, which means that they have a basis of size n.
- However, there are also infinitely dimensional vector spaces which are of extreme importance to engineering.
- The first example of such spaces are the real valued or complex valued functions of a real variable which are $2a$ periodic, where $a \in \mathbb{R}$.
- Often we will pay attention only to one period of such a function $f(t)$, by restricting $f(t)$ to $[-a, a]$.

Infinitely dimensional vector spaces

- So far our vectors were elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ where n is an integer.
- Such spaces are of dimension n, which means that they have a basis of size n.
- However, there are also infinitely dimensional vector spaces which are of extreme importance to engineering.
- The first example of such spaces are the real valued or complex valued functions of a real variable which are $2a$ periodic, where $a \in \mathbb{R}$.
- Often we will pay attention only to one period of such a function $f(t)$, by restricting $f(t)$ to $[-a, a]$.

Infinitely dimensional vector spaces

- So far our vectors were elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ where n is an integer.
- Such spaces are of dimension n, which means that they have a basis of size n.
- However, there are also infinitely dimensional vector spaces which are of extreme importance to engineering.
- The first example of such spaces are the real valued or complex valued functions of a real variable which are $2a$ periodic, where $a \in \mathbb{R}$.
- Often we will pay attention only to one period of such a function $f(t)$, by restricting $f(t)$ to $[-a, a]$.

Infinitely dimensional vector spaces

- So far our vectors were elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ where n is an integer.
- Such spaces are of dimension n, which means that they have a basis of size n.
- However, there are also infinitely dimensional vector spaces which are of extreme importance to engineering.
- The first example of such spaces are the real valued or complex valued functions of a real variable which are $2a$ periodic, where $a \in \mathbb{R}$.
- Often we will pay attention only to one period of such a function $f(t)$, by restricting $f(t)$ to $[-a, a]$.

Infinitely dimensional vector spaces

- So far our vectors were elements of $\mathbb{R}^n$ or $\mathbb{C}^n$ where n is an integer.
- Such spaces are of dimension n, which means that they have a basis of size n.
- However, there are also infinitely dimensional vector spaces which are of extreme importance to engineering.
- The first example of such spaces are the real valued or complex valued functions of a real variable which are $2a$ periodic, where $a \in \mathbb{R}$.
- Often we will pay attention only to one period of such a function $f(t)$, by restricting $f(t)$ to $[-a, a]$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- On the other hand, every function $f : [-a, a] \mapsto \mathbb{R}$ or $f : [-a, a] \mapsto \mathbb{C}$ can be periodically extended to the whole of $\mathbb{R}$.
- While the theory can be formulated for much more general functions, for practical applications we only consider $2a$-periodic functions $f(t)$ which are piece-wise continuous and have at most finitely many discontinuities in $[-a, a]$.
- We will ignore the values of such functions at points of discontinuity; thus, we will identify all functions which are equal at all points of continuity. We denote the collection of such (classes of) functions as $C_{pw}[-a, a]$.
- Since each piece of such a function is continuous, it is bounded over its domain.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- On the other hand, every function $f : [-a, a] \mapsto \mathbb{R}$ or $f : [-a, a] \mapsto \mathbb{C}$ can be periodically extended to the whole of $\mathbb{R}$.
- While the theory can be formulated for much more general functions, for practical applications we only consider $2a$-periodic functions $f(t)$ which are piece-wise continuous and have at most finitely many discontinuities in $[-a, a]$.
- We will ignore the values of such functions at points of discontinuity; thus, we will identify all functions which are equal at all points of continuity. We denote the collection of such (classes of) functions as $C_{pw}[-a, a]$.
- Since each piece of such a function is continuous, it is bounded over its domain.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- On the other hand, every function $f : [-a, a] \mapsto \mathbb{R}$ or $f : [-a, a] \mapsto \mathbb{C}$ can be periodically extended to the whole of $\mathbb{R}$.
- While the theory can be formulated for much more general functions, for practical applications we only consider $2a$-periodic functions $f(t)$ which are piece-wise continuous and have at most finitely many discontinuities in $[-a, a]$.
- We will ignore the values of such functions at points of discontinuity; thus, we will identify all functions which are equal at all points of continuity. We denote the collection of such (classes of) functions as $C_{pw}[-a, a]$.
- Since each piece of such a function is continuous, it is bounded over its domain.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- On the other hand, every function $f : [-a, a] \mapsto \mathbb{R}$ or $f : [-a, a] \mapsto \mathbb{C}$ can be periodically extended to the whole of $\mathbb{R}$.
- While the theory can be formulated for much more general functions, for practical applications we only consider $2a$-periodic functions $f(t)$ which are piece-wise continuous and have at most finitely many discontinuities in $[-a, a]$.
- We will ignore the values of such functions at points of discontinuity; thus, we will identify all functions which are equal at all points of continuity. We denote the collection of such (classes of) functions as $C_{pw}[-a, a]$.
- Since each piece of such a function is continuous, it is bounded over its domain.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Thus, such functions satisfy

$$\int_{-a}^{a} |f(t)|^2 dt < \infty$$

- A more general class of functions with a more general type of an integral (the Lebesgue integral) which satisfy the above inequality is denoted by $L^2[-a, a]$.
- We now introduce a scalar product on $C_{pw}[-a, a]$ by

$$\langle f, g \rangle = \frac{1}{2a} \int_{-a}^{a} f(t)\overline{g(t)} dt$$

- One can verify that the three axioms for a scalar product are satisfied for all $f, g \in C_{pw}$.
- Such a scalar product induces a norm via

$$\|f\|_2 = \sqrt{\int_{-a}^{a} |f(t)|^2 dt}$$

- Note that this is very similar to how scalar product is defined for finitely dimensional vectors, except that summation over finitely many coordinates is replaced by an "infinitely refined" sum, i.e., an integral.

- Thus, such functions satisfy

$$\int_{-a}^{a} |f(t)|^2 dt < \infty$$

- A more general class of functions with a more general type of an integral (the Lebesgue integral) which satisfy the above inequality is denoted by $L^2[-a, a]$.

- We now introduce a scalar product on $C_{pw}[-a, a]$ by

$$\langle f, g \rangle = \frac{1}{2a} \int_{-a}^{a} f(t)\overline{g(t)} dt$$

- One can verify that the three axioms for a scalar product are satisfied for all $f, g \in C_{pw}$.

- Such a scalar product induces a norm via

$$\|f\|_2 = \sqrt{\int_{-a}^{a} |f(t)|^2 dt}$$

- Note that this is very similar to how scalar product is defined for finitely dimensional vectors, except that summation over finitely many coordinates is replaced by an "infinitely refined" sum, i.e., an integral.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Thus, such functions satisfy

$$\int_{-a}^{a} |f(t)|^2 dt < \infty$$

- A more general class of functions with a more general type of an integral (the Lebesgue integral) which satisfy the above inequality is denoted by $L^2[-a, a]$.
- We now introduce a scalar product on $C_{pw}[-a, a]$ by

$$\langle f, g \rangle = \frac{1}{2a} \int_{-a}^{a} f(t)\overline{g(t)} dt$$

- One can verify that the three axioms for a scalar product are satisfied for all $f, g \in C_{pw}$.
- Such a scalar product induces a norm via

$$\|f\|_2 = \sqrt{\int_{-a}^{a} |f(t)|^2 dt}$$

- Note that this is very similar to how scalar product is defined for finitely dimensional vectors, except that summation over finitely many coordinates is replaced by an "infinitely refined" sum, i.e., an integral.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Thus, such functions satisfy

$$\int_{-a}^{a} |f(t)|^2 dt < \infty$$

- A more general class of functions with a more general type of an integral (the Lebesgue integral) which satisfy the above inequality is denoted by $L^2[-a, a]$.
- We now introduce a scalar product on $C_{pw}[-a, a]$ by

$$\langle f, g \rangle = \frac{1}{2a} \int_{-a}^{a} f(t)\overline{g(t)}dt$$

- One can verify that the three axioms for a scalar product are satisfied for all $f, g \in C_{pw}$.
- Such a scalar product induces a norm via

$$\|f\|_2 = \sqrt{\int_{-a}^{a} |f(t)|^2 dt}$$

- Note that this is very similar to how scalar product is defined for finitely dimensional vectors, except that summation over finitely many coordinates is replaced by an "infinitely refined" sum, i.e., an integral.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Thus, such functions satisfy

$$\int_{-a}^{a} |f(t)|^2 dt < \infty$$

- A more general class of functions with a more general type of an integral (the Lebesgue integral) which satisfy the above inequality is denoted by $L^2[-a, a]$.

- We now introduce a scalar product on $C_{pw}[-a, a]$ by

$$\langle f, g \rangle = \frac{1}{2a} \int_{-a}^{a} f(t)\overline{g(t)} dt$$

- One can verify that the three axioms for a scalar product are satisfied for all $f, g \in C_{pw}$.

- Such a scalar product induces a norm via

$$\|f\|_2 = \sqrt{\int_{-a}^{a} |f(t)|^2 dt}$$

- Note that this is very similar to how scalar product is defined for finitely dimensional vectors, except that summation over finitely many coordinates is replaced by an "infinitely refined" sum, i.e., an integral.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Thus, such functions satisfy

$$\int_{-a}^{a} |f(t)|^2 dt < \infty$$

- A more general class of functions with a more general type of an integral (the Lebesgue integral) which satisfy the above inequality is denoted by $L^2[-a, a]$.

- We now introduce a scalar product on $C_{pw}[-a, a]$ by

$$\langle f, g \rangle = \frac{1}{2a} \int_{-a}^{a} f(t)\overline{g(t)} dt$$

- One can verify that the three axioms for a scalar product are satisfied for all $f, g \in C_{pw}$.

- Such a scalar product induces a norm via

$$\|f\|_2 = \sqrt{\int_{-a}^{a} |f(t)|^2 dt}$$

- Note that this is very similar to how scalar product is defined for finitely dimensional vectors, except that summation over finitely many coordinates is replaced by an "infinitely refined" sum, i.e., an integral.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha|\|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions $1 - 3$ are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)| dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha|\|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions $1 - 3$ are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)|dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha|\|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions $1 - 3$ are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)| dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha| \|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions $1 - 3$ are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)| dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha|\|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions $1 - 3$ are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)| dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha|\|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions $1 - 3$ are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)| dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha| \|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions 1 – 3 are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)| dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- In general, the *norm* $\|v\|$ of a vector v is any function (which plays the role of the "length" of such a vector) which satisfies the following intuitively clear conditions for all vectors u, v:
  1. If $v \neq 0$ then $\|v\| > 0$ and $\|0\| = 0$.
  2. For every scalar α, (either from $\mathbb{R}$ or $\mathbb{C}$) $\|\alpha v\| = |\alpha| \|v\|$;
  3. $\|u + v\| \leq \|u\| + \|v\|$.

- The norm also allows us to measure distances; if two points A, B are given by their corresponding vectors $\vec{a}$ and $\vec{b}$, then $d(A, B) = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\|$.
- Thus, the third condition ensures the triangle inequality for the corresponding distance function: $d(A, B) \leq d(A, C) + d(C, B)$.

- The above conditions 1 – 3 are also satisfied with the following norm on all functions for which the following integral is finite:

$$\|f\|_1 = \int_{-a}^{a} |f(t)| dt$$

- However, unlike the norm $\|f\|_2$, the norm $\|f\|_1$ is not induced by any scalar product function.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- We say that a set of functions $\mathcal{B} = \{\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t), \ldots\}$ is a (Schauder) basis of $C_{pw}[-a, a]$ if for every function $f(t) \in C_{pw}$ there exists a *unique* set scalars $c_0, c_1, \ldots, c_n, \ldots$ such that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = 0$$

- We denote this by

$$f = \sum_{i=0}^{\infty} c_i \varphi_i$$

- Note that this DOES NOT mean that $f(t) = \sum_{i=0}^{\infty} c_i \varphi_i(t)$ for all t, but only that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = \lim_{n\to\infty} \sqrt{\int_{-a}^{a} \left| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right|^2 dt} = 0.$$

- A basis $\mathcal{B}$ of $C_{pw}[-a, a]$ is orthonormal if for all $\varphi_i(t), \varphi_j(t) \in \mathcal{B}$

$$\langle \varphi_i, \varphi_j \rangle = \frac{1}{2a} \int_{-a}^{a} \varphi_i(t) \overline{\varphi_j(t)} dt = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Infinitely dimensional vector space $C_{pw}[-a,a]$

- We say that a set of functions $\mathcal{B} = \{\varphi_0(t), \varphi_1(t), \dots, \varphi_n(t), \dots\}$ is a (Schauder) basis of $C_{pw}[-a,a]$ if for every function $f(t) \in C_{pw}$ there exists a *unique* set scalars $c_0, c_1, \dots, c_n, \dots$ such that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = 0$$

- We denote this by

$$f = \sum_{i=0}^{\infty} c_i \varphi_i$$

- Note that this DOES NOT mean that $f(t) = \sum_{i=0}^{\infty} c_i \varphi_i(t)$ for all t, but only that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = \lim_{n\to\infty} \sqrt{\int_{-a}^{a} \left| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right|^2 dt} = 0.$$

- A basis $\mathcal{B}$ of $C_{pw}[-a,a]$ is orthonormal if for all $\varphi_i(t), \varphi_j(t) \in \mathcal{B}$

$$\langle \varphi_i, \varphi_j \rangle = \frac{1}{2a} \int_{-a}^{a} \varphi_i(t) \overline{\varphi_j(t)} dt = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Infinitely dimensional vector space $C_{pw}[-a, a]$

- We say that a set of functions $\mathcal{B} = \{\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t), \ldots\}$ is a (Schauder) basis of $C_{pw}[-a, a]$ if for every function $f(t) \in C_{pw}$ there exists a *unique* set scalars $c_0, c_1, \ldots, c_n, \ldots$ such that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = 0$$

- We denote this by

$$f = \sum_{i=0}^{\infty} c_i \varphi_i$$

- Note that this DOES NOT mean that $f(t) = \sum_{i=0}^{\infty} c_i \varphi_i(t)$ for all t, but only that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = \lim_{n\to\infty} \sqrt{\int_{-a}^{a} \left| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right|^2 dt} = 0.$$

- A basis $\mathcal{B}$ of $C_{pw}[-a, a]$ is orthonormal if for all $\varphi_i(t), \varphi_j(t) \in \mathcal{B}$

$$\langle \varphi_i, \varphi_j \rangle = \frac{1}{2a} \int_{-a}^{a} \varphi_i(t)\overline{\varphi_j(t)}dt = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Infinitely dimensional vector space $C_{pw}[-a, a]$

- We say that a set of functions $\mathcal{B} = \{\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t), \ldots\}$ is a (Schauder) basis of $C_{pw}[-a, a]$ if for every function $f(t) \in C_{pw}$ there exists a *unique* set scalars $c_0, c_1, \ldots, c_n, \ldots$ such that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = 0$$

- We denote this by

$$f = \sum_{i=0}^{\infty} c_i \varphi_i$$

- Note that this DOES NOT mean that $f(t) = \sum_{i=0}^{\infty} c_i \varphi_i(t)$ for all t, but only that

$$\lim_{n\to\infty} \left\| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right\| = \lim_{n\to\infty} \sqrt{\int_{-a}^{a} \left| f(t) - \sum_{i=0}^{n} c_i \varphi_i(t) \right|^2 dt} = 0.$$

- A basis $\mathcal{B}$ of $C_{pw}[-a, a]$ is orthonormal if for all $\varphi_i(t), \varphi_j(t) \in \mathcal{B}$

$$\langle \varphi_i, \varphi_j \rangle = \frac{1}{2a} \int_{-a}^{a} \varphi_i(t)\overline{\varphi_j(t)} dt = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Recall that for $\omega_n^k = e^{i\,\frac{2\pi\,k}{n}}$ and for $\varphi_k = \frac{1}{\sqrt{n}}((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1})$ we have

$$\langle \varphi_k, \varphi_m \rangle = \frac{1}{n}\sum_{j=0}^{n-1}(\omega_n^k)^j\overline{(\omega_n^m)^j} = \frac{1}{n}\sum_{j=0}^{n-1} e^{i\,k\,\frac{2\pi}{n}j}\overline{e^{i\,m\,\frac{2\pi}{n}j}} = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Remarkably, for $\varphi_k(t) = e^{i\,k\,t}$ we have

$$\langle \varphi_k(t), \varphi_m(t) \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i\,k\,t}\overline{e^{i\,m\,t}}dt = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Thus, functions $\varphi_k(t) = e^{i\,k\,t}$ for $0 \leq k < n$ are orthonormal both under:
  1. a discrete time scalar product defined as a sum of values at instants $0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, 3\frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$ spanning the interval $[0, 2\pi]$;
  2. a continuous time scalar product which is an integral over the interval $[0, 2\pi]$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Recall that for $\omega_n^k = e^{i\ \frac{2\pi\ k}{n}}$ and for $\varphi_k = \frac{1}{\sqrt{n}}((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1})$ we have

$$\langle \varphi_k, \varphi_m \rangle = \frac{1}{n}\sum_{j=0}^{n-1}(\omega_n^k)^j \overline{(\omega_n^m)^j} = \frac{1}{n}\sum_{j=0}^{n-1} e^{i\ k\ \frac{2\pi}{n} j}\overline{e^{i\ m\ \frac{2\pi}{n} j}} = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Remarkably, for $\varphi_k(t) = e^{i\ k\ t}$ we have

$$\langle \varphi_k(t), \varphi_m(t) \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i\ k\ t}\overline{e^{i\ m\ t}}dt = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Thus, functions $\varphi_k(t) = e^{i\ k\ t}$ for $0 \leq k < n$ are orthonormal both under:
  1. a discrete time scalar product defined as a sum of values at instants $0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, 3\frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$ spanning the interval $[0, 2\pi]$;
  2. a continuous time scalar product which is an integral over the interval $[0, 2\pi]$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Recall that for $\omega_n^k = e^{i\,\frac{2\pi\,k}{n}}$ and for $\varphi_k = \frac{1}{\sqrt{n}}((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1})$ we have

$$\langle \varphi_k, \varphi_m \rangle = \frac{1}{n}\sum_{j=0}^{n-1} (\omega_n^k)^j \overline{(\omega_n^m)^j} = \frac{1}{n}\sum_{j=0}^{n-1} e^{i\,k\,\frac{2\pi}{n}\,j}\overline{e^{i\,m\,\frac{2\pi}{n}\,j}} = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Remarkably, for $\varphi_k(t) = e^{i\,k\,t}$ we have

$$\langle \varphi_k(t), \varphi_m(t) \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i\,k\,t}\overline{e^{i\,m\,t}}dt = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Thus, functions $\varphi_k(t) = e^{i\,k\,t}$ for $0 \leq k < n$ are orthonormal both under:
  1. a discrete time scalar product defined as a sum of values at instants $0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, 3\frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$ spanning the interval $[0, 2\pi]$;
  2. a continuous time scalar product which is an integral over the interval $[0, 2\pi]$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Recall that for $\omega_n^k = e^{i\,\frac{2\pi\,k}{n}}$ and for $\varphi_k = \frac{1}{\sqrt{n}}((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1})$ we have

$$\langle \varphi_k, \varphi_m \rangle = \frac{1}{n}\sum_{j=0}^{n-1}(\omega_n^k)^j\overline{(\omega_n^m)^j} = \frac{1}{n}\sum_{j=0}^{n-1} e^{i\,k\,\frac{2\pi}{n}\,j}\overline{e^{i\,m\,\frac{2\pi}{n}\,j}} = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Remarkably, for $\varphi_k(t) = e^{i\,k\,t}$ we have

$$\langle \varphi_k(t), \varphi_m(t) \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i\,k\,t}\overline{e^{i\,m\,t}}dt = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Thus, functions $\varphi_k(t) = e^{i\,k\,t}$ for $0 \leq k < n$ are orthonormal both under:
  1. a discrete time scalar product defined as a sum of values at instants $0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, 3\frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$ spanning the interval $[0, 2\pi]$;
  2. a continuous time scalar product which is an integral over the interval $[0, 2\pi]$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Recall that for $\omega_n^k = e^{i\,\frac{2\pi\,k}{n}}$ and for $\varphi_k = \frac{1}{\sqrt{n}}((\omega_n^k)^0, (\omega_n^k)^1, \ldots, (\omega_n^k)^{n-1})$ we have

$$\langle \varphi_k, \varphi_m \rangle = \frac{1}{n}\sum_{j=0}^{n-1} (\omega_n^k)^j \overline{(\omega_n^m)^j} = \frac{1}{n}\sum_{j=0}^{n-1} e^{i\,k\,\frac{2\pi}{n}\,j}\overline{e^{i\,m\,\frac{2\pi}{n}\,j}} = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Remarkably, for $\varphi_k(t) = e^{i\,k\,t}$ we have

$$\langle \varphi_k(t), \varphi_m(t) \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i\,k\,t}\overline{e^{i\,m\,t}}dt = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases}$$

- Thus, functions $\varphi_k(t) = e^{i\,k\,t}$ for $0 \leq k < n$ are orthonormal both under:
  1. a discrete time scalar product defined as a sum of values at instants $0, \frac{2\pi}{n}, 2\frac{2\pi}{n}, 3\frac{2\pi}{n}, \ldots, (n-1)\frac{2\pi}{n}$ spanning the interval $[0, 2\pi]$;
  2. a continuous time scalar product which is an integral over the interval $[0, 2\pi]$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Just as in the finitely dimensional case, we have that for all $f(t) \in C_{pw}$,

$$f(t) = \sum_{n=0}^{\infty} \langle f, \varphi_n \rangle \varphi_n(t)$$

- This is again in the sense that

$$\left\| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right|^2 dt \to 0 \quad \text{as } n \to \infty$$

- At all points t_0 where $f(t)$ is differentiable we also have

$$\sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t_0) \to f(t_0) \quad \text{as } n \to \infty$$

- For functions with period $2a$ the basis functions have to be rescaled accordingly to make their periods equal to an integer fraction of $2a$, by setting $\varphi_n(t) = e^{i\, n\, \frac{\pi}{a} t}$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Just as in the finitely dimensional case, we have that for all $f(t) \in C_{pw}$,

$$f(t) = \sum_{n=0}^{\infty} \langle f, \varphi_n \rangle \varphi_n(t)$$

- This is again in the sense that

$$\left\| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right|^2 dt \to 0 \quad \text{as } n \to \infty$$

- At all points t_0 where $f(t)$ is differentiable we also have

$$\sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t_0) \to f(t_0) \quad \text{as } n \to \infty$$

- For functions with period $2a$ the basis functions have to be rescaled accordingly to make their periods equal to an integer fraction of $2a$, by setting $\varphi_n(t) = e^{i\, n\, \frac{\pi}{a} t}$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Just as in the finitely dimensional case, we have that for all $f(t) \in C_{pw}$,

$$f(t) = \sum_{n=0}^{\infty} \langle f, \varphi_n \rangle \varphi_n(t)$$

- This is again in the sense that

$$\left\| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right|^2 dt \to 0 \quad \text{as } n \to \infty$$

- At all points t_0 where $f(t)$ is differentiable we also have

$$\sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t_0) \to f(t_0) \quad \text{as } n \to \infty$$

- For functions with period $2a$ the basis functions have to be rescaled accordingly to make their periods equal to an integer fraction of $2a$, by setting $\varphi_n(t) = e^{i\, n\, \frac{\pi}{a} t}$.

Infinitely dimensional vector space $C_{pw}[-a, a]$

- Just as in the finitely dimensional case, we have that for all $f(t) \in C_{pw}$,

$$f(t) = \sum_{n=0}^{\infty} \langle f, \varphi_n \rangle \varphi_n(t)$$

- This is again in the sense that

$$\left\| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(t) - \sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t) \right|^2 dt \to 0 \quad \text{as } n \to \infty$$

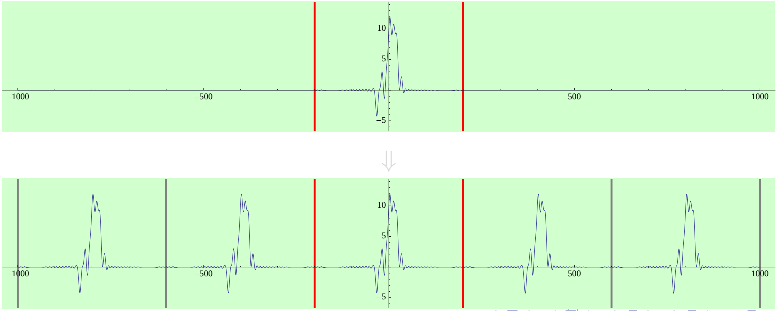
- At all points t_0 where $f(t)$ is differentiable we also have

$$\sum_{k=0}^{n} \langle f, \varphi_n \rangle \varphi_n(t_0) \to f(t_0) \quad \text{as } n \to \infty$$

- For functions with period $2a$ the basis functions have to be rescaled accordingly to make their periods equal to an integer fraction of $2a$, by setting $\varphi_n(t) = e^{i\,n\,\frac{\pi}{a}\,t}$.

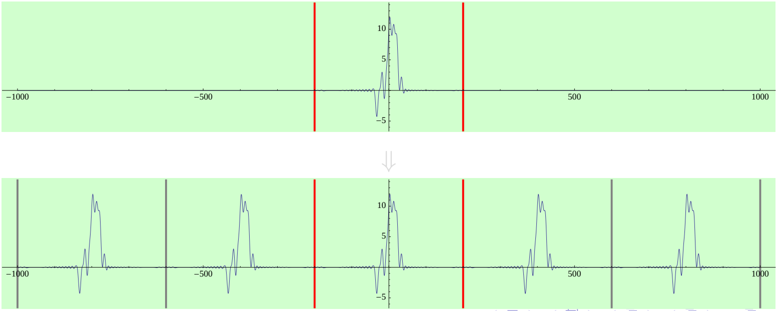
Fourier Transform

- What happens if $f(t)$ is not periodic, but decays with time?
- In this case a representation of a signal in a way which is somewhat analogous to the Fourier series for the periodic functions, is accomplished using the Fourier transform.
- Since the rigorous mathematical development is tricky, rather than doing things rigorously, we will only motivate the Fourier transform heuristically.
- So assume that $f(t)$ is vanishingly small outside $[-L, L]$, where L is chosen sufficiently large, and then extend $f(t)$ periodically outside $(-L, L]$, simply by making shifted copies of $f(t)$ over $(-L, L]$, thus obtaining $f_L(t)$ which is $2L$ periodic; see Figure below.



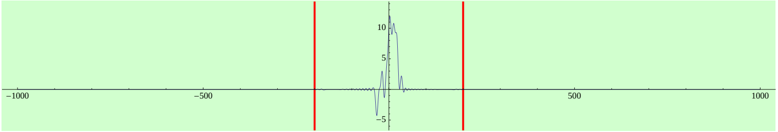
Fourier Transform

- What happens if $f(t)$ is not periodic, but decays with time?
- In this case a representation of a signal in a way which is somewhat analogous to the Fourier series for the periodic functions, is accomplished using the Fourier transform.
- Since the rigorous mathematical development is tricky, rather than doing things rigorously, we will only motivate the Fourier transform heuristically.
- So assume that $f(t)$ is vanishingly small outside $[-L, L]$, where L is chosen sufficiently large, and then extend $f(t)$ periodically outside $(-L, L]$, simply by making shifted copies of $f(t)$ over $(-L, L]$, thus obtaining $f_L(t)$ which is $2L$ periodic; see Figure below.

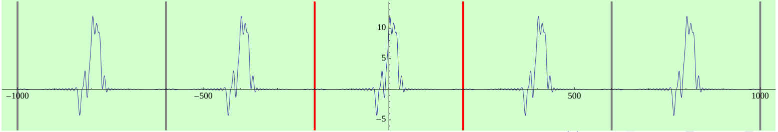


Fourier Transform

- What happens if $f(t)$ is not periodic, but decays with time?
- In this case a representation of a signal in a way which is somewhat analogous to the Fourier series for the periodic functions, is accomplished using the Fourier transform.
- Since the rigorous mathematical development is tricky, rather than doing things rigorously, we will only motivate the Fourier transform heuristically.
- So assume that $f(t)$ is vanishingly small outside $[-L, L]$, where L is chosen sufficiently large, and then extend $f(t)$ periodically outside $(-L, L]$, simply by making shifted copies of $f(t)$ over $(-L, L]$, thus obtaining $f_L(t)$ which is $2L$ periodic; see Figure below.

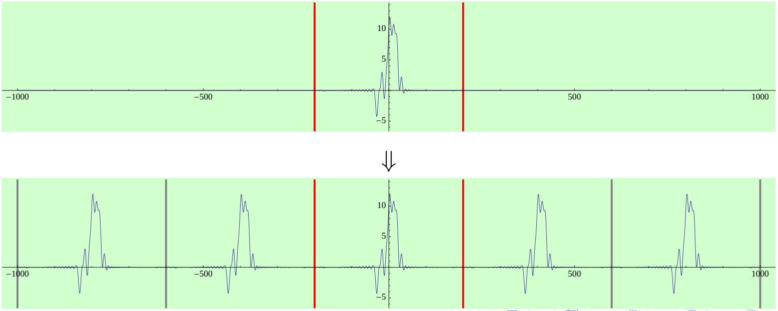


⇓



Fourier Transform

- What happens if $f(t)$ is not periodic, but decays with time?
- In this case a representation of a signal in a way which is somewhat analogous to the Fourier series for the periodic functions, is accomplished using the Fourier transform.
- Since the rigorous mathematical development is tricky, rather than doing things rigorously, we will only motivate the Fourier transform heuristically.
- So assume that $f(t)$ is vanishingly small outside $[-L, L]$, where L is chosen sufficiently large, and then extend $f(t)$ periodically outside $(-L, L]$, simply by making shifted copies of $f(t)$ over $(-L, L]$, thus obtaining $f_L(t)$ which is $2L$ periodic; see Figure below.



Fourier Transform

- We can now expand such a $2L$-periodic function $f_L(t)$ into Fourier series in the basis consisting of functions $\varphi_k(t) = e^{i\,\frac{k\pi}{L}\,t}$:

$$f_L(t) = \sum_{k=-\infty}^{\infty} c_k^L e^{i\,\frac{k\pi}{L}\,t} \tag{1}$$

where

$$c_k^L = \langle f, e^{i\,k\,\frac{\pi}{L}\,t}\rangle = \frac{1}{2L}\int_{-L}^{L} f_L(t)e^{-\,i\,\frac{k\pi}{L}\,t}dt \approx \frac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-\,i\,\frac{k\pi}{L}\,t}dt, \tag{2}$$

because $f(t)$ is small outside $[-L, L]$.

- Let us now define the *Fourier transform* $\hat{f}(\omega)$ of $f(t)$ as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\,i\,\omega t}dt. \tag{3}$$

- Then, comparing (2) and (3), we see that $c_k^L \approx \frac{1}{2L}\hat{f}(k\pi/L)$, and substituting in (1) we get

$$f_L(t) \approx \sum_{k=-\infty}^{\infty} \frac{1}{2L}\hat{f}(k\pi/L)e^{i\,k\,\frac{\pi}{L}\,t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{\pi}{L}\hat{f}(k\pi/L)e^{i\,k\,\frac{\pi}{L}\,t}.$$

Fourier Transform

- We can now expand such a $2L$-periodic function $f_L(t)$ into Fourier series in the basis consisting of functions $\varphi_k(t) = e^{i\,\frac{k\pi}{L}\,t}$:

$$f_L(t) = \sum_{k=-\infty}^{\infty} c_k^L e^{i\,\frac{k\pi}{L}\,t} \tag{1}$$

where

$$c_k^L = \langle f, e^{i\,k\,\frac{\pi}{L}\,t}\rangle = \frac{1}{2L}\int_{-L}^{L} f_L(t)e^{-\,i\,\frac{k\pi}{L}\,t}dt \approx \frac{1}{2L}\int_{-\infty}^{\infty} f(t)e^{-\,i\,\frac{k\pi}{L}\,t}dt, \tag{2}$$

because $f(t)$ is small outside $[-L, L]$.

- Let us now define the *Fourier transform* $\hat{f}(\omega)$ of $f(t)$ as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\,i\,\omega t}dt. \tag{3}$$

- Then, comparing (2) and (3), we see that $c_k^L \approx \frac{1}{2L}\hat{f}(k\pi/L)$, and substituting in (1) we get

$$f_L(t) \approx \sum_{k=-\infty}^{\infty} \frac{1}{2L}\hat{f}(k\pi/L)e^{i\,k\,\frac{\pi}{L}\,t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{\pi}{L}\hat{f}(k\pi/L)e^{i\,k\,\frac{\pi}{L}\,t}.$$

- We can now expand such a $2L$-periodic function $f_L(t)$ into Fourier series in the basis consisting of functions $\varphi_k(t) = e^{i\,\frac{k\pi}{L}\,t}$:

$$f_L(t) = \sum_{k=-\infty}^{\infty} c_k^L\, e^{i\,\frac{k\pi}{L}\,t} \tag{1}$$

where

$$c_k^L = \langle f, e^{i\;k\,\frac{\pi}{L}\,t}\rangle = \frac{1}{2L}\int_{-L}^{L} f_L(t) e^{-\,i\,\frac{k\pi}{L}\,t} dt \approx \frac{1}{2L}\int_{-\infty}^{\infty} f(t) e^{-\,i\,\frac{k\pi}{L}\,t} dt, \tag{2}$$

because $f(t)$ is small outside $[-L, L]$.

- Let us now define the *Fourier transform* $\hat{f}(\omega)$ of $f(t)$ as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\,i\,\omega t} dt. \tag{3}$$

- Then, comparing (2) and (3), we see that $c_k^L \approx \frac{1}{2L}\hat{f}(k\pi/L)$, and substituting in (1) we get

$$f_L(t) \approx \sum_{k=-\infty}^{\infty} \frac{1}{2L}\hat{f}(k\pi/L) e^{i\,k\,\frac{\pi}{L}\,t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \frac{\pi}{L}\hat{f}(k\pi/L) e^{i\,k\,\frac{\pi}{L}\,t}.$$

Fourier Transform

- However,

$$f_L(t) \approx \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\pi}{L} \hat{f}\left(k\frac{\pi}{L}\right) e^{i\,k\frac{\pi}{L}t}.$$

  is the Riemann sum of an integral:

$$f_L(t) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega,$$

  with $\Delta x = \pi/L$ and with approximations becoming better as L becomes larger.
- Thus, we can expect that in the limit

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

- However,

$$f_L(t) \approx \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\pi}{L} \hat{f}\left(k\frac{\pi}{L}\right) e^{i\,k\,\frac{\pi}{L}\,t}.$$

  is the Riemann sum of an integral:

$$f_L(t) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega,$$

  with $\Delta x = \pi/L$ and with approximations becoming better as L becomes larger.
- Thus, we can expect that in the limit

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

Fourier Transform

- Compare now

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\,\omega t} dt \qquad \text{with} \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

- Just as in the case of the Discrete Fourier Transform the "forward" transform and the inverse transform differ only in the sign of the exponent $e^{\pm\,i\,\omega t}$ and the normalising factor $1/2\pi$, but the second one can be fixed by defining

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\,i\,\omega t} dt \qquad \Leftrightarrow \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

- If we represent $\widehat{f}(\omega)$ in polar form, $\widehat{f}(\omega) = \rho(\omega) e^{i\,\theta(\omega)}$ we obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i\,\theta(\omega)} e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i(\omega t + \theta(\omega))} d\omega$$

- Thus, just as for the Discrete Fourier Transform, $\rho(\omega)$ is the amplitude and $\theta(\omega)$ is the phase shift for $e^{i\,\omega t}$ harmonic oscillation of frequency ω.
- So the signal $f(t)$ is represented as an "infinitely refined" sum (i.e., an integral) of pure harmonic oscillations.

Fourier Transform

- Compare now

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad \text{with} \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

- Just as in the case of the Discrete Fourier Transform the "forward" transform and the inverse transform differ only in the sign of the exponent $e^{\pm i\omega t}$ and the normalising factor $1/2\pi$, but the second one can be fixed by defining

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad \Leftrightarrow \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

- If we represent $\widehat{f}(\omega)$ in polar form, $\widehat{f}(\omega) = \rho(\omega) e^{i\theta(\omega)}$ we obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i\theta(\omega)} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i(\omega t + \theta(\omega))} d\omega$$

- Thus, just as for the Discrete Fourier Transform, $\rho(\omega)$ is the amplitude and $\theta(\omega)$ is the phase shift for $e^{i\omega t}$ harmonic oscillation of frequency ω.
- So the signal $f(t)$ is represented as an "infinitely refined" sum (i.e., an integral) of pure harmonic oscillations.

Fourier Transform

- Compare now

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad \text{with} \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

- Just as in the case of the Discrete Fourier Transform the "forward" transform and the inverse transform differ only in the sign of the exponent $e^{\pm i\omega t}$ and the normalising factor $1/2\pi$, but the second one can be fixed by defining

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad \Leftrightarrow \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega.$$

- If we represent $\widehat{f}(\omega)$ in polar form, $\widehat{f}(\omega) = \rho(\omega) e^{i\theta(\omega)}$ we obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i\theta(\omega)} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i(\omega t + \theta(\omega))} d\omega$$

- Thus, just as for the Discrete Fourier Transform, $\rho(\omega)$ is the amplitude and $\theta(\omega)$ is the phase shift for $e^{i\omega t}$ harmonic oscillation of frequency ω.
- So the signal $f(t)$ is represented as an "infinitely refined" sum (i.e., an integral) of pure harmonic oscillations.

Fourier Transform

- Compare now

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\,i\,\omega t} dt \qquad \text{with} \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

- Just as in the case of the Discrete Fourier Transform the "forward" transform and the inverse transform differ only in the sign of the exponent $e^{\pm\,i\,\omega t}$ and the normalising factor $1/2\pi$, but the second one can be fixed by defining

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\,i\,\omega t} dt \qquad \Leftrightarrow \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

- If we represent $\widehat{f}(\omega)$ in polar form, $\widehat{f}(\omega) = \rho(\omega) e^{i\,\theta(\omega)}$ we obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i\,\theta(\omega)} e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i(\omega t + \theta(\omega))} d\omega$$

- Thus, just as for the Discrete Fourier Transform, $\rho(\omega)$ is the amplitude and $\theta(\omega)$ is the phase shift for $e^{i\,\omega t}$ harmonic oscillation of frequency ω.
- So the signal $f(t)$ is represented as an "infinitely refined" sum (i.e., an integral) of pure harmonic oscillations.

Fourier Transform

- Compare now

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\,\omega t} dt \qquad \text{with} \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

- Just as in the case of the Discrete Fourier Transform the "forward" transform and the inverse transform differ only in the sign of the exponent $e^{\pm i\,\omega t}$ and the normalising factor $1/2\pi$, but the second one can be fixed by defining

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\,\omega t} dt \qquad \Leftrightarrow \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega.$$

- If we represent $\widehat{f}(\omega)$ in polar form, $\widehat{f}(\omega) = \rho(\omega) e^{i\,\theta(\omega)}$ we obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i\,\theta(\omega)} e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega) e^{i(\omega t + \theta(\omega))} d\omega$$

- Thus, just as for the Discrete Fourier Transform, $\rho(\omega)$ is the amplitude and $\theta(\omega)$ is the phase shift for $e^{i\,\omega t}$ harmonic oscillation of frequency ω.
- So the signal $f(t)$ is represented as an "infinitely refined" sum (i.e., an integral) of pure harmonic oscillations.

Bandlimited signals of finite energy

- Most signals of engineering interest, due to physical limitations, cannot contain arbitrarily high frequencies.
- Sometimes we are simply not interested in frequencies above certain limit, treating them as unwanted noise.
- For example, we cannot hear sounds of frequencies above $20kHz$ so they are irrelevant for audio applications and thus they can be safely removed from an audio signal without producing any noticeable side effects.
- Assume that $f(t)$ has a *finite bandwidth*; we will chose a unit interval of time so that *all frequencies present in the signal are smaller than* $\pm\pi$, i.e., that

$$\hat{f}(\omega) = 0 \quad \text{if} \quad |\omega| \geq \pi.$$

- It might look paradoxical that frequencies can be negative. While this is mainly a technical convenience which makes signal representation considerably simpler, one can also remember that a wheel can spin with certain angular velocity in two opposite directions.

Bandlimited signals of finite energy

- Most signals of engineering interest, due to physical limitations, cannot contain arbitrarily high frequencies.
- Sometimes we are simply not interested in frequencies above certain limit, treating them as unwanted noise.
- For example, we cannot hear sounds of frequencies above $20kHz$ so they are irrelevant for audio applications and thus they can be safely removed from an audio signal without producing any noticeable side effects.
- Assume that $f(t)$ has a *finite bandwidth*; we will chose a unit interval of time so that *all frequencies present in the signal are smaller than* $\pm\pi$, i.e., that

$$\hat{f}(\omega) = 0 \quad \text{if} \quad |\omega| \geq \pi.$$

- It might look paradoxical that frequencies can be negative. While this is mainly a technical convenience which makes signal representation considerably simpler, one can also remember that a wheel can spin with certain angular velocity in two opposite directions.

Bandlimited signals of finite energy

- Most signals of engineering interest, due to physical limitations, cannot contain arbitrarily high frequencies.
- Sometimes we are simply not interested in frequencies above certain limit, treating them as unwanted noise.
- For example, we cannot hear sounds of frequencies above $20kHz$ so they are irrelevant for audio applications and thus they can be safely removed from an audio signal without producing any noticeable side effects.
- Assume that $f(t)$ has a *finite bandwidth*; we will chose a unit interval of time so that *all frequencies present in the signal are smaller than* $\pm\pi$, i.e., that

$$\hat{f}(\omega) = 0 \quad \text{if} \quad |\omega| \geq \pi.$$

- It might look paradoxical that frequencies can be negative. While this is mainly a technical convenience which makes signal representation considerably simpler, one can also remember that a wheel can spin with certain angular velocity in two opposite directions.

Bandlimited signals of finite energy

- Most signals of engineering interest, due to physical limitations, cannot contain arbitrarily high frequencies.
- Sometimes we are simply not interested in frequencies above certain limit, treating them as unwanted noise.
- For example, we cannot hear sounds of frequencies above $20kHz$ so they are irrelevant for audio applications and thus they can be safely removed from an audio signal without producing any noticeable side effects.
- Assume that $f(t)$ has a *finite bandwidth*; we will chose a unit interval of time so that *all frequencies present in the signal are smaller than* $\pm\pi$, i.e., that

$$\hat{f}(\omega) = 0 \quad \text{if} \quad |\omega| \geq \pi.$$

- It might look paradoxical that frequencies can be negative. While this is mainly a technical convenience which makes signal representation considerably simpler, one can also remember that a wheel can spin with certain angular velocity in two opposite directions.

Bandlimited signals of finite energy

- Most signals of engineering interest, due to physical limitations, cannot contain arbitrarily high frequencies.
- Sometimes we are simply not interested in frequencies above certain limit, treating them as unwanted noise.
- For example, we cannot hear sounds of frequencies above $20kHz$ so they are irrelevant for audio applications and thus they can be safely removed from an audio signal without producing any noticeable side effects.
- Assume that $f(t)$ has a *finite bandwidth*; we will chose a unit interval of time so that *all frequencies present in the signal are smaller than* $\pm\pi$, i.e., that

$$\hat{f}(\omega) = 0 \quad \text{if} \quad |\omega| \geq \pi.$$

- It might look paradoxical that frequencies can be negative. While this is mainly a technical convenience which makes signal representation considerably simpler, one can also remember that a wheel can spin with certain angular velocity in two opposite directions.

Bandlimited signals of finite energy

- We will also assume that signals have *finite energy*, i.e., that their usual L^2 norm is finite:

$$\|f\|_2 = \sqrt{\int_{-\infty}^{\infty} f(t)^2 \, dt} < \infty$$

- The space of such signals, with the usual scalar product and the norm of L^2 are called *the space of band limited signals of finite energy* or the Paley-Wiener space and is usually denoted by $BL(\pi)$ or PW_π.

Bandlimited signals of finite energy

- We will also assume that signals have *finite energy*, i.e., that their usual L^2 norm is finite:

$$\|f\|_2 = \sqrt{\int_{-\infty}^{\infty} f(t)^2 \, dt} < \infty$$

- The space of such signals, with the usual scalar product and the norm of L^2 are called *the space of band limited signals of finite energy* or the Paley-Wiener space and is usually denoted by $BL(\pi)$ or PW_π.

Bandlimited signals of finite energy

- Since $\hat{f}(\omega) = 0$ outside $(-\pi, \pi)$, we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\,\omega t}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)e^{i\,\omega t}d\omega \qquad (4)$$

- We can now represent $\hat{f}(\omega)$ over the interval $(-\pi, \pi)$ by its Fourier series:

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{i\,k\omega},$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)e^{-\,i\,k\omega}d\omega.$$

- Comparing this equation with (4), we see that $c_k = f(-k)$, i.e., the Fourier coefficients are just the samples of the signal!

Bandlimited signals of finite energy

- Since $\hat{f}(\omega) = 0$ outside $(-\pi, \pi)$, we have

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\,\omega t} d\omega \tag{4}$$

- We can now represent $\hat{f}(\omega)$ over the interval $(-\pi, \pi)$ by its Fourier series:

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{i\,k\omega},$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{-\,i\,k\omega} d\omega.$$

- Comparing this equation with (4), we see that $c_k = f(-k)$, i.e., the Fourier coefficients are just the samples of the signal!

Bandlimited signals of finite energy

- Since $\hat{f}(\omega) = 0$ outside $(-\pi, \pi)$, we have

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\,\omega t} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\,\omega t} d\omega \tag{4}$$

- We can now represent $\hat{f}(\omega)$ over the interval $(-\pi, \pi)$ by its Fourier series:

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{i\,k\omega},$$

  where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{-\,i\,k\omega} d\omega.$$

- Comparing this equation with (4), we see that $c_k = f(-k)$, i.e., the Fourier coefficients are just the samples of the signal!

Bandlimited signals of finite energy

- Thus, the Fourier transform of a π-band limited signal of finite energy is

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} f(-k)e^{i\,k\omega}. \tag{5}$$

- We can now substitute $\hat{f}(\omega)$ in

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)e^{i\,\omega t}d\omega$$

with its Fourier series and obtain

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} f(-k)e^{i\,k\omega}\right)e^{i\,\omega t}d\omega$$

- We use the substitution $k = -n$ and obtain

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} f(n)e^{i\,\omega(t-n)}d\omega$$

Bandlimited signals of finite energy

- Thus, the Fourier transform of a π-band limited signal of finite energy is

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} f(-k)e^{i\,k\omega}. \tag{5}$$

- We can now substitute $\hat{f}(\omega)$ in

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)e^{i\,\omega t}d\omega$$

with its Fourier series and obtain

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} f(-k)e^{i\,k\omega}\right) e^{i\,\omega t}d\omega$$

- We use the substitution $k = -n$ and obtain

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(n)e^{i\,\omega(t-n)}d\omega$$

Bandlimited signals of finite energy

- Thus, the Fourier transform of a π-band limited signal of finite energy is

$$\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} f(-k) e^{i\,k\omega}. \tag{5}$$

- We can now substitute $\hat{f}(\omega)$ in

$$f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{i\,\omega t} d\omega$$

with its Fourier series and obtain

$$f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} f(-k) e^{i\,k\omega} \right) e^{i\,\omega t} d\omega$$

- We use the substitution $k = -n$ and obtain

$$f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(n) e^{i\,\omega(t-n)} d\omega$$

Bandlimited signals of finite energy

- From

$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(n)e^{i\,\omega(t-n)}d\omega$$

integrating term by term we obtain

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\left[\frac{e^{i\,\omega(t-n)}}{i(t-n)}\right]_{\omega=-\pi}^{\omega=\pi} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\left[\frac{\cos\omega(t-n)+i\sin\omega(t-n)}{i(t-n)}\right]_{\omega=-\pi}^{\omega=\pi} \end{aligned}$$

- Note that $\cos\pi(t-n) - \cos(-\pi(t-n)) = 0$ while $\sin\pi(t-n) - \sin(-\pi(t-n)) = 2\sin\pi(t-n)$ and so we get

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}.$$

Bandlimited signals of finite energy

- From
$$f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} f(n)e^{i\,\omega(t-n)}d\omega$$
integrating term by term we obtain
$$\begin{aligned} f(t) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\left[\frac{e^{i\,\omega(t-n)}}{i(t-n)}\right]_{\omega=-\pi}^{\omega=\pi} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(n)\left[\frac{\cos\omega(t-n)+i\sin\omega(t-n)}{i(t-n)}\right]_{\omega=-\pi}^{\omega=\pi} \end{aligned}$$
- Note that $\cos\pi(t-n) - \cos(-\pi(t-n)) = 0$ while $\sin\pi(t-n) - \sin(-\pi(t-n)) = 2\sin\pi(t-n)$ and so we get
$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}.$$

Bandlimited signals of finite energy

- The formula we have just obtained,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}.$$

is usually called the Shannon Sampling Theorem, even though he did not discovered it but has only popularised it.

- A more proper name should be "Whittaker - Nyquist - Kotelnikov - Shannon Sampling Theorem".
- It asserts that it is possible to obtain a perfect reconstruction of a band limited signal from its discrete samples, sampled at intervals which are half the period of the sinusoid of the maximal allowed frequency, because the sine wave $\sin \pi t$ of frequency π has period equal to 2, and we sample the signal at all integers.
- This is usually expressed as "*the sampling frequency should be twice the maximal frequency present in the signal*".
- This makes digital signal processing (DSP) possible, because in principle, a continuous time signal can be perfectly captured by its (sufficiently dense) discrete samples.

Bandlimited signals of finite energy

- The formula we have just obtained,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}.$$

  is usually called the Shannon Sampling Theorem, even though he did not discovered it but has only popularised it.
- A more proper name should be "Whittaker - Nyquist - Kotelnikov - Shannon Sampling Theorem".
- It asserts that it is possible to obtain a perfect reconstruction of a band limited signal from its discrete samples, sampled at intervals which are half the period of the sinusoid of the maximal allowed frequency, because the sine wave $\sin \pi t$ of frequency π has period equal to 2, and we sample the signal at all integers.
- This is usually expressed as "*the sampling frequency should be twice the maximal frequency present in the signal*".
- This makes digital signal processing (DSP) possible, because in principle, a continuous time signal can be perfectly captured by its (sufficiently dense) discrete samples.

Bandlimited signals of finite energy

- The formula we have just obtained,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}.$$

  is usually called the Shannon Sampling Theorem, even though he did not discovered it but has only popularised it.
- A more proper name should be "Whittaker - Nyquist - Kotelnikov - Shannon Sampling Theorem".
- It asserts that it is possible to obtain a perfect reconstruction of a band limited signal from its discrete samples, sampled at intervals which are half the period of the sinusoid of the maximal allowed frequency, because the sine wave $\sin \pi t$ of frequency π has period equal to 2, and we sample the signal at all integers.
- This is usually expressed as "*the sampling frequency should be twice the maximal frequency present in the signal*".
- This makes digital signal processing (DSP) possible, because in principle, a continuous time signal can be perfectly captured by its (sufficiently dense) discrete samples.

Bandlimited signals of finite energy

- The formula we have just obtained,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}.$$

  is usually called the Shannon Sampling Theorem, even though he did not discovered it but has only popularised it.
- A more proper name should be "Whittaker - Nyquist - Kotelnikov - Shannon Sampling Theorem".
- It asserts that it is possible to obtain a perfect reconstruction of a band limited signal from its discrete samples, sampled at intervals which are half the period of the sinusoid of the maximal allowed frequency, because the sine wave $\sin \pi t$ of frequency π has period equal to 2, and we sample the signal at all integers.
- This is usually expressed as "*the sampling frequency should be twice the maximal frequency present in the signal*".
- This makes digital signal processing (DSP) possible, because in principle, a continuous time signal can be perfectly captured by its (sufficiently dense) discrete samples.

Bandlimited signals of finite energy

- The formula we have just obtained,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}.$$

  is usually called the Shannon Sampling Theorem, even though he did not discovered it but has only popularised it.
- A more proper name should be "Whittaker - Nyquist - Kotelnikov - Shannon Sampling Theorem".
- It asserts that it is possible to obtain a perfect reconstruction of a band limited signal from its discrete samples, sampled at intervals which are half the period of the sinusoid of the maximal allowed frequency, because the sine wave $\sin \pi t$ of frequency π has period equal to 2, and we sample the signal at all integers.
- This is usually expressed as "*the sampling frequency should be twice the maximal frequency present in the signal*".
- This makes digital signal processing (DSP) possible, because in principle, a continuous time signal can be perfectly captured by its (sufficiently dense) discrete samples.

Bandlimited signals of finite energy

- Function $\dfrac{\sin t}{t}$ is usually called *the cardinal sine function* or *the sinc function* and is denoted by $\operatorname{sinc}(t)$.
- Thus we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc} \pi(t-n) \tag{6}$$

- Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega t} d\omega = \frac{1}{2\pi}\frac{e^{i\,\omega t}}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{1}{2\pi}\frac{\cos\omega t + i\sin\omega t}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{\sin\pi t}{\pi t} = \operatorname{sinc}\pi t \tag{7}$$

we see that the Fourier transform $\widehat{\operatorname{sinc}}(\omega)$ of $\operatorname{sinc}\pi t$ is given by

$$\widehat{\operatorname{sinc}}(\omega) = \begin{cases} 1 & \text{if } |\omega| < \pi \\ 0 & \text{if } |\omega| > \pi \end{cases} \tag{8}$$

Bandlimited signals of finite energy

- Function $\frac{\sin t}{t}$ is usually called *the cardinal sine function* or *the sinc function* and is denoted by $\mathrm{sinc}(t)$.
- Thus we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,\pi(t-n) \tag{6}$$

- Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega t} d\omega = \frac{1}{2\pi}\frac{e^{i\,\omega t}}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{1}{2\pi}\frac{\cos\omega t + i\sin\omega t}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{\sin\pi t}{\pi t} = \mathrm{sinc}\,\pi t \tag{7}$$

we see that the Fourier transform $\widehat{\mathrm{sinc}}(\omega)$ of $\mathrm{sinc}\,\pi t$ is given by

$$\widehat{\mathrm{sinc}}(\omega) = \begin{cases} 1 & \text{if } |\omega| < \pi \\ 0 & \text{if } |\omega| > \pi \end{cases} \tag{8}$$

Bandlimited signals of finite energy

- Function $\dfrac{\sin t}{t}$ is usually called *the cardinal sine function* or *the sinc function* and is denoted by $\mathrm{sinc}(t)$.
- Thus we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,\pi(t-n) \tag{6}$$

- Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega t} d\omega = \frac{1}{2\pi}\,\frac{e^{i\,\omega t}}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{1}{2\pi}\,\frac{\cos\omega t + i\sin\omega t}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{\sin\pi t}{\pi t} = \mathrm{sinc}\,\pi t \tag{7}$$

we see that the Fourier transform $\widehat{\mathrm{sinc}}(\omega)$ of $\mathrm{sinc}\,\pi t$ is given by

$$\widehat{\mathrm{sinc}}(\omega) = \begin{cases} 1 & \text{if } |\omega| < \pi \\ 0 & \text{if } |\omega| > \pi \end{cases} \tag{8}$$

Bandlimited signals of finite energy

- Function $\dfrac{\sin t}{t}$ is usually called *the cardinal sine function* or *the sinc function* and is denoted by $\operatorname{sinc}(t)$.
- Thus we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc} \pi(t-n) \tag{6}$$

- Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega t} d\omega = \frac{1}{2\pi}\frac{e^{i\,\omega t}}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{1}{2\pi}\frac{\cos\omega t + i\sin\omega t}{i\,t}\Big|_{\omega=-\pi}^{\pi} = \frac{\sin\pi t}{\pi t} = \operatorname{sinc}\pi t \tag{7}$$

we see that the Fourier transform $\widehat{\operatorname{sinc}}(\omega)$ of $\operatorname{sinc}\pi t$ is given by

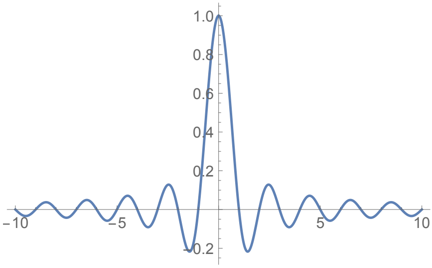
$$\widehat{\operatorname{sinc}}(\omega) = \begin{cases} 1 & \text{if} \quad |\omega| < \pi \\ 0 & \text{if} \quad |\omega| > \pi \end{cases} \tag{8}$$

Bandlimited signals of finite energy

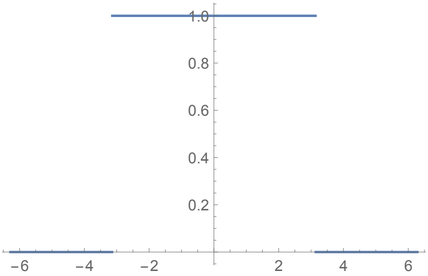
- If we substitute $t = 0$ in (7) then, since $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega = 1$, we get $\operatorname{sinc} 0 = 1$; at all other integers $n \neq 0$ we have

$$\operatorname{sinc}(\pi\, n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega n} d\omega = 0$$

  because we have n many complete periods of $e^{i\,\omega n}$ within $[-\pi, \pi]$.
- Thus, if we sample $\operatorname{sinc} \pi t$ at integers we get 1 at $t = 0$ and 0 at all other integer sampling points.



$\operatorname{sinc}\pi t$ function in the time domain



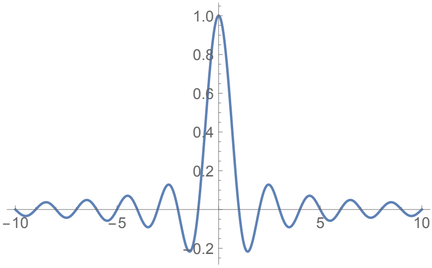
$\operatorname{sinc}\pi t$ function in the frequency domain

Bandlimited signals of finite energy

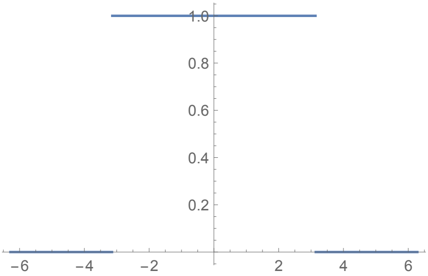
- If we substitute $t = 0$ in (7) then, since $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega = 1$, we get $\operatorname{sinc} 0 = 1$; at all other integers $n \neq 0$ we have

$$\operatorname{sinc}(\pi\, n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega n} d\omega = 0$$

  because we have n many complete periods of $e^{i\,\omega n}$ within $[-\pi, \pi]$.
- Thus, if we sample $\operatorname{sinc} \pi t$ at integers we get 1 at $t = 0$ and 0 at all other integer sampling points.



$\operatorname{sinc}\pi t$ function in the time domain



$\operatorname{sinc}\pi t$ function in the frequency domain

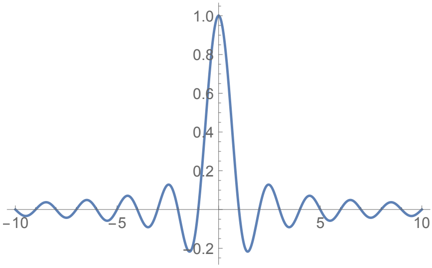
Bandlimited signals of finite energy

- If we substitute $t = 0$ in (7) then, since $\frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega = 1$, we get $\operatorname{sinc} 0 = 1$; at all other integers $n \neq 0$ we have

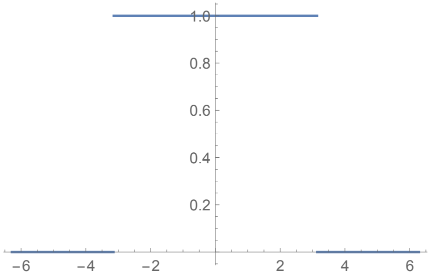
$$\operatorname{sinc}(\pi\, n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega n} d\omega = 0$$

because we have n many complete periods of $e^{i\,\omega n}$ within $[-\pi, \pi]$.

- Thus, if we sample $\operatorname{sinc} \pi t$ at integers we get 1 at $t = 0$ and 0 at all other integer sampling points.



sincπt function in the time domain



sincπt function in the frequency domain

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n\in\mathbb{N}}$

- Every function $f(x)$ which is square integrable, i.e., such that

$$\int_{-\infty}^{\infty} f^2(t)dt < \infty$$

(this is usually denoted by $f \in L^2$) has a Fourier transform $\widehat{f}(\omega)$ such that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\,\omega t}d\omega;$$

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\,i\,\omega t}dt.$$

- One can show that the mapping $\mathcal{F} : f(t) \mapsto \widehat{f}(\omega)$ has the following important property of *isometry*: for every two functions $f(t), g(t) \in L^2$

$$\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}d\omega = \langle \widehat{f}(\omega), \widehat{g}(\omega)\rangle \tag{9}$$

and in particular,

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)\overline{\widehat{f}(\omega)}d\omega = \|\widehat{f}(\omega)\|^2 \tag{10}$$

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n \in \mathbb{N}}$

- Every function $f(x)$ which is square integrable, i.e., such that

$$\int_{-\infty}^{\infty} f^2(t)dt < \infty$$

(this is usually denoted by $f \in L^2$) has a Fourier transform $\widehat{f}(\omega)$ such that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\,\omega t} d\omega;$$

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\,i\,\omega t} dt.$$

- One can show that the mapping $\mathcal{F} : f(t) \mapsto \widehat{f}(\omega)$ has the following important property of *isometry*: for every two functions $f(t), g(t) \in L^2$

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}d\omega = \langle \widehat{f}(\omega), \widehat{g}(\omega) \rangle \tag{9}$$

and in particular,

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{f}(\omega)\overline{\widehat{f}(\omega)}d\omega = \|\widehat{f}(\omega)\|^2 \tag{10}$$

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n\in\mathbb{N}}$

- Every function $f(x)$ which is square integrable, i.e., such that

$$\int_{-\infty}^{\infty} f^2(t)dt < \infty$$

(this is usually denoted by $f \in L^2$) has a Fourier transform $\widehat{f}(\omega)$ such that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\,\omega t}d\omega;$$

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\,i\,\omega t}dt.$$

- One can show that the mapping $\mathcal{F} : f(t) \mapsto \widehat{f}(\omega)$ has the following important property of *isometry*: for every two functions $f(t), g(t) \in L^2$

$$\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}d\omega = \langle \widehat{f}(\omega), \widehat{g}(\omega)\rangle \qquad (9)$$

and in particular,

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)\overline{\widehat{f}(\omega)}d\omega = \|\widehat{f}(\omega)\|^2 \qquad (10)$$

Fourier series in the basis $\{\operatorname{sinc}\pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, $\mathcal{F}$ preserves both the "lengths" (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the "geometry" of the two spaces of signals, one in the time domain, the other in the "frequency" domain.
- Note that (7) implies

$$\operatorname{sinc}\pi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,\omega t}d\omega;$$

Thus,

$$[\widehat{\operatorname{sinc}\,\pi(t-n)}](\omega) = \begin{cases} e^{-\,i\,n\omega} & \text{if } \ |\omega| < \pi \\ 0 & \text{if } \ |\omega| > \pi \end{cases}$$

and consequently

$$\int_{-\infty}^{\infty}\operatorname{sinc}\pi(t-n)\operatorname{sinc}\pi(t-m)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,m\omega}d\omega = \begin{cases} 1 & \text{if } \ n=m \\ 0 & \text{if } \ n\neq m \end{cases}$$

- This means that functions $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$ are orthonormal.
- In fact, they form an orthonormal basis of the vector space of band limited functions of finite energy.
- The Shannon expansion (6) simply represents a signal $f(t)$ in that basis, i.e., is the Fourier series of $f(t)$ with respect to the basis $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$.

Fourier series in the basis $\{\operatorname{sinc}\pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, $\mathcal{F}$ preserves both the "lengths" (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the "geometry" of the two spaces of signals, one in the time domain, the other in the "frequency" domain.
- Note that (7) implies

$$\operatorname{sinc}\pi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,\omega t}d\omega;$$

Thus,

$$[\widehat{\operatorname{sinc}}\,\pi(t-n)](\omega) = \begin{cases} e^{-\,i\,n\omega} & \text{if } \ |\omega| < \pi \\ 0 & \text{if } \ |\omega| > \pi \end{cases}$$

and consequently

$$\int_{-\infty}^{\infty}\operatorname{sinc}\pi(t-n)\operatorname{sinc}\pi(t-m)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,m\omega}d\omega = \begin{cases} 1 & \text{if } \ n = m \\ 0 & \text{if } \ n \neq m \end{cases}$$

- This means that functions $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$ are orthonormal.
- In fact, they form an orthonormal basis of the vector space of band limited functions of finite energy.
- The Shannon expansion (6) simply represents a signal $f(t)$ in that basis, i.e., is the Fourier series of $f(t)$ with respect to the basis $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$.

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, $\mathcal{F}$ preserves both the "lengths" (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the "geometry" of the two spaces of signals, one in the time domain, the other in the "frequency" domain.
- Note that (7) implies

$$\operatorname{sinc} \pi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,\omega t}d\omega;$$

Thus,

$$[\widehat{\operatorname{sinc}}\,\pi(t-n)](\omega) = \begin{cases} e^{-\,i\,n\omega} & \text{if } \; |\omega| < \pi \\ 0 & \text{if } \; |\omega| > \pi \end{cases}$$

and consequently

$$\int_{-\infty}^{\infty}\operatorname{sinc}\pi(t-n)\operatorname{sinc}\pi(t-m)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,m\omega}d\omega = \begin{cases} 1 & \text{if } \; n = m \\ 0 & \text{if } \; n \neq m \end{cases}$$

- This means that functions $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$ are orthonormal.
- In fact, they form an orthonormal basis of the vector space of band limited functions of finite energy.
- The Shannon expansion (6) simply represents a signal $f(t)$ in that basis, i.e., is the Fourier series of $f(t)$ with respect to the basis $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$.

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, $\mathcal{F}$ preserves both the "lengths" (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the "geometry" of the two spaces of signals, one in the time domain, the other in the "frequency" domain.
- Note that (7) implies

$$\operatorname{sinc} \pi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,\omega t}d\omega;$$

Thus,

$$[\widehat{\operatorname{sinc}}\, \pi(t-n)](\omega) = \begin{cases} e^{-\,i\,n\omega} & \text{if } \;|\omega| < \pi \\ 0 & \text{if } \;|\omega| > \pi \end{cases}$$

and consequently

$$\int_{-\infty}^{\infty} \operatorname{sinc}\pi(t-n)\operatorname{sinc}\pi(t-m)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,m\omega}d\omega = \begin{cases} 1 & \text{if } \;n = m \\ 0 & \text{if } \;n \neq m \end{cases}$$

- This means that functions $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$ are orthonormal.
- In fact, they form an orthonormal basis of the vector space of band limited functions of finite energy.
- The Shannon expansion (6) simply represents a signal $f(t)$ in that basis, i.e., is the Fourier series of $f(t)$ with respect to the basis $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$.

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, $\mathcal{F}$ preserves both the "lengths" (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the "geometry" of the two spaces of signals, one in the time domain, the other in the "frequency" domain.
- Note that (7) implies

$$\operatorname{sinc} \pi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega(t-n)} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega} e^{i\,\omega t} d\omega;$$

Thus,

$$[\widehat{\operatorname{sinc}\, \pi(t-n)}](\omega) = \begin{cases} e^{-\,i\,n\omega} & \text{if } \ |\omega| < \pi \\ 0 & \text{if } \ |\omega| > \pi \end{cases}$$

and consequently

$$\int_{-\infty}^{\infty} \operatorname{sinc}\pi(t-n)\operatorname{sinc}\pi(t-m) dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega} e^{i\,m\omega} d\omega = \begin{cases} 1 & \text{if } \ n = m \\ 0 & \text{if } \ n \neq m \end{cases}$$

- This means that functions $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$ are orthonormal.
- In fact, they form an orthonormal basis of the vector space of band limited functions of finite energy.
- The Shannon expansion (6) simply represents a signal $f(t)$ in that basis, i.e., is the Fourier series of $f(t)$ with respect to the basis $\{\operatorname{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$.

Fourier series in the basis $\{\text{sinc}\,\pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, $\mathcal{F}$ preserves both the "lengths" (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the "geometry" of the two spaces of signals, one in the time domain, the other in the "frequency" domain.
- Note that (7) implies

$$\text{sinc}\,\pi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,\omega t}d\omega;$$

Thus,

$$[\widehat{\text{sinc}}\,\pi(t-n)](\omega) = \begin{cases} e^{-\,i\,n\omega} & \text{if } \; |\omega| < \pi \\ 0 & \text{if } \; |\omega| > \pi \end{cases}$$

and consequently

$$\int_{-\infty}^{\infty}\text{sinc}\pi(t-n)\text{sinc}\pi(t-m)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,m\omega}d\omega = \begin{cases} 1 & \text{if } \; n = m \\ 0 & \text{if } \; n \neq m \end{cases}$$

- This means that functions $\{\text{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$ are orthonormal.
- In fact, they form an orthonormal basis of the vector space of band limited functions of finite energy.
- The Shannon expansion (6) simply represents a signal $f(t)$ in that basis, i.e., is the Fourier series of $f(t)$ with respect to the basis $\{\text{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$.

Fourier series in the basis $\{\mathrm{sinc}\,\pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, $\mathcal{F}$ preserves both the "lengths" (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the "geometry" of the two spaces of signals, one in the time domain, the other in the "frequency" domain.
- Note that (7) implies

$$\mathrm{sinc}\,\pi(t-n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\,\omega(t-n)}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,\omega t}d\omega;$$

Thus,

$$[\widehat{\mathrm{sinc}}\,\pi(t-n)](\omega) = \begin{cases} e^{-\,i\,n\omega} & \text{if } |\omega| < \pi \\ 0 & \text{if } |\omega| > \pi \end{cases}$$

and consequently

$$\int_{-\infty}^{\infty} \mathrm{sinc}\pi(t-n)\mathrm{sinc}\pi(t-m)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\,i\,n\omega}e^{i\,m\omega}d\omega = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

- This means that functions $\{\mathrm{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$ are orthonormal.
- In fact, they form an orthonormal basis of the vector space of band limited functions of finite energy.
- The Shannon expansion (6) simply represents a signal $f(t)$ in that basis, i.e., is the Fourier series of $f(t)$ with respect to the basis $\{\mathrm{sinc}\pi(t-k)\}_{k\in\mathbf{Z}}$.

Fourier series in the basis $\{\operatorname{sinc}\pi(t-n)\}_{n\in\mathbb{N}}$

- To see that, using the isometry property, we have that the values of the projections of $f(t)$ onto the basis vectors $\operatorname{sinc}\pi(t-n)$ are equal to:

$$\int_{-\infty}^{\infty} f(t)\operatorname{sinc}\pi(t-n)dt = \langle f, \operatorname{sinc}\pi(t-n)\rangle = \langle \widehat{f}, \widehat{\operatorname{sinc}}\,\pi(t-n)\rangle$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{f}(\omega)e^{-i(-n)\omega}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{f}(\omega)e^{i\,n\omega}d\omega$$
$$= f(n)$$

- Since for any orthonormal basis $\{b_n\}$ the corresponding Fourier series expansion is of the form

$$f = \sum \langle f, b_n\rangle b_n$$

we get that

$$f(t) = \sum_{n=-\infty}^{\infty}\langle f, \operatorname{sinc}\pi(t-n)\rangle \operatorname{sinc}\pi(t-n) = \sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}\pi(t-n)$$

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n \in \mathbb{N}}$

- To see that, using the isometry property, we have that the values of the projections of $f(t)$ onto the basis vectors $\operatorname{sinc} \pi(t-n)$ are equal to:

$$\begin{aligned}\int_{-\infty}^{\infty} f(t) \operatorname{sinc} \pi(t-n) dt = \langle f, \operatorname{sinc} \pi(t-n)\rangle &= \langle \widehat{f}, \widehat{\operatorname{sinc}\, \pi(t-n)}\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega) e^{-i(-n)\omega} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega) e^{i n\omega} d\omega \\ &= f(n)\end{aligned}$$

- Since for any orthonormal basis $\{b_n\}$ the corresponding Fourier series expansion is of the form

$$f = \sum \langle f, b_n\rangle b_n$$

we get that

$$f(t) = \sum_{n=-\infty}^{\infty} \langle f, \operatorname{sinc} \pi(t-n)\rangle \operatorname{sinc} \pi(t-n) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc} \pi(t-n)$$

Fourier series in the basis $\{\operatorname{sinc}\pi(t-n)\}_{n\in\mathbb{N}}$

- To see that, using the isometry property, we have that the values of the projections of $f(t)$ onto the basis vectors $\operatorname{sinc}\pi(t-n)$ are equal to:

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\operatorname{sinc}\pi(t-n)dt &= \langle f, \operatorname{sinc}\pi(t-n)\rangle = \langle \widehat{f}, \widehat{\operatorname{sinc}}\,\pi(t-n)\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{f}(\omega)e^{-i(-n)\omega}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{f}(\omega)e^{i\,n\omega}d\omega \\ &= f(n)\end{aligned}$$

- Since for any orthonormal basis $\{b_n\}$ the corresponding Fourier series expansion is of the form

$$f = \sum \langle f, b_n\rangle b_n$$

we get that

$$f(t) = \sum_{n=-\infty}^{\infty}\langle f, \operatorname{sinc}\pi(t-n)\rangle \operatorname{sinc}\pi(t-n) = \sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}\pi(t-n)$$

Fourier series in the basis $\{\operatorname{sinc}\pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, we see that the values of a π-band limited signal at all integers, i.e., $\{\dots, f(-n), \dots, f(-1), f(0), f(1), \dots, f(n), \dots\}$, are the coefficients of two Fourier series:
  1. In the frequency domain (i.e., in the space of the Fourier transforms of all π-band limited signals) they are the Fourier coefficients of $\widehat{f}(\omega)$ in the usual basis of the complex exponentials $\{e^{i\,n\omega} : n \in \mathbb{Z}\}$:

$$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(-n)e^{i\,n\omega} = \sum_{n=-\infty}^{\infty} f(n)e^{-\,i\,n\omega} \tag{11}$$

  2. In the time domain, taking the Inverse Fourier Transform of both sides, we get the Shannon formula which is just the (generalised) Fourier series in the basis $\{\operatorname{sinc}\pi(t-n)\}_{n\in\mathbf{Z}}$:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}\pi(t-n) \tag{12}$$

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n\in\mathbb{N}}$

- Thus, we see that the values of a π-band limited signal at all integers, i.e., $\{\ldots, f(-n), \ldots, f(-1), f(0), f(1), \ldots, f(n), \ldots\}$, are the coefficients of two Fourier series:
  1. In the frequency domain (i.e., in the space of the Fourier transforms of all π-band limited signals) they are the Fourier coefficients of $\widehat{f}(\omega)$ in the usual basis of the complex exponentials $\{e^{i\,n\omega} \;:\; n \in \mathbb{Z}\}$:

$$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(-n)e^{i\,n\omega} = \sum_{n=-\infty}^{\infty} f(n)e^{-\,i\,n\omega} \tag{11}$$

  2. In the time domain, taking the Inverse Fourier Transform of both sides, we get the Shannon formula which is just the (generalised) Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n\in\mathbf{Z}}$:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc} \pi(t-n) \tag{12}$$

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n \in \mathbb{N}}$

- Thus, we see that the values of a π-band limited signal at all integers, i.e., $\{\dots, f(-n), \dots, f(-1), f(0), f(1), \dots, f(n), \dots\}$, are the coefficients of two Fourier series:
  1. In the frequency domain (i.e., in the space of the Fourier transforms of all π-band limited signals) they are the Fourier coefficients of $\widehat{f}(\omega)$ in the usual basis of the complex exponentials $\{e^{i\,n\omega} \;:\; n \in \mathbb{Z}\}$:

$$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(-n) e^{i\,n\omega} = \sum_{n=-\infty}^{\infty} f(n) e^{-\,i\,n\omega} \tag{11}$$

  2. In the time domain, taking the Inverse Fourier Transform of both sides, we get the Shannon formula which is just the (generalised) Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n \in \mathbf{Z}}$:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc} \pi(t-n) \tag{12}$$

Fourier series in the basis $\{\operatorname{sinc} \pi(t-n)\}_{n \in \mathbb{N}}$

- As mentioned, the Fourier transform defines an isometry between the two domains: the "length" of a vector (i.e., the total energy of a band-limited signal) $f(t)$ is the same if measured in either time or in the frequency domain, in the sense that

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{f}(\omega)|^2 dt = \|\widehat{f}(\omega)\|^2$$

- Also, for any two band limited signals $\langle f(t), g(t) \rangle = \langle \widehat{f}(\omega), \widehat{g}(\omega) \rangle$.
- For a band limited signal $f(t)$ we let

$$\vec{f} = (\ldots, f(-n), \ldots, f(-1), f(0), f(1), \ldots, f(n), \ldots),$$

which is a square summable sequence, $\sum_{n=-\infty}^{\infty} |f(n)|^2 < \infty$.

Fourier series in the basis $\{\mathrm{sinc}\, \pi(t-n)\}_{n \in \mathbb{N}}$

- As mentioned, the Fourier transform defines an isometry between the two domains: the "length" of a vector (i.e., the total energy of a band-limited signal) $f(t)$ is the same if measured in either time or in the frequency domain, in the sense that

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{f}(\omega)|^2 dt = \|\widehat{f}(\omega)\|^2$$

- Also, for any two band limited signals $\langle f(t), g(t) \rangle = \langle \widehat{f}(\omega), \widehat{g}(\omega) \rangle$.
- For a band limited signal $f(t)$ we let

$$\vec{f} = (\ldots, f(-n), \ldots, f(-1), f(0), f(1), \ldots, f(n), \ldots),$$

which is a square summable sequence, $\sum_{n=-\infty}^{\infty} |f(n)|^2 < \infty$.

Discretisation of band limited signals

- We now establish yet another important isometry:

$$\begin{aligned}
\langle f(t), g(t)\rangle &= \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \\
&= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}\pi(t-n)\right)\overline{\left(\sum_{m=-\infty}^{\infty} g(m)\operatorname{sinc}\pi(t-m)\right)}dt \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n)\overline{g(m)}\int_{-\infty}^{\infty}\operatorname{sinc}\pi(t-n)\operatorname{sinc}\pi(t-m)dt \\
&= \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)} \\
&= \langle \vec{f}, \vec{g}\rangle.
\end{aligned}$$

- In the step next to the last one we used the fact that, by orhonormality of the basis $\{\operatorname{sinc}\pi(t-n) \ : \ n \in \mathbb{Z}\}$, we have

$$\int_{-\infty}^{\infty}\operatorname{sinc}\pi(t-n)\operatorname{sinc}\pi(t-m)dt = \begin{cases} 1 & \text{if } \ m = n \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Discretisation of band limited signals

- We now establish yet another important isometry:

$$\begin{aligned}\langle f(t), g(t)\rangle &= \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \\ &= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,\pi(t-n)\right)\overline{\left(\sum_{m=-\infty}^{\infty} g(m)\,\mathrm{sinc}\,\pi(t-m)\right)}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n)\overline{g(m)}\int_{-\infty}^{\infty}\mathrm{sinc}\,\pi(t-n)\,\mathrm{sinc}\,\pi(t-m)dt \\ &= \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)} \\ &= \langle \vec{f}, \vec{g}\rangle.\end{aligned}$$

- In the step next to the last one we used the fact that, by orhonormality of the basis $\{\mathrm{sinc}\,\pi(t-n) \;:\; n \in \mathbb{Z}\}$, we have

$$\int_{-\infty}^{\infty}\mathrm{sinc}\,\pi(t-n)\,\mathrm{sinc}\,\pi(t-m)dt = \begin{cases}1 & \text{if } \; m = n \\ 0 & \text{otherwise}\end{cases} \tag{13}$$

Discretisation of band limited signals

- Thus,

$$\langle f(t), g(t)\rangle = \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)} = \langle \vec{f}, \vec{g}\rangle$$

- In particular, we also get

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |f(n)|^2 = \|\vec{f}\|^2$$

- Thus, the space of π-band limited signals of finite energy is isometric to the space l^2 of square summable sequences, $l^2 = \{a_n : \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}$, where the numbers a_n can be interpreted as samples of a π-band limited signal of finite energy.
- Note that in such isometry function $\operatorname{sinc} \pi(t-n)$ is mapped to the sequence

$$e_n(m) = \begin{cases} 1 & \text{if } \; m = n \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Discretisation of band limited signals

- Thus,

$$\langle f(t), g(t)\rangle = \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)} = \langle \vec{f}, \vec{g}\rangle$$

- In particular, we also get

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |f(n)|^2 = \|\vec{f}\|^2$$

- Thus, the space of π-band limited signals of finite energy is isometric to the space l^2 of square summable sequences, $l^2 = \{a_n : \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}$, where the numbers a_n can be interpreted as samples of a π-band limited signal of finite energy.
- Note that in such isometry function $\operatorname{sinc} \pi(t-n)$ is mapped to the sequence

$$e_n(m) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Discretisation of band limited signals

- Thus,

$$\langle f(t), g(t)\rangle = \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)} = \langle \vec{f}, \vec{g}\rangle$$

- In particular, we also get

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |f(n)|^2 = \|\vec{f}\|^2$$

- Thus, the space of π-band limited signals of finite energy is isometric to the space l^2 of square summable sequences, $l^2 = \{a_n : \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}$, where the numbers a_n can be interpreted as samples of a π-band limited signal of finite energy.
- Note that in such isometry function $\operatorname{sinc} \pi(t-n)$ is mapped to the sequence

$$e_n(m) = \begin{cases} 1 & \text{if } \; m = n \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Discretisation of band limited signals

- Thus,

$$\langle f(t), g(t)\rangle = \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)} = \langle \vec{f}, \vec{g}\rangle$$

- In particular, we also get

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |f(n)|^2 = \|\vec{f}\|^2$$

- Thus, the space of π-band limited signals of finite energy is isometric to the space l^2 of square summable sequences, $l^2 = \{a_n : \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}$, where the numbers a_n can be interpreted as samples of a π-band limited signal of finite energy.
- Note that in such isometry function $\operatorname{sinc}\pi(t-n)$ is mapped to the sequence

$$e_n(m) = \begin{cases} 1 & \text{if } \ m = n \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Discretisation of band limited signals

- Thus, we get the following isometric spaces, with their bases and the corresponding Fourier series:

<u>frequency domain:</u>

$$\widehat{f}(\omega) \qquad \{e^{i\,n\omega} \;:\; n \in \mathbb{Z}\} \qquad \widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-\,i\,n\omega}$$

<u>time domain:</u>

$$f(t) \qquad \{\operatorname{sinc}\pi(t-n) \;:\; n \in \mathbb{Z}\} \qquad f(t) = \sum_{n=-\infty}^{\infty} f(n)\operatorname{sinc}\pi(t-n)$$

<u>sequences from</u> l^2

$$\{f(n)\} \qquad \{e_n = (\ldots 0, 1, 0 \ldots) \;:\; n \in \mathbb{Z}\} \qquad \vec{f} = \sum_{n=-\infty}^{\infty} f(n)\, e_n$$

Filters

- So we now know how to represent band-limited signals, and would like to process them.
- For example, we might want to remove certain frequencies, such as the 50 Hz hum which could have come from the mains power supply, or the "hissing" from an old vinyl LP record.
- Such operations are examples of *filtering*.
- In general, a filter L acts on signals $f(t)$ producing an output signal $L[f](t)$.
- Note that both the input signal $f(t)$ and the output signal $L[f](t)$ are taken as "wholes", in their entire duration.
- Filters do NOT produce outputs $L[f](t_0)$ at any given instant of time t_0 from the value of $f(t_0)$ alone.

Filters

- So we now know how to represent band-limited signals, and would like to process them.
- For example, we might want to remove certain frequencies, such as the 50 Hz hum which could have come from the mains power supply, or the "hissing" from an old vinyl LP record.
- Such operations are examples of *filtering*.
- In general, a filter L acts on signals $f(t)$ producing an output signal $L[f](t)$.
- Note that both the input signal $f(t)$ and the output signal $L[f](t)$ are taken as "wholes", in their entire duration.
- Filters do NOT produce outputs $L[f](t_0)$ at any given instant of time t_0 from the value of $f(t_0)$ alone.

Filters

- So we now know how to represent band-limited signals, and would like to process them.
- For example, we might want to remove certain frequencies, such as the 50 Hz hum which could have come from the mains power supply, or the "hissing" from an old vinyl LP record.
- Such operations are examples of *filtering*.
- In general, a filter L acts on signals $f(t)$ producing an output signal $L[f](t)$.
- Note that both the input signal $f(t)$ and the output signal $L[f](t)$ are taken as "wholes", in their entire duration.
- Filters do NOT produce outputs $L[f](t_0)$ at any given instant of time t_0 from the value of $f(t_0)$ alone.

Filters

- So we now know how to represent band-limited signals, and would like to process them.
- For example, we might want to remove certain frequencies, such as the 50 Hz hum which could have come from the mains power supply, or the "hissing" from an old vinyl LP record.
- Such operations are examples of *filtering*.
- In general, a filter L acts on signals $f(t)$ producing an output signal $L[f](t)$.
- Note that both the input signal $f(t)$ and the output signal $L[f](t)$ are taken as "wholes", in their entire duration.
- Filters do NOT produce outputs $L[f](t_0)$ at any given instant of time t_0 from the value of $f(t_0)$ alone.

Filters

- So we now know how to represent band-limited signals, and would like to process them.
- For example, we might want to remove certain frequencies, such as the 50 Hz hum which could have come from the mains power supply, or the “hissing” from an old vinyl LP record.
- Such operations are examples of *filtering*.
- In general, a filter L acts on signals $f(t)$ producing an output signal $L[f](t)$.
- Note that both the input signal $f(t)$ and the output signal $L[f](t)$ are taken as “wholes”, in their entire duration.
- Filters do NOT produce outputs $L[f](t_0)$ at any given instant of time t_0 from the value of $f(t_0)$ alone.

Filters

- Filters have to satisfy the following three properties:

1. L is **continuous**, i.e., if a sequence of signals $f_n(t)$ converges to a signal $f(t)$ in the sense of the L^2 norm, then the image of the limit signal is the limit of the images of $f_n(t)$:

$$\text{if} \quad \lim_{n\to\infty} \|f_n(t) - f(t)\| = 0, \ \text{then} \quad \lim_{n\to\infty} \|L[f_n](t) - L[f](t)\| = 0.$$

2. L is a **linear operator**, i.e., for arbitrary scalars c_1, c_2 and arbitrary two band limited signals $f(t), g(t)$,

$$L[c_1 f + c_2 g](t) = c_1 L[f](t) + c_2 L[g](t)$$

3. L is **time invariant** (also called *shift invariant*) The image of a time shifted signal is the just an equally shifted image of the original signal:

$$L[f(t+\tau)] = L[f(t)](t+\tau)$$

Filters

- Filters have to satisfy the following three properties:

1. L is **continuous**, i.e., if a sequence of signals $f_n(t)$ converges to a signal $f(t)$ in the sense of the L^2 norm, then the image of the limit signal is the limit of the images of $f_n(t)$:

$$\text{if} \quad \lim_{n\to\infty} \|f_n(t) - f(t)\| = 0, \;\; \text{then} \;\; \lim_{n\to\infty} \|L[f_n](t) - L[f](t)\| = 0.$$

2. L is a **linear operator**, i.e., for arbitrary scalars c_1, c_2 and arbitrary two band limited signals $f(t), g(t)$,

$$L[c_1 f + c_2 g](t) = c_1 L[f](t) + c_2 L[g](t)$$

3. L is **time invariant** (also called *shift invariant*) The image of a time shifted signal is the just an equally shifted image of the original signal:

$$L[f(t+\tau)] = L[f(t)](t+\tau)$$

Filters

- Filters have to satisfy the following three properties:

1. L is **continuous**, i.e., if a sequence of signals $f_n(t)$ converges to a signal $f(t)$ in the sense of the L^2 norm, then the image of the limit signal is the limit of the images of $f_n(t)$:

$$\text{if} \quad \lim_{n\to\infty} \|f_n(t) - f(t)\| = 0, \quad \text{then} \quad \lim_{n\to\infty} \|L[f_n](t) - L[f](t)\| = 0.$$

2. L is a **linear operator**, i.e., for arbitrary scalars c_1, c_2 and arbitrary two band limited signals $f(t), g(t)$,

$$L[c_1 f + c_2 g](t) = c_1 L[f](t) + c_2 L[g](t)$$

3. L is **time invariant** (also called *shift invariant*) The image of a time shifted signal is the just an equally shifted image of the original signal:

$$L[f(t+\tau)] = L[f(t)](t+\tau)$$

Filters

- Filters have to satisfy the following three properties:
  1. L is **continuous**, i.e., if a sequence of signals $f_n(t)$ converges to a signal $f(t)$ in the sense of the L^2 norm, then the image of the limit signal is the limit of the images of $f_n(t)$:

     $$\text{if} \quad \lim_{n\to\infty} \|f_n(t) - f(t)\| = 0, \quad \text{then} \quad \lim_{n\to\infty} \|L[f_n](t) - L[f](t)\| = 0.$$

  2. L is a **linear operator**, i.e., for arbitrary scalars c_1, c_2 and arbitrary two band limited signals $f(t), g(t)$,

     $$L[c_1 f + c_2 g](t) = c_1 L[f](t) + c_2 L[g](t)$$

  3. L is **time invariant** (also called *shift invariant*) The image of a time shifted signal is the just an equally shifted image of the original signal:

     $$L[f(t+\tau)] = L[f(t)](t+\tau)$$

Filters

- What a filter L does to an arbitrary signal in $BL(\pi)$ is completely determined by what L does to signal $\text{sinc}(\pi t)$:

$$
\begin{aligned}
L(f(t)) &= L\left[\sum_{k=-\infty}^{\infty} f(k)\frac{\sin\pi(t-k)}{\pi(t-k)}\right] \\
&= L\left[\lim_{N\to\infty}\sum_{k=-N}^{N} f(k)\frac{\sin\pi(t-k)}{\pi(t-k)}\right] \\
&= \lim_{N\to\infty} L\left[\sum_{k=-N}^{N} f(k)\frac{\sin\pi(t-k)}{\pi(t-k)}\right] \qquad \text{(by continuity of } L\text{)} \\
&= \lim_{N\to\infty}\sum_{k=-N}^{N} f(k)L\left[\frac{\sin\pi(t-k)}{\pi(t-k)}\right] \qquad \text{(by linearity of } L\text{)} \\
&= \lim_{N\to\infty}\sum_{k=-N}^{N} f(k)L\left[\frac{\sin\pi u}{\pi u}\right]_{u=(t-k)} \qquad \text{(by time invariance of } L\text{)} \\
&= \sum_{k=-\infty}^{\infty} f(k)L[\text{sinc}\ \pi u](t-k)
\end{aligned}
$$

Filters

- So, to summarise:

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\text{sinc } \pi(t-k)$$

$$L[f(t)] = \sum_{k=-\infty}^{\infty} f(k)L[\text{sinc } \pi u](t-k)$$

- Here $L[\text{sinc } \pi u](t-k)$ stands for first obtaining the image $l(u) = L[\text{sinc } \pi u]$ and then replacing u in $l(u)$ with the shifted variable $t-k$, i.e., $l(t-k) = L[\text{sinc } \pi u](t-k)$.

Filters

- So, to summarise:

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\text{sinc }\pi(t-k)$$

$$L[f(t)] = \sum_{k=-\infty}^{\infty} f(k)L[\text{sinc }\pi u](t-k)$$

- Here $L[\text{sinc }\pi u](t-k)$ stands for first obtaining the image $l(u) = L[\text{sinc }\pi u]$ and then replacing u in $l(u)$ with the shifted variable $t-k$, i.e., $l(t-k) = L[\text{sinc }\pi u](t-k)$.

Filters

- So, to summarise:

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\text{sinc }\pi(t-k)$$

$$L[f(t)] = \sum_{k=-\infty}^{\infty} f(k)L[\text{sinc }\pi u](t-k)$$

- Here $L[\text{sinc }\pi u](t-k)$ stands for first obtaining the image $l(u) = L[\text{sinc }\pi u]$ and then replacing u in $l(u)$ with the shifted variable $t-k$, i.e., $l(t-k) = L[\text{sinc }\pi u](t-k)$.

Filters

- Consider the response of L for input $\operatorname{sinc}\pi t$,

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right]$$

  and let its Fourier transform be $\hat{L}(\omega)$.

- Then

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega t} d\omega$$

  and

$$l(t-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega (t-k)} d\omega.$$

- Combining this with

$$L[f(t)] = \sum_{k=-\infty}^{\infty} f(k) L[\operatorname{sinc}\,\pi u](t-k) = \sum_{k=-\infty}^{\infty} f(k) l(t-k)$$

  we obtain

Filters

- Consider the response of L for input $\operatorname{sinc} \pi t$,

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right]$$

and let its Fourier transform be $\hat{L}(\omega)$.

- Then

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega t} d\omega$$

and

$$l(t-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega (t-k)} d\omega.$$

- Combining this with

$$L[f(t)] = \sum_{k=-\infty}^{\infty} f(k) L[\operatorname{sinc} \pi u](t-k) = \sum_{k=-\infty}^{\infty} f(k) l(t-k)$$

we obtain

Filters

- Consider the response of L for input $\operatorname{sinc} \pi t$,

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right]$$

  and let its Fourier transform be $\hat{L}(\omega)$.

- Then

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega t} d\omega$$

  and

$$l(t-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega (t-k)} d\omega.$$

- Combining this with

$$L[f(t)] = \sum_{k=-\infty}^{\infty} f(k) L[\operatorname{sinc} \pi u](t-k) = \sum_{k=-\infty}^{\infty} f(k) l(t-k)$$

  we obtain

Filters

- Consider the response of L for input $\operatorname{sinc}\pi t$,

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right]$$

  and let its Fourier transform be $\hat{L}(\omega)$.
- Then

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega t} d\omega$$

  and

$$l(t-k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega (t-k)} d\omega.$$

- Combining this with

$$L[f(t)] = \sum_{k=-\infty}^{\infty} f(k) L[\operatorname{sinc} \pi u](t-k) = \sum_{k=-\infty}^{\infty} f(k) l(t-k)$$

  we obtain

Filters

$$\begin{aligned} L[f(t)] &= \sum_{k=-\infty}^{\infty} f(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega(t-k)} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} f(k) e^{-\,i\,\omega k} \right) \hat{L}(\omega) e^{i\,\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \hat{L}(\omega) e^{i\,\omega t} d\omega. \end{aligned}$$

- This implies

$$\widehat{L[f(t)]} = \hat{f}(\omega)\hat{L}(\omega)$$

- So a linear operator acts on a signal by multiplying the Fourier transform of the signal by the *transfer function* of the operator, i.e. by multiplying $\widehat{f}(\omega)$ by the Fourier transform $\widehat{L}(\omega)$ of $L\,[\mathrm{sinc}(\pi t)]$.

Filters

$$\begin{aligned} L[f(t)] &= \sum_{k=-\infty}^{\infty} f(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega(t-k)} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} f(k) e^{-\,i\,\omega k} \right) \hat{L}(\omega) e^{i\,\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \hat{L}(\omega) e^{i\,\omega t} d\omega. \end{aligned}$$

- This implies

$$\widehat{L[f(t)]} = \hat{f}(\omega)\hat{L}(\omega)$$

- So a linear operator acts on a signal by multiplying the Fourier transform of the signal by the *transfer function* of the operator, i.e. by multiplying $\widehat{f}(\omega)$ by the Fourier transform $\widehat{L}(\omega)$ of $L\,[\text{sinc}(\pi t)]$.

Filters

$$
\begin{aligned}
L[f(t)] &= \sum_{k=-\infty}^{\infty} f(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\,\omega(t-k)} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} f(k) e^{-\,i\,\omega k} \right) \hat{L}(\omega) e^{i\,\omega t} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \hat{L}(\omega) e^{i\,\omega t} d\omega.
\end{aligned}
$$

- This implies

$$
\widehat{L[f(t)]} = \hat{f}(\omega)\hat{L}(\omega)
$$

- So a linear operator acts on a signal by multiplying the Fourier transform of the signal by the *transfer function* of the operator, i.e. by multiplying $\widehat{f}(\omega)$ by the Fourier transform $\widehat{L}(\omega)$ of $L\,[\mathrm{sinc}(\pi t)]$.

Filters

- Note that

$$\begin{aligned} L[f](t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\hat{L}(\omega)e^{i\,\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} f(u)e^{-\,i\,\omega u}\hat{L}(\omega)e^{i\,\omega t}dud\omega \\ &= \int_{-\infty}^{\infty} f(u)\frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{L}(\omega)e^{i\,\omega(t-u)}d\omega du \\ &= \int_{-\infty}^{\infty} f(u)l(t-u)du \end{aligned}$$

where

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right].$$

- Integral

$$\int_{-\infty}^{\infty} f(u)l(t-u)du$$

is called the continuous time convolution $f * l$ of signals $f(t)$ and $l(t)$.

- Note that

$$\begin{aligned} L[f](t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\hat{L}(\omega)e^{i\,\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} f(u)e^{-\,i\,\omega u}\hat{L}(\omega)e^{i\,\omega t}dud\omega \\ &= \int_{-\infty}^{\infty} f(u)\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{L}(\omega)e^{i\,\omega(t-u)}d\omega du \\ &= \int_{-\infty}^{\infty} f(u)l(t-u)du \end{aligned}$$

  where

$$l(t) = L\left[\frac{\sin \pi t}{\pi t}\right].$$

- Integral

$$\int_{-\infty}^{\infty} f(u)l(t-u)du$$

  is called the continuous time convolution $f * l$ of signals $f(t)$ and $l(t)$.

Filters

- Finally, by expanding the two Fourier transforms into their Fourier series we obtain that for all integers k

$$\begin{aligned} L[f](k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\hat{L}(\omega)e^{i\,\omega k}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} f(n)e^{-\,i\,n\omega}\right)\left(\sum_{m=-\infty}^{\infty} l(m)e^{-\,i\,m\omega}\right)e^{i\,\omega k}d\omega \\ &= \sum_{n,m=-\infty}^{\infty} f(n)l(m)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-(n+m))\omega}d\omega \end{aligned}$$

- However,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-(n+m))\omega}d\omega = \begin{cases} 1 & \text{if } k = m+n \\ 0 & \text{otherwise} \end{cases}$$

- Thus, each summand is non zero only for $k = m+n$, i.e., only for $m = k-n$, and we get:

$$L[f](k) = \sum_{n=-\infty}^{\infty} f(n)l(k-n) = \sum_{m=-\infty}^{\infty} f(n-m)l(m).$$

Filters

- Finally, by expanding the two Fourier transforms into their Fourier series we obtain that for all integers k

$$\begin{aligned} L[f](k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\hat{L}(\omega)e^{i\,\omega k}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}f(n)e^{-\,i\,n\omega}\right)\left(\sum_{m=-\infty}^{\infty}l(m)e^{-\,i\,m\omega}\right)e^{i\,\omega k}d\omega \\ &= \sum_{n,m=-\infty}^{\infty}f(n)l(m)\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(k-(n+m))\omega}d\omega \end{aligned}$$

- However,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(k-(n+m))\omega}d\omega = \begin{cases} 1 & \text{if } k = m+n \\ 0 & \text{otherwise} \end{cases}$$

- Thus, each summand is non zero only for $k = m + n$, i.e., only for $m = k - n$, and we get:

$$L[f](k) = \sum_{n=-\infty}^{\infty}f(n)l(k-n) = \sum_{m=-\infty}^{\infty}f(n-m)l(m).$$

Filters

- Finally, by expanding the two Fourier transforms into their Fourier series we obtain that for all integers k

$$
\begin{aligned}
L[f](k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(\omega)\hat{L}(\omega)e^{i\,\omega k}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} f(n)e^{-\,i\,n\omega}\right)\left(\sum_{m=-\infty}^{\infty} l(m)e^{-\,i\,m\omega}\right)e^{i\,\omega k}d\omega \\
&= \sum_{n,m=-\infty}^{\infty} f(n)l(m)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-(n+m))\omega}d\omega
\end{aligned}
$$

- However,

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-(n+m))\omega}d\omega = \begin{cases} 1 & \text{if } k = m+n \\ 0 & \text{otherwise} \end{cases}
$$

- Thus, each summand is non zero only for $k = m + n$, i.e., only for $m = k - n$, and we get:

$$
L[f](k) = \sum_{n=-\infty}^{\infty} f(n)l(k-n) = \sum_{m=-\infty}^{\infty} f(n-m)l(m).
$$

Filters

Thus, to summarise:

1. Every π-band limited signal of finite energy is uniquely determined by its samples on integers:
$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \text{sinc}\, \pi(t-k)$$
2. Every filter $L[f]$ acting on such signals is uniquely determined by its output $l(t)$ with the $\text{sinc}(\pi\, t)$ as input, $l(t) = L[\text{sinc}(\pi\, t)]$, called the impulse response of L:
$$L[f](t) = \int_{-\infty}^{\infty} f(u)l(t-u)du = \int_{-\infty}^{\infty} f(t-u)l(u)du.$$
3. The Fourier transform $\widehat{l}(\omega)$ of such $l(t)$ is called the transfer function of filter L and the Fourier transform $\widehat{L[f]}(\omega)$ of $L[f(t)]$ is obtained as
$$\widehat{L[f]}(\omega) = \widehat{f}(\omega)\widehat{l}(\omega).$$
4. The samples $\{L[f](n) \;:\; n \in \mathbb{Z}\}$ of the output $L[f](t)$ are obtained as a discrete linear convolution of the samples $\{f(n) \;:\; n \in \mathbb{Z}\}$ of the input signal $f(t)$ and the samples $\{l(n) \;:\; n \in \mathbb{Z}\}$ of the impulse response $l(t)$ of the filter L,
$$L[f](n) = \sum_{k=-\infty}^{\infty} f(k)l(n-k) = \sum_{k=-\infty}^{\infty} f(n-k)l(k)$$

Filters

Thus, to summarise:

1. Every π-band limited signal of finite energy is uniquely determined by its samples on integers:
$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\text{sinc}\,\pi(t-k)$$
2. Every filter $L[f]$ acting on such signals is uniquely determined by its output $l(t)$ with the $\text{sinc}(\pi\,t)$ as input, $l(t) = L[\text{sinc}(\pi\,t)]$, called the impulse response of L:
$$L[f](t) = \int_{-\infty}^{\infty} f(u)l(t-u)du = \int_{-\infty}^{\infty} f(t-u)l(u)du.$$
3. The Fourier transform $\widehat{l}(\omega)$ of such $l(t)$ is called the transfer function of filter L and the Fourier transform $\widehat{L[f]}(\omega)$ of $L[f(t)]$ is obtained as
$$\widehat{L[f]}(\omega) = \widehat{f}(\omega)\widehat{l}(\omega).$$
4. The samples $\{L[f](n) \, : \, n \in \mathbb{Z}\}$ of the output $L[f](t)$ are obtained as a discrete linear convolution of the samples $\{f(n) \, : \, n \in \mathbb{Z}\}$ of the input signal $f(t)$ and the samples $\{l(n) \, : \, n \in \mathbb{Z}\}$ of the impulse response $l(t)$ of the filter L,
$$L[f](n) = \sum_{k=-\infty}^{\infty} f(k)l(n-k) = \sum_{k=-\infty}^{\infty} f(n-k)l(k)$$

Filters

Thus, to summarise:

1. Every π-band limited signal of finite energy is uniquely determined by its samples on integers:
$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\operatorname{sinc}\pi(t-k)$$
2. Every filter $L[f]$ acting on such signals is uniquely determined by its output $l(t)$ with the $\operatorname{sinc}(\pi\, t)$ as input, $l(t) = L[\operatorname{sinc}(\pi\, t)]$, called the impulse response of L:
$$L[f](t) = \int_{-\infty}^{\infty} f(u)l(t-u)du = \int_{-\infty}^{\infty} f(t-u)l(u)du.$$
3. The Fourier transform $\widehat{l}(\omega)$ of such $l(t)$ is called the transfer function of filter L and the Fourier transform $\widehat{L[f]}(\omega)$ of $L[f(t)]$ is obtained as
$$\widehat{L[f]}(\omega) = \widehat{f}(\omega)\widehat{l}(\omega).$$
4. The samples $\{L[f](n) : n \in \mathbb{Z}\}$ of the output $L[f](t)$ are obtained as a discrete linear convolution of the samples $\{f(n) : n \in \mathbb{Z}\}$ of the input signal $f(t)$ and the samples $\{l(n) : n \in \mathbb{Z}\}$ of the impulse response $l(t)$ of the filter L,
$$L[f](n) = \sum_{k=-\infty}^{\infty} f(k)l(n-k) = \sum_{k=-\infty}^{\infty} f(n-k)l(k)$$

Filters

Thus, to summarise:

1. Every π-band limited signal of finite energy is uniquely determined by its samples on integers:
$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\mathrm{sinc}\,\pi(t-k)$$
2. Every filter $L[f]$ acting on such signals is uniquely determined by its output $l(t)$ with the $\mathrm{sinc}(\pi t)$ as input, $l(t) = L[\mathrm{sinc}(\pi t)]$, called the impulse response of L:
$$L[f](t) = \int_{-\infty}^{\infty} f(u)l(t-u)du = \int_{-\infty}^{\infty} f(t-u)l(u)du.$$
3. The Fourier transform $\widehat{l}(\omega)$ of such $l(t)$ is called the transfer function of filter L and the Fourier transform $\widehat{L[f]}(\omega)$ of $L[f(t)]$ is obtained as
$$\widehat{L[f]}(\omega) = \widehat{f}(\omega)\widehat{l}(\omega).$$
4. The samples $\{L[f](n) : n \in \mathbb{Z}\}$ of the output $L[f](t)$ are obtained as a discrete linear convolution of the samples $\{f(n) : n \in \mathbb{Z}\}$ of the input signal $f(t)$ and the samples $\{l(n) : n \in \mathbb{Z}\}$ of the impulse response $l(t)$ of the filter L,
$$L[f](n) = \sum_{k=-\infty}^{\infty} f(k)l(n-k) = \sum_{k=-\infty}^{\infty} f(n-k)l(k)$$

Filters

Thus, to summarise:

1. Every π-band limited signal of finite energy is uniquely determined by its samples on integers:
$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\text{sinc}\, \pi(t-k)$$
2. Every filter $L[f]$ acting on such signals is uniquely determined by its output $l(t)$ with the $\text{sinc}(\pi\, t)$ as input, $l(t) = L[\text{sinc}(\pi\, t)]$, called the impulse response of L:
$$L[f](t) = \int_{-\infty}^{\infty} f(u)l(t-u)du = \int_{-\infty}^{\infty} f(t-u)l(u)du.$$
3. The Fourier transform $\widehat{l}(\omega)$ of such $l(t)$ is called the transfer function of filter L and the Fourier transform $\widehat{L[f]}(\omega)$ of $L[f(t)]$ is obtained as
$$\widehat{L[f]}(\omega) = \widehat{f}(\omega)\widehat{l}(\omega).$$
4. The samples $\{L[f](n) \ : \ n \in \mathbb{Z}\}$ of the output $L[f](t)$ are obtained as a discrete linear convolution of the samples $\{f(n) \ : \ n \in \mathbb{Z}\}$ of the input signal $f(t)$ and the samples $\{l(n) \ : \ n \in \mathbb{Z}\}$ of the impulse response $l(t)$ of the filter L,
$$L[f](n) = \sum_{k=-\infty}^{\infty} f(k)l(n-k) = \sum_{k=-\infty}^{\infty} f(n-k)l(k)$$

Filters

- In practice, all signals have only finitely many samples, but can be rather long.
- For example, a 5 minute song would typically have have $5 \times 60 \times 44\,100 \approx 13$ million samples.
- The impulse response $l(k)$ of a filter would typically have up to a thousand or so non zero samples.
- Thus, the output signal $L[f](n)$ would be of the form

$$L[f](n) = \sum_{k=-N/2}^{N/2} f(n-k)l(k)$$

  where N is the length of the filter.
- To compute the convolution we would need to pad both the signal and the impulse response of the filter to the size equal to the number of samples of the signal plus the length of the impulse response of the filter minus 1, which is impractical.
- Thus, the signal is broken into frames of length about the length of the impulse response of the filter and the convolution with the impulse response of the filter is computed separately for each frame.
- The outputs of every two consecutive frames partially overlap and are appropriately added together with the proper offset to get the output of the filter for the whole input signal.

Filters

- In practice, all signals have only finitely many samples, but can be rather long.
- For example, a 5 minute song would typically have have $5 \times 60 \times 44\,100 \approx 13$ million samples.
- The impulse response $l(k)$ of a filter would typically have up to a thousand or so non zero samples.
- Thus, the output signal $L[f](n)$ would be of the form

$$L[f](n) = \sum_{k=-N/2}^{N/2} f(n-k)l(k)$$

  where N is the length of the filter.
- To compute the convolution we would need to pad both the signal and the impulse response of the filter to the size equal to the number of samples of the signal plus the length of the impulse response of the filter minus 1, which is impractical.
- Thus, the signal is broken into frames of length about the length of the impulse response of the filter and the convolution with the impulse response of the filter is computed separately for each frame.
- The outputs of every two consecutive frames partially overlap and are appropriately added together with the proper offset to get the output of the filter for the whole input signal.

Filters

- In practice, all signals have only finitely many samples, but can be rather long.
- For example, a 5 minute song would typically have have $5 \times 60 \times 44\,100 \approx 13$ million samples.
- The impulse response $l(k)$ of a filter would typically have up to a thousand or so non zero samples.
- Thus, the output signal $L[f](n)$ would be of the form

$$L[f](n) = \sum_{k=-N/2}^{N/2} f(n-k)l(k)$$

  where N is the length of the filter.
- To compute the convolution we would need to pad both the signal and the impulse response of the filter to the size equal to the number of samples of the signal plus the length of the impulse response of the filter minus 1, which is impractical.
- Thus, the signal is broken into frames of length about the length of the impulse response of the filter and the convolution with the impulse response of the filter is computed separately for each frame.
- The outputs of every two consecutive frames partially overlap and are appropriately added together with the proper offset to get the output of the filter for the whole input signal.

Filters

- In practice, all signals have only finitely many samples, but can be rather long.
- For example, a 5 minute song would typically have have $5 \times 60 \times 44\,100 \approx 13$ million samples.
- The impulse response $l(k)$ of a filter would typically have up to a thousand or so non zero samples.
- Thus, the output signal $L[f](n)$ would be of the form

$$L[f](n) = \sum_{k=-N/2}^{N/2} f(n-k)l(k)$$

  where N is the length of the filter.
- To compute the convolution we would need to pad both the signal and the impulse response of the filter to the size equal to the number of samples of the signal plus the length of the impulse response of the filter minus 1, which is impractical.
- Thus, the signal is broken into frames of length about the length of the impulse response of the filter and the convolution with the impulse response of the filter is computed separately for each frame.
- The outputs of every two consecutive frames partially overlap and are appropriately added together with the proper offset to get the output of the filter for the whole input signal.

Filters

- In practice, all signals have only finitely many samples, but can be rather long.
- For example, a 5 minute song would typically have have $5 \times 60 \times 44\,100 \approx 13$ million samples.
- The impulse response $l(k)$ of a filter would typically have up to a thousand or so non zero samples.
- Thus, the output signal $L[f](n)$ would be of the form

$$L[f](n) = \sum_{k=-N/2}^{N/2} f(n-k)l(k)$$

  where N is the length of the filter.
- To compute the convolution we would need to pad both the signal and the impulse response of the filter to the size equal to the number of samples of the signal plus the length of the impulse response of the filter minus 1, which is impractical.
- Thus, the signal is broken into frames of length about the length of the impulse response of the filter and the convolution with the impulse response of the filter is computed separately for each frame.
- The outputs of every two consecutive frames partially overlap and are appropriately added together with the proper offset to get the output of the filter for the whole input signal.

Filters

- In practice, all signals have only finitely many samples, but can be rather long.
- For example, a 5 minute song would typically have have $5 \times 60 \times 44\,100 \approx 13$ million samples.
- The impulse response $l(k)$ of a filter would typically have up to a thousand or so non zero samples.
- Thus, the output signal $L[f](n)$ would be of the form

$$L[f](n) = \sum_{k=-N/2}^{N/2} f(n-k)l(k)$$

  where N is the length of the filter.
- To compute the convolution we would need to pad both the signal and the impulse response of the filter to the size equal to the number of samples of the signal plus the length of the impulse response of the filter minus 1, which is impractical.
- Thus, the signal is broken into frames of length about the length of the impulse response of the filter and the convolution with the impulse response of the filter is computed separately for each frame.
- The outputs of every two consecutive frames partially overlap and are appropriately added together with the proper offset to get the output of the filter for the whole input signal.

Filters

- In practice, all signals have only finitely many samples, but can be rather long.
- For example, a 5 minute song would typically have have $5 \times 60 \times 44\,100 \approx 13$ million samples.
- The impulse response $l(k)$ of a filter would typically have up to a thousand or so non zero samples.
- Thus, the output signal $L[f](n)$ would be of the form

$$L[f](n) = \sum_{k=-N/2}^{N/2} f(n-k)l(k)$$

  where N is the length of the filter.
- To compute the convolution we would need to pad both the signal and the impulse response of the filter to the size equal to the number of samples of the signal plus the length of the impulse response of the filter minus 1, which is impractical.
- Thus, the signal is broken into frames of length about the length of the impulse response of the filter and the convolution with the impulse response of the filter is computed separately for each frame.
- The outputs of every two consecutive frames partially overlap and are appropriately added together with the proper offset to get the output of the filter for the whole input signal.

MP3 sound compression algorithm

- Unfortunately, and perhaps unexpectedly, the audio compression algorithm MP3 is a far more complex algorithm than JPEG, so here we can afford only a **very rough** sketch of it.
- Why compressing audio?
- As we saw, in order to keep the Shannon formula applicable and thus ensure a perfect recovery of the continuous input waveform from its discrete samples, we have to sample at a rate which is at least twice the frequency of the highest frequency component, which for sound is 20kHz.
- Since it is impossible to perfectly remove all frequencies higher than 20 kHz present in the sound, we sample the signal at a slightly higher rate than twice the sound bandwidth, namely 44,100 times a second.
- To preserve the sound dynamic accurately (the range of loudness), we have to sample the sound with quantisation of 16 bits (including the sign bit).

MP3 sound compression algorithm

- Unfortunately, and perhaps unexpectedly, the audio compression algorithm MP3 is a far more complex algorithm than JPEG, so here we can afford only a **very rough** sketch of it.
- Why compressing audio?
- As we saw, in order to keep the Shannon formula applicable and thus ensure a perfect recovery of the continuous input waveform from its discrete samples, we have to sample at a rate which is at least twice the frequency of the highest frequency component, which for sound is 20kHz.
- Since it is impossible to perfectly remove all frequencies higher than 20 kHz present in the sound, we sample the signal at a slightly higher rate than twice the sound bandwidth, namely 44,100 times a second.
- To preserve the sound dynamic accurately (the range of loudness), we have to sample the sound with quantisation of 16 bits (including the sign bit).

MP3 sound compression algorithm

- Unfortunately, and perhaps unexpectedly, the audio compression algorithm MP3 is a far more complex algorithm than JPEG, so here we can afford only a **very rough** sketch of it.
- Why compressing audio?
- As we saw, in order to keep the Shannon formula applicable and thus ensure a perfect recovery of the continuous input waveform from its discrete samples, we have to sample at a rate which is at least twice the frequency of the highest frequency component, which for sound is 20kHz.
- Since it is impossible to perfectly remove all frequencies higher than 20 kHz present in the sound, we sample the signal at a slightly higher rate than twice the sound bandwidth, namely 44,100 times a second.
- To preserve the sound dynamic accurately (the range of loudness), we have to sample the sound with quantisation of 16 bits (including the sign bit).

MP3 sound compression algorithm

- Unfortunately, and perhaps unexpectedly, the audio compression algorithm MP3 is a far more complex algorithm than JPEG, so here we can afford only a **very rough** sketch of it.
- Why compressing audio?
- As we saw, in order to keep the Shannon formula applicable and thus ensure a perfect recovery of the continuous input waveform from its discrete samples, we have to sample at a rate which is at least twice the frequency of the highest frequency component, which for sound is 20kHz.
- Since it is impossible to perfectly remove all frequencies higher than 20 kHz present in the sound, we sample the signal at a slightly higher rate than twice the sound bandwidth, namely 44,100 times a second.
- To preserve the sound dynamic accurately (the range of loudness), we have to sample the sound with quantisation of 16 bits (including the sign bit).

MP3 sound compression algorithm

- Unfortunately, and perhaps unexpectedly, the audio compression algorithm MP3 is a far more complex algorithm than JPEG, so here we can afford only a **very rough** sketch of it.
- Why compressing audio?
- As we saw, in order to keep the Shannon formula applicable and thus ensure a perfect recovery of the continuous input waveform from its discrete samples, we have to sample at a rate which is at least twice the frequency of the highest frequency component, which for sound is 20kHz.
- Since it is impossible to perfectly remove all frequencies higher than 20 kHz present in the sound, we sample the signal at a slightly higher rate than twice the sound bandwidth, namely 44,100 times a second.
- To preserve the sound dynamic accurately (the range of loudness), we have to sample the sound with quantisation of 16 bits (including the sign bit).

MP3 sound compression algorithm

- Thus, in raw format, for a single second of a single channel of sound, we would need 705,600 bits.
- A two hour movie with 5.1 audio channels would require 3.86GB, which could barely fit to a DVD (4.7 GB capacity).
- The only solution is sound compression by a factor of order of about 10.
- Unlike JPEG which encodes psycho-visual features of our visual sensory system in a single matrix Q, MP3 relies on a far more detailed features of our auditory system.
- First of all, our auditory system is not equally sensitive for all audible frequencies:

MP3 sound compression algorithm

- Thus, in raw format, for a single second of a single channel of sound, we would need 705,600 bits.
- A two hour movie with 5.1 audio channels would require 3.86GB, which could barely fit to a DVD (4.7 GB capacity).
- The only solution is sound compression by a factor of order of about 10.
- Unlike JPEG which encodes psycho-visual features of our visual sensory system in a single matrix Q, MP3 relies on a far more detailed features of our auditory system.
- First of all, our auditory system is not equally sensitive for all audible frequencies:

MP3 sound compression algorithm

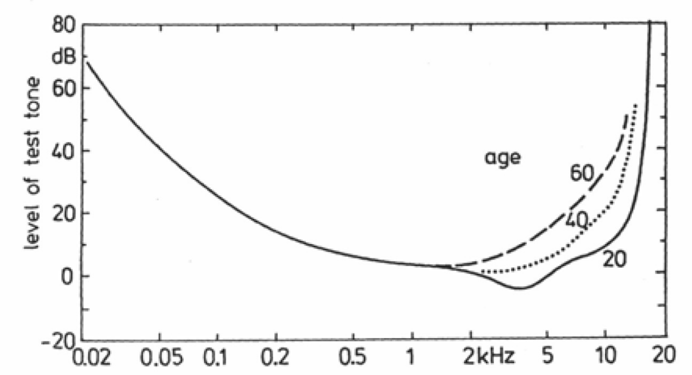
- Thus, in raw format, for a single second of a single channel of sound, we would need 705,600 bits.
- A two hour movie with 5.1 audio channels would require 3.86GB, which could barely fit to a DVD (4.7 GB capacity).
- The only solution is sound compression by a factor of order of about 10.
- Unlike JPEG which encodes psycho-visual features of our visual sensory system in a single matrix Q, MP3 relies on a far more detailed features of our auditory system.
- First of all, our auditory system is not equally sensitive for all audible frequencies:

MP3 sound compression algorithm

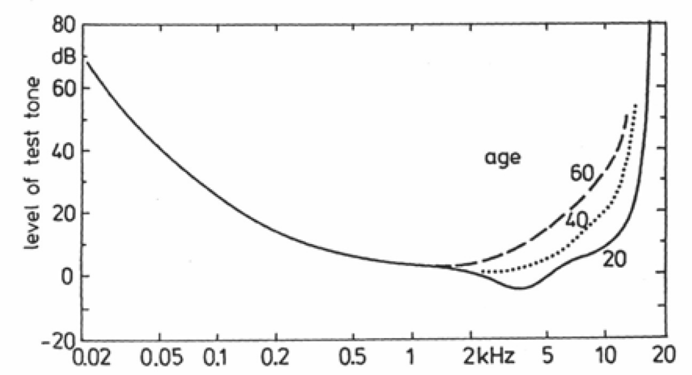
- Thus, in raw format, for a single second of a single channel of sound, we would need 705,600 bits.
- A two hour movie with 5.1 audio channels would require 3.86GB, which could barely fit to a DVD (4.7 GB capacity).
- The only solution is sound compression by a factor of order of about 10.
- Unlike JPEG which encodes psycho-visual features of our visual sensory system in a single matrix Q, MP3 relies on a far more detailed features of our auditory system.
- First of all, our auditory system is not equally sensitive for all audible frequencies:

MP3 sound compression algorithm

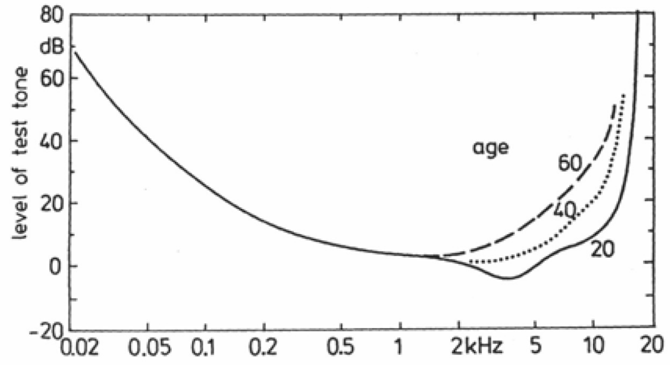
- Thus, in raw format, for a single second of a single channel of sound, we would need 705,600 bits.
- A two hour movie with 5.1 audio channels would require 3.86GB, which could barely fit to a DVD (4.7 GB capacity).
- The only solution is sound compression by a factor of order of about 10.
- Unlike JPEG which encodes psycho-visual features of our visual sensory system in a single matrix Q, MP3 relies on a far more detailed features of our auditory system.
- First of all, our auditory system is not equally sensitive for all audible frequencies:



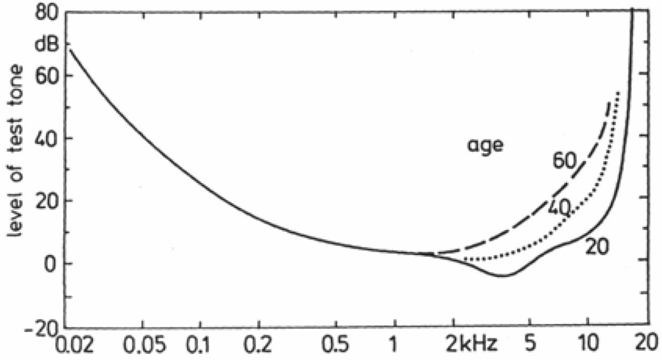
- Thus, just like in the case of JPEG, not all frequencies have to be encoded with equal quantisation step to keep the artefact unnoticeable.
- Also, higher amplitude can be quantised more coarsely, because we perceive only relative signal to noise ratio (SNR), not the absolute error.
- Such quantisation, which has progressively larger quantisation steps as the amplitude increases is called the power-law quantisation.



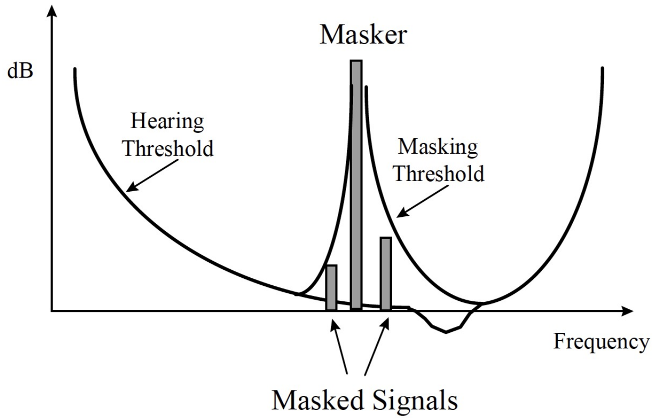
- Thus, just like in the case of JPEG, not all frequencies have to be encoded with equal quantisation step to keep the artefact unnoticeable.
- Also, higher amplitude can be quantised more coarsely, because we perceive only relative signal to noise ratio (SNR), not the absolute error.
- Such quantisation, which has progressively larger quantisation steps as the amplitude increases is called the power-law quantisation.



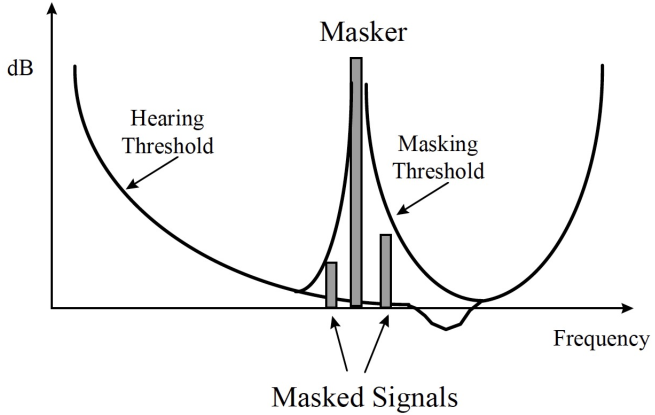
- Thus, just like in the case of JPEG, not all frequencies have to be encoded with equal quantisation step to keep the artefact unnoticeable.
- Also, higher amplitude can be quantised more coarsely, because we perceive only relative signal to noise ratio (SNR), not the absolute error.
- Such quantisation, which has progressively larger quantisation steps as the amplitude increases is called the power-law quantisation.



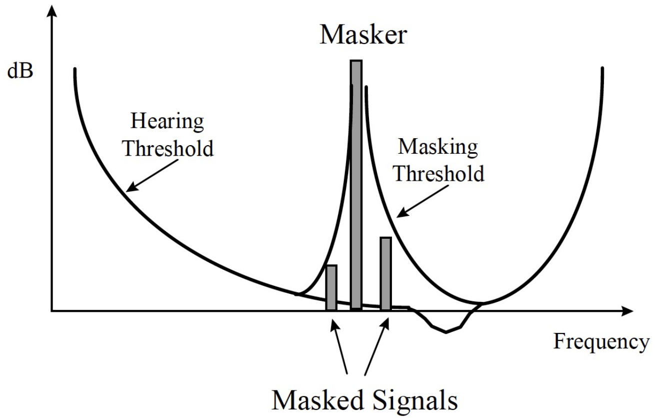
- Thus, just like in the case of JPEG, not all frequencies have to be encoded with equal quantisation step to keep the artefact unnoticeable.
- Also, higher amplitude can be quantised more coarsely, because we perceive only relative signal to noise ratio (SNR), not the absolute error.
- Such quantisation, which has progressively larger quantisation steps as the amplitude increases is called the power-law quantisation.



- The main psycho-acoustic feature of the human auditory system which is used in sound compression is called *frequency masking*.
- If there is a large amplitude component at a particular frequency, then components at the neighbouring frequencies, which would otherwise be above the hearing threshold, are masked by such a large amplitude component and cannot be perceived.
- Thus, there is no need to encode them in the compressed sound.

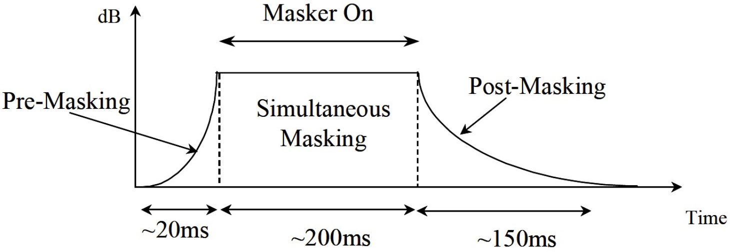


- The main psycho-acoustic feature of the human auditory system which is used in sound compression is called *frequency masking*.
- If there is a large amplitude component at a particular frequency, then components at the neighbouring frequencies, which would otherwise be above the hearing threshold, are masked by such a large amplitude component and cannot be perceived.
- Thus, there is no need to encode them in the compressed sound.



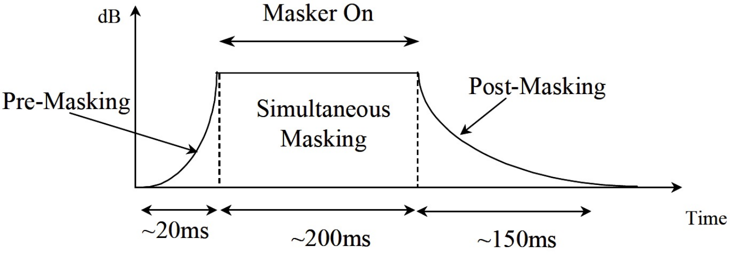
- The main psycho-acoustic feature of the human auditory system which is used in sound compression is called *frequency masking*.
- If there is a large amplitude component at a particular frequency, then components at the neighbouring frequencies, which would otherwise be above the hearing threshold, are masked by such a large amplitude component and cannot be perceived.
- Thus, there is no need to encode them in the compressed sound.

MP3 sound compression algorithm



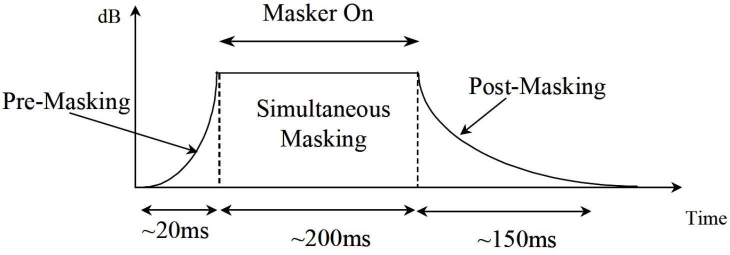
- Moreover, such masking takes place not only for the duration of the masker component, but also about 150 milliseconds after the masker has ended and, surprisingly, about 20 milliseconds *before* the masker starts!
- All of the above facts are taken into account by a rather complex encoding MP3 algorithm.

MP3 sound compression algorithm

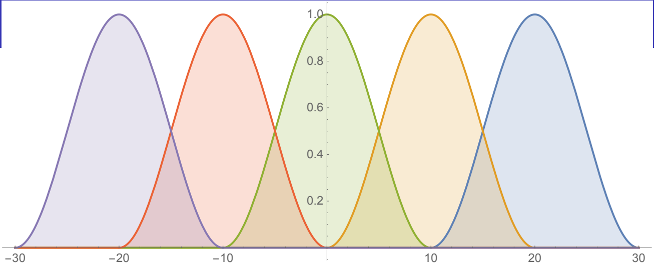


- Moreover, such masking takes place not only for the duration of the masker component, but also about 150 milliseconds after the masker has ended and, surprisingly, about 20 milliseconds *before* the masker starts!
- All of the above facts are taken into account by a rather complex encoding MP3 algorithm.

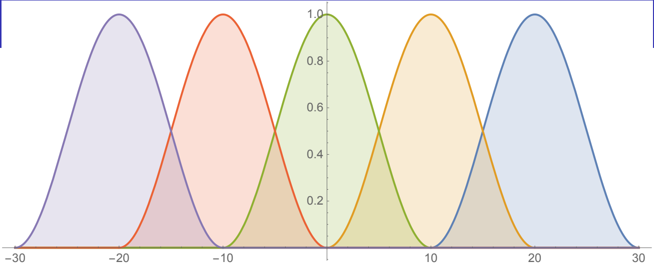
MP3 sound compression algorithm



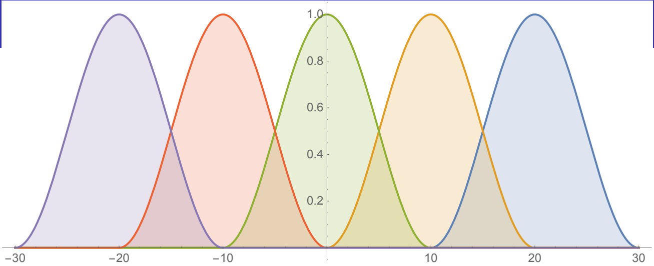
- Moreover, such masking takes place not only for the duration of the masker component, but also about 150 milliseconds after the masker has ended and, surprisingly, about 20 milliseconds *before* the masker starts!
- All of the above facts are taken into account by a rather complex encoding MP3 algorithm.



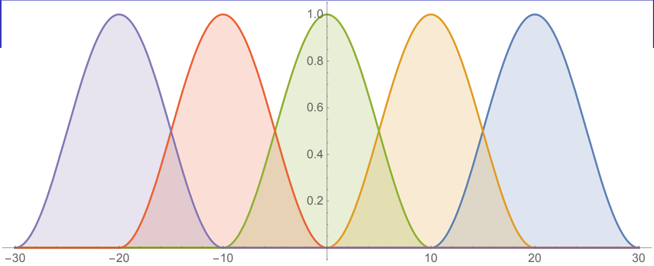
- To start, the algorithm uses windows (of several different durations) to break the signal into 50% overlapping frames containing about 1000 samples; the widows sum up to 1.
- Windows overlap to make sure transitions between the windows are smooth and unnoticeable.
- Each window is analysed for frequency content.
- Such frequency content of each window is compared with the frequency content of the previous and the subsequent window to decide which frequency components need to be encoded and with what number of bits, using the psycho-acoustic features we have discussed.



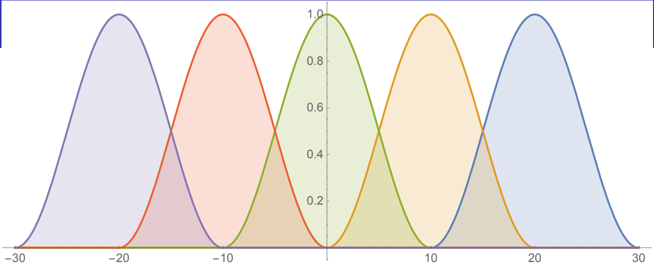
- To start, the algorithm uses windows (of several different durations) to break the signal into 50% overlapping frames containing about 1000 samples; the widows sum up to 1.
- Windows overlap to make sure transitions between the windows are smooth and unnoticeable.
- Each window is analysed for frequency content.
- Such frequency content of each window is compared with the frequency content of the previous and the subsequent window to decide which frequency components need to be encoded and with what number of bits, using the psycho-acoustic features we have discussed.



- To start, the algorithm uses windows (of several different durations) to break the signal into 50% overlapping frames containing about 1000 samples; the widows sum up to 1.
- Windows overlap to make sure transitions between the windows are smooth and unnoticeable.
- Each window is analysed for frequency content.
- Such frequency content of each window is compared with the frequency content of the previous and the subsequent window to decide which frequency components need to be encoded and with what number of bits, using the psycho-acoustic features we have discussed.

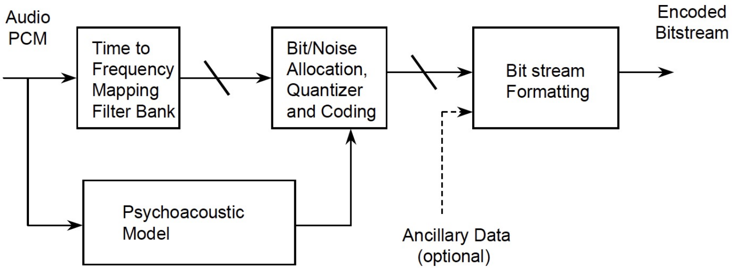


- To start, the algorithm uses windows (of several different durations) to break the signal into 50% overlapping frames containing about 1000 samples; the widows sum up to 1.
- Windows overlap to make sure transitions between the windows are smooth and unnoticeable.
- Each window is analysed for frequency content.
- Such frequency content of each window is compared with the frequency content of the previous and the subsequent window to decide which frequency components need to be encoded and with what number of bits, using the psycho-acoustic features we have discussed.

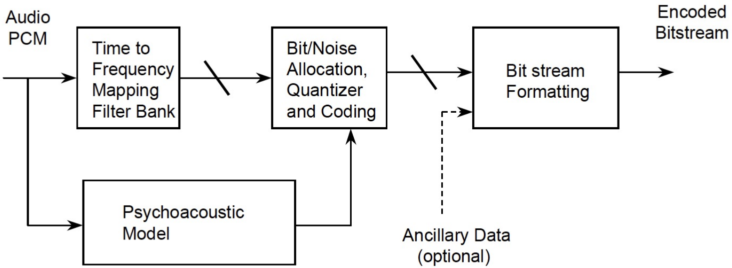


- To start, the algorithm uses windows (of several different durations) to break the signal into 50% overlapping frames containing about 1000 samples; the widows sum up to 1.
- Windows overlap to make sure transitions between the windows are smooth and unnoticeable.
- Each window is analysed for frequency content.
- Such frequency content of each window is compared with the frequency content of the previous and the subsequent window to decide which frequency components need to be encoded and with what number of bits, using the psycho-acoustic features we have discussed.

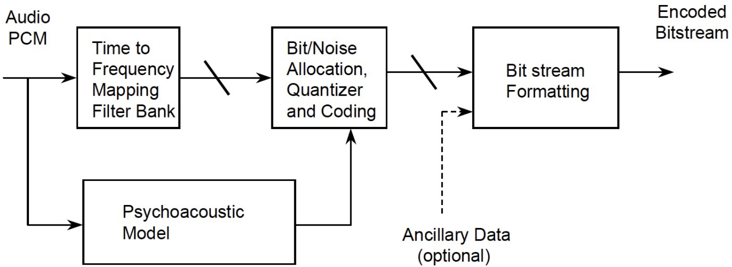
MP3



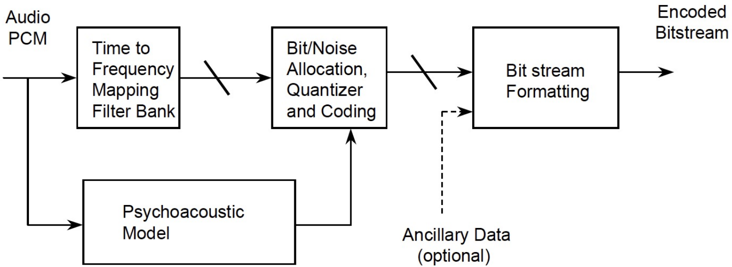
- The signal is then passed through a bank of filters (using a fast, FFT implementation of convolution).
- These filters can be of several kinds including DCT, so you can think of them as producing the DCT of the signal contained in each window.
- Only the coefficients of the relevant, audible frequency components will be encoded with minimal resolution sufficient to ensure that the quantisation noise is below the threshold of hearing for such a frequency component.
- The algorithm iteratively tries to allocate the bit budget optimally by comparing the IDCT of the quantised signal with the original to check if all the quantisation noise and dropped components stay below the audible threshold.
- It finally uses the Huffman encoding to encode the DCT coefficients of each frame as well as the information how the bits are distributed across the DCT coefficients.



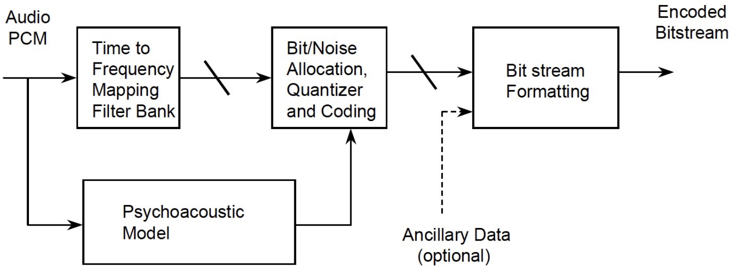
- The signal is then passed through a bank of filters (using a fast, FFT implementation of convolution).
- These filters can be of several kinds including DCT, so you can think of them as producing the DCT of the signal contained in each window.
- Only the coefficients of the relevant, audible frequency components will be encoded with minimal resolution sufficient to ensure that the quantisation noise is below the threshold of hearing for such a frequency component.
- The algorithm iteratively tries to allocate the bit budget optimally by comparing the IDCT of the quantised signal with the original to check if all the quantisation noise and dropped components stay below the audible threshold.
- It finally uses the Huffman encoding to encode the DCT coefficients of each frame as well as the information how the bits are distributed across the DCT coefficients.



- The signal is then passed through a bank of filters (using a fast, FFT implementation of convolution).
- These filters can be of several kinds including DCT, so you can think of them as producing the DCT of the signal contained in each window.
- Only the coefficients of the relevant, audible frequency components will be encoded with minimal resolution sufficient to ensure that the quantisation noise is below the threshold of hearing for such a frequency component.
- The algorithm iteratively tries to allocate the bit budget optimally by comparing the IDCT of the quantised signal with the original to check if all the quantisation noise and dropped components stay below the audible threshold.
- It finally uses the Huffman encoding to encode the DCT coefficients of each frame as well as the information how the bits are distributed across the DCT coefficients.

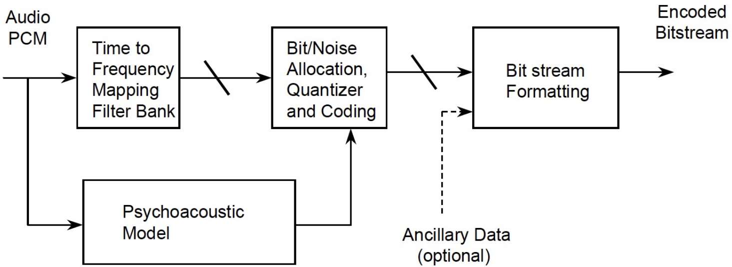


- The signal is then passed through a bank of filters (using a fast, FFT implementation of convolution).
- These filters can be of several kinds including DCT, so you can think of them as producing the DCT of the signal contained in each window.
- Only the coefficients of the relevant, audible frequency components will be encoded with minimal resolution sufficient to ensure that the quantisation noise is below the threshold of hearing for such a frequency component.
- The algorithm iteratively tries to allocate the bit budget optimally by comparing the IDCT of the quantised signal with the original to check if all the quantisation noise and dropped components stay below the audible threshold.
- It finally uses the Huffman encoding to encode the DCT coefficients of each frame as well as the information how the bits are distributed across the DCT coefficients.



- The signal is then passed through a bank of filters (using a fast, FFT implementation of convolution).
- These filters can be of several kinds including DCT, so you can think of them as producing the DCT of the signal contained in each window.
- Only the coefficients of the relevant, audible frequency components will be encoded with minimal resolution sufficient to ensure that the quantisation noise is below the threshold of hearing for such a frequency component.
- The algorithm iteratively tries to allocate the bit budget optimally by comparing the IDCT of the quantised signal with the original to check if all the quantisation noise and dropped components stay below the audible threshold.
- It finally uses the Huffman encoding to encode the DCT coefficients of each frame as well as the information how the bits are distributed across the DCT coefficients.

MP3



- The signal is then passed through a bank of filters (using a fast, FFT implementation of convolution).
- These filters can be of several kinds including DCT, so you can think of them as producing the DCT of the signal contained in each window.
- Only the coefficients of the relevant, audible frequency components will be encoded with minimal resolution sufficient to ensure that the quantisation noise is below the threshold of hearing for such a frequency component.
- The algorithm iteratively tries to allocate the bit budget optimally by comparing the IDCT of the quantised signal with the original to check if all the quantisation noise and dropped components stay below the audible threshold.
- It finally uses the Huffman encoding to encode the DCT coefficients of each frame as well as the information how the bits are distributed across the DCT coefficients.

MP3 sound compression algorithm

- Decoder is MUCH simpler; it uses such an information to invert the process and recover an approximation of the original waveform which then goes to a Digital to Analog Converter (DAC) to produce continuous time (voltage) signal which is amplified and fed into the speaker.
- So now hopefully you have a reasonably good idea how both still images and the sounds are encoded.
- Motion pictures (movies) supplement encoding of each frame of a movie via the JPEG with further compression by using the fact that consecutive frames are (most of the time) extremely correlated, so the past frames can be used to predict the subsequent frame and then encode only the error of the prediction, but this is a huge topic in itself.
- You can find more about the MP3 in the article “Audio Coding: Basic Principles And Recent Developments” by Marina Bosi from which the plots on this slides were taken.

MP3 sound compression algorithm

- Decoder is MUCH simpler; it uses such an information to invert the process and recover an approximation of the original waveform which then goes to a Digital to Analog Converter (DAC) to produce continuous time (voltage) signal which is amplified and fed into the speaker.
- So now hopefully you have a reasonably good idea how both still images and the sounds are encoded.
- Motion pictures (movies) supplement encoding of each frame of a movie via the JPEG with further compression by using the fact that consecutive frames are (most of the time) extremely correlated, so the past frames can be used to predict the subsequent frame and then encode only the error of the prediction, but this is a huge topic in itself.
- You can find more about the MP3 in the article "Audio Coding: Basic Principles And Recent Developments" by Marina Bosi from which the plots on this slides were taken.

MP3 sound compression algorithm

- Decoder is MUCH simpler; it uses such an information to invert the process and recover an approximation of the original waveform which then goes to a Digital to Analog Converter (DAC) to produce continuous time (voltage) signal which is amplified and fed into the speaker.
- So now hopefully you have a reasonably good idea how both still images and the sounds are encoded.
- Motion pictures (movies) supplement encoding of each frame of a movie via the JPEG with further compression by using the fact that consecutive frames are (most of the time) extremely correlated, so the past frames can be used to predict the subsequent frame and then encode only the error of the prediction, but this is a huge topic in itself.
- You can find more about the MP3 in the article "Audio Coding: Basic Principles And Recent Developments" by Marina Bosi from which the plots on this slides were taken.

MP3 sound compression algorithm

- Decoder is MUCH simpler; it uses such an information to invert the process and recover an approximation of the original waveform which then goes to a Digital to Analog Converter (DAC) to produce continuous time (voltage) signal which is amplified and fed into the speaker.
- So now hopefully you have a reasonably good idea how both still images and the sounds are encoded.
- Motion pictures (movies) supplement encoding of each frame of a movie via the JPEG with further compression by using the fact that consecutive frames are (most of the time) extremely correlated, so the past frames can be used to predict the subsequent frame and then encode only the error of the prediction, but this is a huge topic in itself.
- You can find more about the MP3 in the article "Audio Coding: Basic Principles And Recent Developments" by Marina Bosi from which the plots on this slides were taken.