

Where we're at

- We refined the abstract M-Machine to a **C-Machine**, with explicit stacks:

$$s \succ e \quad s \succ v$$

- Function application is still executed via substitution:

$$(\text{Apply } \langle\langle f.x. e \rangle\rangle \square) \triangleright s \succ v \mapsto_C s \succ e[x := v, f := (\text{Fun } (f.x.e))]$$

- We're going to extend our C-Machine to replace substitutions with an **environment**, giving us a new **E-Machine**

Environments

Definition

An *environment* is a *context* containing the values of variables.

It is like the *states* of TinyImp, except the value of a variable never changes.

$$\frac{}{\bullet \text{ Env}} \quad \frac{\eta \text{ Env}}{x = v, \eta \text{ Env}}$$

$\eta(x)$ denotes the *leftmost* value bound to x in η .

Let's change our machine states to include an *environment*:

$$s \mid \eta \succ e \quad s \mid \eta \prec v$$

First Attempt

First, we'll add a rule for consulting the environment if we encounter a free variable:

$$\frac{}{s \mid \eta \succ x \quad \mapsto_E \quad s \mid \eta \prec \eta(x)}$$

Then, we just need to handle function application.

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One broken attempt:

$$\frac{}{(\text{Apply } \langle\langle f.x. e \rangle\rangle \square) \triangleright s \mid \eta \prec v \mapsto_E s \mid (x = v, f = \langle\langle f.x. e \rangle\rangle, \eta) \succ e}$$

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We don't know when to remove the variables again!

Second Attempt

We will extend our **stacks** to allow us to **save** the old environment to it.

$$\frac{\eta \text{ Env } s \text{ Stack}}{\eta \triangleright s \text{ Stack}}$$

When we call a function, we save the environment to the stack.

$$(\text{Apply } \langle\langle f.x. e \rangle\rangle \square) \triangleright s \mid \eta \prec v \mapsto E$$

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When the function returns, we restore the old environment, clearing out the new bindings:

$$\overline{\eta \triangleright s \mid \eta' \prec v \mapsto_E s \mid \eta \prec v}$$

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$$\overline{\eta \triangleright s \mid \eta' \prec v \mapsto_E s \mid \eta \prec v}$$

This attempt is also broken (we'll see why soon)

Simple Example

○ | ● \succ (Ap (Fun (*f*.x. (Plus x (N 1))))

Simple Example

$$(\text{Ap } \square (\text{N } 3)) \triangleright \begin{array}{c} \circ \\ \circ \end{array} \mid \begin{array}{c} \bullet \\ \bullet \end{array}$$

$$\begin{array}{l} \succ (\text{Ap } (\text{Fun } (f.x. (\text{Plus } \times (\text{N } 1)))) \\ \succ (\text{Fun } (f.x. (\text{Plus } \times (\text{N } 1)))) \end{array}$$

Simple Example

○ | ●
 (Ap □ (N 3)) ▷ ○ | ●
 (Ap □ (N 3)) ▷ ○ | ●

⋈ (Ap (Fun (f.x. (Plus x (N 1))))
 ⋈ (Fun (f.x. (Plus x (N 1))))
 ⋈ ⟨⟨f.x. (Plus x (N 1))⟩⟩

Simple Example

	○		●
(Ap □ (N 3)) ▷	○		●
(Ap □ (N 3)) ▷	○		●
(Ap «···» □) ▷	○		●

⌊	(Ap (Fun (f.x. (Plus x (N 1))))
⌊	(Fun (f.x. (Plus x (N 1))))
⌊	«f.x. (Plus x (N 1))»
⌊	(N 3)

Simple Example

	○		●	γ	(Ap (Fun (<i>f.x.</i> (Plus × (N 1))))))
(Ap □ (N 3)) ▷	○		●	γ	(Fun (<i>f.x.</i> (Plus × (N 1))))
(Ap □ (N 3)) ▷	○		●	λ	⟨⟨ <i>f.x.</i> (Plus × (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	γ	(N 3)
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	λ	3

Simple Example

○		●	⋈	(Ap (Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷ ○		●	⋈	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷ ○		●	⋈	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □) ▷ ○		●	⋈	(N 3)
(Ap ⟨⟨...⟩⟩ □) ▷ ○		●	⋈	3
● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	(Plus x (N 1))

Simple Example

	○		●	⌞	(Ap (Fun (f.x. (Plus x (N 1)))))
(Ap □ (N 3)) ▷	○		●	⌞	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	⌞	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	⌞	(N 3)
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	⌞	3
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(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	⌞	x

Simple Example

	○		●	↘	(Ap (Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	↘	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	↘	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	↘	(N 3)
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	↘	3
	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	(Plus x (N 1))
(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	x
(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	3

Simple Example

	○		●	⋈	(Ap (Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	⋈	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	⋈	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	⋈	(N 3)
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	⋈	3
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(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	x
(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	3
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(Ap □ (N 3))▷	○		●	⋈	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3))▷	○		●	⋈	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □)▷	○		●	⋈	(N 3)
(Ap ⟨⟨...⟩⟩ □)▷	○		●	⋈	3
	●▷○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	(Plus x (N 1))
(Plus □ (N 1))▷	●▷○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	x
(Plus □ (N 1))▷	●▷○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	3
(Plus 3 □)▷	●▷○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	(N 1)
(Plus 3 □)▷	●▷○		x = 3, f = ⟨⟨...⟩⟩, ●	⋈	1

Simple Example

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(Ap □ (N 3)) ▷	○		●	↘	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	↘	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	↘	(N 3)
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	↘	3
	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	(Plus x (N 1))
(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	x
(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	3
(Plus 3 □) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	(N 1)
(Plus 3 □) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	1
	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	4

Simple Example

	○		●	↘	(Ap (Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	↘	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3)) ▷	○		●	↘	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	↘	(N 3)
(Ap ⟨⟨...⟩⟩ □) ▷	○		●	↘	3
	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	(Plus x (N 1))
(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	x
(Plus □ (N 1)) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	3
(Plus 3 □) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	(N 1)
(Plus 3 □) ▷	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	1
	● ▷ ○		x = 3, f = ⟨⟨...⟩⟩, ●	↘	4
	○		●	↘	4

Simple Example

	○		●	↗	(Ap (Fun (f.x. (Plus x (N 1)))))
(Ap □ (N 3))▷	○		●	↗	(Fun (f.x. (Plus x (N 1))))
(Ap □ (N 3))▷	○		●	↘	⟨⟨f.x. (Plus x (N 1))⟩⟩
(Ap ⟨⟨...⟩⟩ □)▷	○		●	↗	(N 3)
(Ap ⟨⟨...⟩⟩ □)▷	○		●	↘	3
	●▷	○	x = 3, f = ⟨⟨...⟩⟩, ●	↗	(Plus x (N 1))
(Plus □ (N 1))▷	●▷	○	x = 3, f = ⟨⟨...⟩⟩, ●	↗	x
(Plus □ (N 1))▷	●▷	○	x = 3, f = ⟨⟨...⟩⟩, ●	↘	3
(Plus 3 □)▷	●▷	○	x = 3, f = ⟨⟨...⟩⟩, ●	↗	(N 1)
(Plus 3 □)▷	●▷	○	x = 3, f = ⟨⟨...⟩⟩, ●	↘	1
	●▷	○	x = 3, f = ⟨⟨...⟩⟩, ●	↘	4
	○		●	↘	4

Seems to work for **basic examples**, but is there some way to break it?

Closure Capture

○ | ● λ (Ap (Ap (Fun ($f.x.$ (Fun ($g.y.$ x)))) (N 3)) (N 4))

Closure Capture

$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$

$\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$

Closure Capture

$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$

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$\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x))))$

Closure Capture

$$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x))))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle$$

Closure Capture

$$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x))))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle$$

$$\mapsto_E (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3)$$

Closure Capture

$$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))))) (\text{N } 3)) (\text{N } 4))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$$

$$\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x))))$$

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$$\mapsto_E (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3)$$

$$\mapsto_E (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ 3$$

Closure Capture

$$\begin{aligned}
& \circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4)) \\
\mapsto_E & (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) \\
\mapsto_E & (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) \\
\mapsto_E & (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle \\
\mapsto_E & (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3) \\
\mapsto_E & (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec 3 \\
\mapsto_E & \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \succ (\text{Fun } (g.y. x))
\end{aligned}$$

Closure Capture

$$\begin{aligned}
& \circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))))) (\text{N } 3)) (\text{N } 4)) \\
\mapsto_E & (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) \\
\mapsto_E & (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) \\
\mapsto_E & (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle \\
\mapsto_E & (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3) \\
\mapsto_E & (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec 3 \\
\mapsto_E & \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \succ (\text{Fun } (g.y. x)) \\
\mapsto_E & \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \prec \langle\langle g.y. x \rangle\rangle
\end{aligned}$$

Closure Capture

$$\begin{aligned} & \circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4)) \\ \mapsto_E & (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) \\ \mapsto_E & (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) \\ \mapsto_E & (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle \\ \mapsto_E & (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3) \\ \mapsto_E & (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec 3 \\ \mapsto_E & \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \succ (\text{Fun } (g.y. x)) \\ \mapsto_E & \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \prec \langle\langle g.y. x \rangle\rangle \\ \mapsto_E & (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle g.y. x \rangle\rangle \end{aligned}$$

Closure Capture

$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$
 $\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$
 $\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x))))$
 $\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle$
 $\mapsto_E (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3)$
 $\mapsto_E (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec 3$
 $\mapsto_E \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \succ (\text{Fun } (g.y. x))$
 $\mapsto_E \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \prec \langle\langle g.y. x \rangle\rangle$
 $\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle g.y. x \rangle\rangle$
 $\mapsto_E (\text{Ap } \langle\langle g.y. x \rangle\rangle \square) \triangleright \circ \mid \bullet \succ (\text{N } 4)$

Closure Capture

$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$
 $\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$
 $\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x))))$
 $\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle$
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 $\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle g.y. x \rangle\rangle$
 $\mapsto_E (\text{Ap } \langle\langle g.y. x \rangle\rangle \square) \triangleright \circ \mid \bullet \succ (\text{N } 4)$
 $\mapsto_E (\text{Ap } \langle\langle g.y. x \rangle\rangle \square) \triangleright \circ \mid \bullet \prec 4$

Closure Capture

$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$
 $\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$
 $\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Fun } (f.x. (\text{Fun } (g.y. x))))$
 $\mapsto_E (\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle f.x. (\text{Fun } (g.y. x)) \rangle\rangle$
 $\mapsto_E (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3)$
 $\mapsto_E (\text{Ap } \langle\langle f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec 3$
 $\mapsto_E \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \succ (\text{Fun } (g.y. x))$
 $\mapsto_E \bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle\langle f \dots \rangle\rangle, \bullet \prec \langle\langle g.y. x \rangle\rangle$
 $\mapsto_E (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle g.y. x \rangle\rangle$
 $\mapsto_E (\text{Ap } \langle\langle g.y. x \rangle\rangle \square) \triangleright \circ \mid \bullet \succ (\text{N } 4)$
 $\mapsto_E (\text{Ap } \langle\langle g.y. x \rangle\rangle \square) \triangleright \circ \mid \bullet \prec 4$
 $\mapsto_E \bullet \triangleright \circ \mid y = 4, g = \langle\langle g.y. x \rangle\rangle, \bullet \succ x$

Oh no! We're stuck!

Something went wrong!

When we return functions, the function's body escapes the scope of bound variables from where it was defined:

```
(let x = 3 in recfun f y = x + y) 5
```

The function value $\langle\langle f.y. x + y \rangle\rangle$, when it is applied, does not “remember” that $x = 3$.

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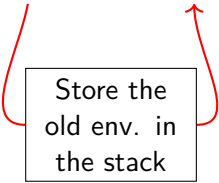
The function value $\langle\langle f.y. x + y \rangle\rangle$, when it is applied, does not “remember” that $x = 3$.

Solution: Store the environment inside the function value!

$$\frac{}{s \mid \eta \succ (\text{Recfun } (f.x. e)) \mapsto_E s \mid \eta \prec \langle\langle \eta, f.x. e \rangle\rangle}$$

This type of function value is called a *closure*.

$(\text{Apply } \langle\langle \eta', f.x. e \rangle\rangle \square) \triangleright s \mid \eta \prec v \mapsto_E \eta \triangleright s \mid (x = v, f = \langle\langle f.x. e \rangle\rangle, \eta') \succ e$

$$(\text{Apply } \langle\langle \eta', f.x. e \rangle\rangle \square) \triangleright s \mid \eta \prec v \mapsto_E \eta \triangleright s \mid (x = v, f = \langle\langle f.x. e \rangle\rangle, \eta') \succ e$$


Store the
old env. in
the stack

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old env. in
the stack

Retrieve the new
env. from the closure

$\circ \mid \bullet \succ (\text{Ap } (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3)) (\text{N } 4))$
 $(\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$
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 $(\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{Ap } (\text{Fun } (f.x. (\text{Fun } (g.y. x)))) (\text{N } 3))$
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 $(\text{Ap } \square (\text{N } 3)) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec \langle\langle \bullet, f.x. (\text{Fun } (g.y. x)) \rangle\rangle$
 $(\text{Ap } \langle\langle \bullet, f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \succ (\text{N } 3)$
 $(\text{Ap } \langle\langle \bullet, f \dots \rangle\rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec 3$

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 $(\text{Ap } \langle \langle \bullet, f \dots \rangle \rangle \square) \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid \bullet \prec 3$
 $\bullet \triangleright (\text{Ap } \square (\text{N } 4)) \triangleright \circ \mid x = 3, f = \langle \langle f \dots \rangle \rangle, \bullet \succ (\text{Fun } (g.y. x))$

- | ● \succ (Ap (Ap (Fun (f.x. (Fun (g.y. x)))) (N 3)) (N 4))
- (Ap □ (N 4)) \triangleright ○ | ● \succ (Ap (Fun (f.x. (Fun (g.y. x)))) (N 3))
- (Ap □ (N 3)) \triangleright (Ap □ (N 4)) \triangleright ○ | ● \succ (Fun (f.x. (Fun (g.y. x))))
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- (Ap $\langle\langle$ ●, f \dots $\rangle\rangle$ □) \triangleright (Ap □ (N 4)) \triangleright ○ | ● \succ (N 3)
- (Ap $\langle\langle$ ●, f \dots $\rangle\rangle$ □) \triangleright (Ap □ (N 4)) \triangleright ○ | ● \prec 3
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- \triangleright (Ap □ (N 4)) \triangleright ○ | x = 3, f = $\langle\langle$ f \dots $\rangle\rangle$, ● \prec $\langle\langle$ (x = 3, f = \dots , ●), g.y. x $\rangle\rangle$

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 $(\text{Ap } \langle \langle (x = 3, f = \dots, \bullet), g.y. x \rangle \rangle \square) \triangleright \circ \mid \bullet \succ (\text{N } 4)$
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- | ● \prec 3

Refinement

- We already sketched a proof that each C-machine execution has a corresponding M-machine execution (**refinement**).
- This means any functional correctness (not security or cost) property we prove about all M-machine executions of a program apply just as well to any C-machine executions of the same program.
- Now we want to prove that each E-machine execution has a corresponding C-machine execution (and therefore an M-machine execution).

Ingredients for Refinement

Once again, we want an **abstraction function** \mathcal{A} that converts E-machine states to C-machine states, such that:

- Each initial state in the E-machine is mapped to an initial state in the C-Machine.
- Each final state in the E-machine is mapped to a final state in the C-Machine.
- For each E-machine transition, either there is a corresponding C-Machine transition, or the two E-machine states map to the same C-machine state.

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- E-Machine values are converted to C-Machine values merely by applying the environment inside closures as a substitution to the expression inside the closure.

With such a function definition, it is trivial to prove that each E-Machine transition has a corresponding transition in the C-Machine, as it is 1:1.

Except!

There is one rule which is not 1:1. **Which one?**

Semantics Refinement to Program Refinement

The C-Machine and M-machine on these slides have been presented in a very abstract way.

However our semantics is now closer to a program implementation:

- The states (with \square gaps) of the C-Machine are the nodes of the control-flow graph of the program.
- The environments of our E-machine become concrete objects:
 - Stack frames
 - Closure objects (also called thunks)

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- The next step is to adjust the program to make this semantics explicit.

We have also learned how to *prove* a step to a lower-level representation is correct. We could re-use these ideas when studying a compiler.

An Alternative: A Normalisation

Here is an alternative approach to the C-Machine construction.

First, transform the program into **A Normal** form:

$$\begin{aligned} 2 + (3 * (4 + x)) &= \text{let } v_1 = 4 + x \text{ in} \\ &\quad \text{let } v_2 = 3 + v_1 \text{ in} \\ &\quad 2 + v_2 \end{aligned}$$

We can prove this transformation is correct, for instance by refinement.

- The need for fresh names v_1, v_2, \dots is a nuisance.

An Alternative: A Normalisation

Here is an alternative approach to the C-Machine construction.

First, transform the program into **A Normal** form:

$$\begin{aligned} 2 + (3 * (4 + x)) &= \text{let } v_1 = 4 + x \text{ in} \\ &\quad \text{let } v_2 = 3 + v_1 \text{ in} \\ &\quad 2 + v_2 \end{aligned}$$

We can prove this transformation is correct, for instance by refinement.

- The need for fresh names v_1, v_2, \dots is a nuisance.

We can now simplify the use of \square in our C-Machine. How?

Order of Evaluation

In either the C-Machine or A-normal form, we commit to the evaluation order of our language.

$$2 + (3 * (4 + x)) = \text{let } v_1 = 4 + x \text{ in} \\ \text{let } v_2 = 3 + v_1 \text{ in} \\ 2 + v_2$$

Now is a good time to think about different styles of evaluation order, especially Haskell's call-by-need or lazy evaluation.