

## Syntax

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# Concrete Syntax

## Arithmetic Expressions

$$\begin{array}{c}
 \frac{i \in \mathbb{Z}}{i \text{ Atom}} \\
 \frac{a \text{ SExp}}{(a) \text{ Atom}} \\
 \frac{e \text{ Atom}}{e \text{ PExp}} \\
 \frac{e \text{ PExp}}{e \text{ SExp}} \\
 \frac{a \text{ Atom} \quad b \text{ PExp}}{a \times b \text{ PExp}} \\
 \frac{a \text{ PExp} \quad b \text{ SExp}}{a + b \text{ SExp}}
 \end{array}$$

All the syntax we have seen so far is *concrete syntax*. Concrete syntax is described by judgements on *strings*.

# Abstract Syntax

Working with concrete syntax directly is *unsuitable* for both compiler implementation and proofs. Consider:

- $3 + (4 \times 5)$
- $3 + 4 \times 5$
- $(3 + (4 \times 5))$

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<sup>1</sup>“There is more than one way to do it”.

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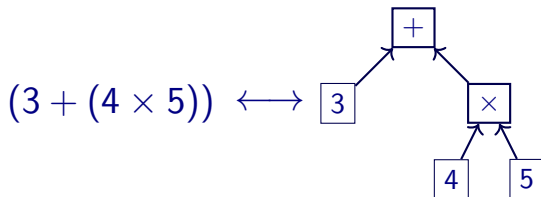
**TIMTOWTDI**<sup>1</sup> makes life harder for us. Different derivations represent the same semantic program. We would like a representation of programs that is as simple as possible, removing any extraneous information. Such a representation is called *abstract syntax*.

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<sup>1</sup>“There is more than one way to do it”.

# Abstract Syntax

Typically, the *abstract syntax* of a program is represented as a **tree** rather than as a string.



Writing trees in our inference rules would become unwieldy. We shall define a **term** language in which to express trees.

# Terms

## Definition

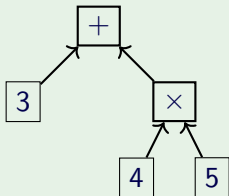
In this course, a *term* is a structure that can either be a *symbol*, like `Plus` or `Times` or `3`; or a *compound*, which consists of an *symbol* followed by one or more *argument subterms*, all in parentheses.

$$t ::= \text{Symbol} \mid (\text{Symbol } t_1 t_2 \dots)$$

These particular terms are also known as *s-expressions*. Terms can equivalently be thought of a subset of `Haskell` where the only kinds of expressions allowed are literals and data constructors.

## Term Examples

### Example



(Plus (Num 3) (Times (Num 4) (Num 5)))

Armed with an appropriate Haskell data declaration, this can be implemented straightforwardly:

```
data Exp = Plus Exp Exp  
         | Times Exp Exp  
         | Num Int
```

# Concrete to Abstract

## Concrete Syntax

$$\begin{array}{c}
 \frac{i \in \mathbb{Z}}{i \text{ Atom}} \quad \frac{a \text{ SExp}}{(a) \text{ Atom}} \quad \frac{e \text{ Atom}}{e \text{ PExp}} \quad \frac{e \text{ PExp}}{e \text{ SExp}} \\
 \frac{a \text{ Atom} \quad b \text{ PExp}}{a \times b \text{ PExp}} \quad \frac{a \text{ PExp} \quad b \text{ SExp}}{a + b \text{ SExp}}
 \end{array}$$

## Abstract Syntax

$$\frac{i \in \mathbb{Z}}{(\text{Num } i) \text{ AST}} \quad \frac{a \text{ AST} \quad b \text{ AST}}{(\text{Plus } a \ b) \text{ AST}} \quad \frac{a \text{ AST} \quad b \text{ AST}}{(\text{Times } a \ b) \text{ AST}}$$

Now we have to specify a *relation* to connect the two!

# Relations

Up until now, most judgements we have used have been *unary* — corresponding to a set of satisfying objects.

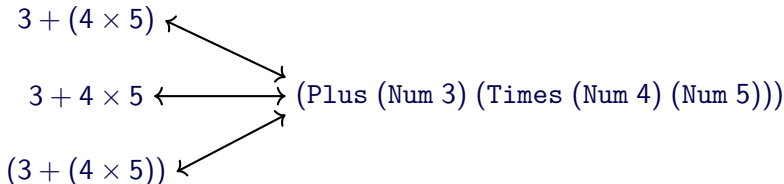
A judgement can also express a relationship between *two objects* (a *binary* judgement) or a *number of objects* (an *n-ary* judgement).

## Example (Relations)

- 4 *divides* 16 (binary)
- mail *is an anagram of* liam (binary)
- 3 *plus* 5 *equals* 8 (ternary)

*n*-ary judgements where  $n \geq 2$  are sometimes called *relations*, and correspond to an *n*-tuple of satisfying objects.

## Parsing Relation



$$i \in \mathbb{Z}$$


---


$$i \text{ Atom}$$

$$a \text{ Atom}$$

$$b \text{ PExp}$$


---


$$a \times b \text{ PExp}$$

$$a \text{ PExp}$$

$$b \text{ SExp}$$


---


$$a + b \text{ SExp}$$

$$e \text{ SExp}$$

$$e \text{ Atom}$$

$$e \text{ PExp}$$


---


$$(e) \text{ Atom}$$

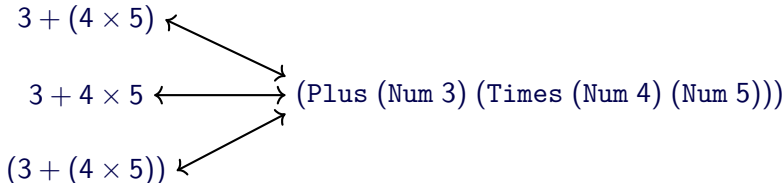

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---


$$e \text{ SExp}$$

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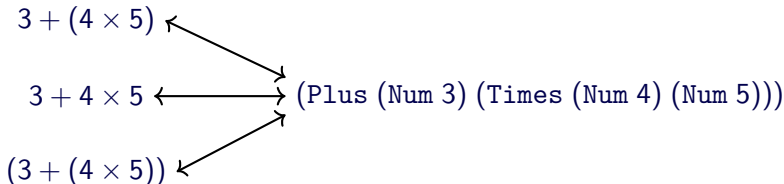


$$i \in \mathbb{Z}$$

$$i \text{ Atom} \longleftrightarrow (\text{Num } i) \text{ AST}$$

 $a \text{ Atom}$ 
 $b \text{ PExp}$ 
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## Parsing Relation



$$i \in \mathbb{Z}$$

$$\frac{}{i \text{ Atom} \longleftrightarrow (\text{Num } i) \text{ AST}}$$

$$\frac{a \text{ Atom} \longleftrightarrow a' \text{ AST} \quad b \text{ PExp} \longleftrightarrow b' \text{ AST}}{a \times b \text{ PExp}}$$

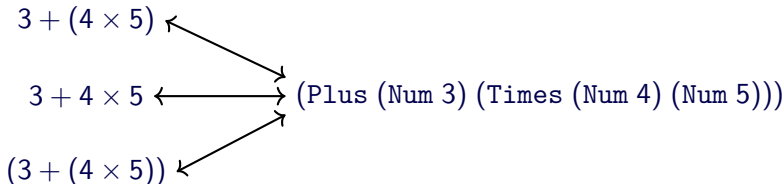
$$\frac{a \text{ PExp} \quad b \text{ SExp}}{a + b \text{ SExp}}$$

$$\frac{e \text{ SExp}}{(e) \text{ Atom}}$$

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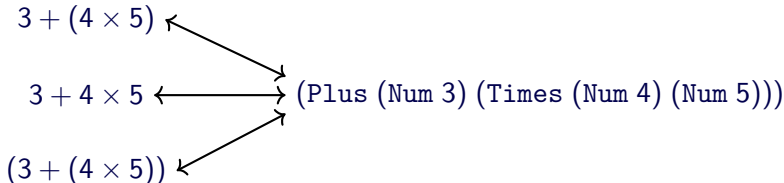
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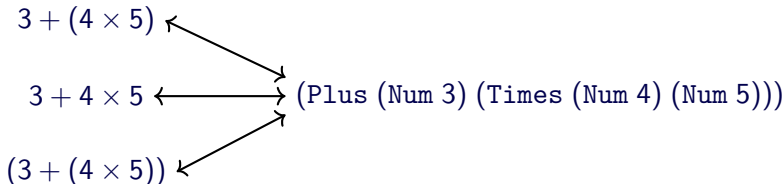
$$\frac{}{a + b \text{ SExp} \longleftrightarrow (\text{Plus } a' b') \text{ AST}}$$

$$\frac{e \text{ SExp}}{(e) \text{ Atom}}$$

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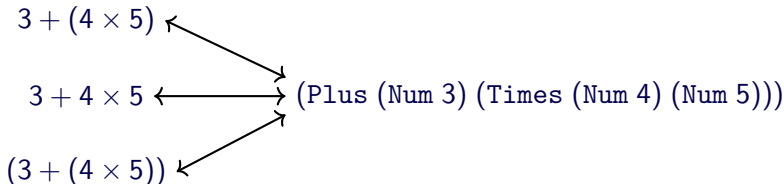
$$\frac{e \text{ SExp} \longleftrightarrow a' \text{ AST} \quad e \text{ Atom}}{e \text{ Atom} \longleftrightarrow a' \text{ AST}}$$

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$$\frac{}{e \text{ SExp} \longleftrightarrow a' \text{ AST}} \quad \frac{}{e \text{ Atom} \longleftrightarrow a \text{ AST}} \quad \frac{}{e \text{ PExp} \longleftrightarrow a \text{ AST}}$$

$$\frac{}{(e) \text{ Atom} \longleftrightarrow a' \text{ AST}} \quad \frac{}{e \text{ PExp} \longleftrightarrow a \text{ AST}} \quad \frac{}{e \text{ SExp} \longleftrightarrow a \text{ AST}}$$

## Relations as Algorithms

The *parsing relation*  $\longleftrightarrow$  is an extension of our existing concrete syntax rules. Therefore it is **unambiguous**, just as those rules are. Furthermore, the abstract syntax can be **unambiguously** determined **solely** by looking at the left hand side of  $\longleftrightarrow$ .

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The *parsing relation*  $\longleftrightarrow$  is an extension of our existing concrete syntax rules. Therefore it is **unambiguous**, just as those rules are. Furthermore, the abstract syntax can be **unambiguously** determined **solely** by looking at the left hand side of  $\longleftrightarrow$ .

### An Algorithm

To determine the term corresponding to a particular string:

- 1 Derive the left hand side of the  $\longleftrightarrow$  (the concrete syntax) **bottom-up** until reaching axioms.
- 2 Fill in the right hand side of the  $\longleftrightarrow$  (the abstract syntax) **top-down**, starting at the axioms.

This process is called *parsing*.

# Example

## Rules

$$\begin{array}{c}
 \frac{i \in \mathbb{Z}}{i \mathbf{A}} \\
 \\
 \frac{a \mathbf{A}}{a \times b \mathbf{P}} \\
 \\
 \frac{a \mathbf{S}}{(a) \mathbf{A}} \\
 \\
 \frac{e \mathbf{A}}{e \mathbf{P}} \\
 \\
 \frac{e \mathbf{P}}{e \mathbf{S}} \\
 \\
 \frac{a \mathbf{P}}{a + b \mathbf{S}} \\
 \\
 \frac{b \mathbf{S}}{b \mathbf{S}}
 \end{array}$$

---


$$1 + 2 \times 3 \mathbf{S}$$

# Example

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 \frac{e \mathbf{P}}{e \mathbf{S}} \\
 \frac{a \mathbf{P}}{a + b \mathbf{S}} \\
 \frac{b \mathbf{S}}{b \mathbf{S}}
 \end{array}$$

$$\frac{\frac{1 \mathbf{P}}{1 + 2 \times 3 \mathbf{S}} \quad \frac{2 \times 3 \mathbf{S}}{2 \times 3 \mathbf{S}}}{1 + 2 \times 3 \mathbf{S}}$$

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 \frac{a \mathbf{A}}{a \times b \mathbf{P}} \\
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 \end{array}$$

$$\frac{\frac{1 \mathbf{A}}{1 \mathbf{P}}}{1 + 2 \times 3 \mathbf{S}}$$

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## Rules

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 \frac{i \in \mathbb{Z}}{i \mathbf{A} \longleftrightarrow (\text{Num } i)} \quad \frac{a \mathbf{S} \longleftrightarrow a'}{(a) \mathbf{A} \longleftrightarrow a'} \quad \frac{e \mathbf{A} \longleftrightarrow a}{e \mathbf{P} \longleftrightarrow a} \quad \frac{e \mathbf{P} \longleftrightarrow a}{e \mathbf{S} \longleftrightarrow a} \\
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 \end{array}$$

$$\begin{array}{c}
 \frac{1 \mathbf{A} \longleftrightarrow (\text{Num } 1) \text{ AST}}{1 \mathbf{P}} \quad \frac{\frac{2 \mathbf{A}}{2 \times 3 \mathbf{P}}}{2 \times 3 \mathbf{S}} \quad \frac{\frac{3 \mathbf{A}}{3 \mathbf{P}}}{1 + 2 \times 3 \mathbf{S}}
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 \frac{1 \mathbf{A} \longleftrightarrow (\text{Num } 1) \text{ AST}}{1 \mathbf{P} \longleftrightarrow (\text{Num } 1) \text{ AST}} \quad \frac{2 \mathbf{A} \quad \frac{3 \mathbf{A}}{3 \mathbf{P}}}{2 \times 3 \mathbf{P}} \\
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## Rules

$$\begin{array}{c}
 \frac{i \in \mathbb{Z}}{i \mathbf{A} \longleftrightarrow (\text{Num } i)} \quad \frac{a \mathbf{S} \longleftrightarrow a'}{(a) \mathbf{A} \longleftrightarrow a'} \quad \frac{e \mathbf{A} \longleftrightarrow a}{e \mathbf{P} \longleftrightarrow a} \quad \frac{e \mathbf{P} \longleftrightarrow a}{e \mathbf{S} \longleftrightarrow a} \\
 \\
 \frac{a \mathbf{A} \longleftrightarrow a' \quad b \mathbf{P} \longleftrightarrow b'}{a \times b \mathbf{P} \longleftrightarrow (\text{Times } a' b')} \quad \frac{a \mathbf{P} \longleftrightarrow a' \quad b \mathbf{S} \longleftrightarrow b'}{a + b \mathbf{S} \longleftrightarrow (\text{Plus } a' b')}
 \end{array}$$

$$\begin{array}{c}
 \frac{1 \mathbf{A} \longleftrightarrow (\text{Num } 1) \mathbf{AST}}{1 \mathbf{P} \longleftrightarrow (\text{Num } 1) \mathbf{AST}} \quad \frac{\frac{2 \mathbf{A} \longleftrightarrow (\text{Num } 2) \mathbf{AST}}{2 \times 3 \mathbf{P} \longleftrightarrow (\text{Times } (\text{Num } 2) (\text{Num } 3)) \mathbf{AST}}{2 \times 3 \mathbf{S} \longleftrightarrow (\text{Times } (\text{Num } 2) (\text{Num } 3)) \mathbf{AST}} \quad \frac{3 \mathbf{A} \longleftrightarrow (\text{Num } 3) \mathbf{AST}}{3 \mathbf{P} \longleftrightarrow (\text{Num } 3) \mathbf{AST}}}{1 + 2 \times 3 \mathbf{S} \longleftrightarrow (\text{Plus } (\text{Num } 1) (\text{Times } (\text{Num } 2) (\text{Num } 3))) \mathbf{AST}}
 \end{array}$$

## The Inverse

What about the inverse operation to **parsing**?

### Unparsing

Unparsing, also called *pretty-printing*, is the process of starting with the **term** on the right hand side of  $\longleftrightarrow$  and attempting to synthesise a string on the left.

## The Inverse

What about the inverse operation to **parsing**?

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Unparsing, also called *pretty-printing*, is the process of starting with the **term** on the right hand side of  $\longleftrightarrow$  and attempting to synthesise a string on the left.

### Problem

There are **many** concrete strings for a given abstract syntax term. The algorithm is *non-deterministic*.

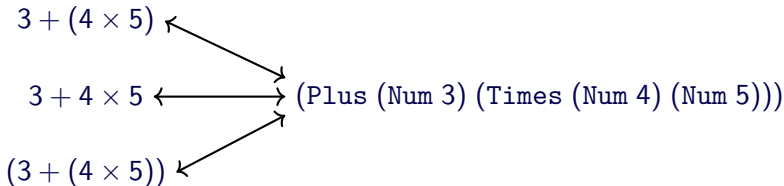
While it is desirable to have:

$$\text{parse} \circ \text{unparse} = \text{id}$$

It is not usually true that:

$$\text{unparse} \circ \text{parse} = \text{id}$$

## Example



Going from **right to left** requires some formatting guesswork to produce readable code.

Algorithms to do this can get quite involved!

Let's implement a parser for arithmetic. to coding

## Adding Let

Let us extend our arithmetic expression language with **variables**, including a `let` construct to give them values.

### Concrete Syntax

$$\frac{x \text{ Ident}}{x \text{ Atom}} \quad \frac{x \text{ Ident} \quad e_1 \text{ SExp} \quad e_2 \text{ SExp}}{\text{let } x = e_1 \text{ in } e_2 \text{ end Atom}}$$

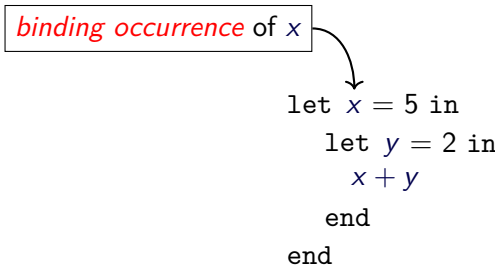
### Example

```
let x = 3 in
  x + 4
end
```

```
let x = 3 in
  let y = 4 in x + y end
end
```

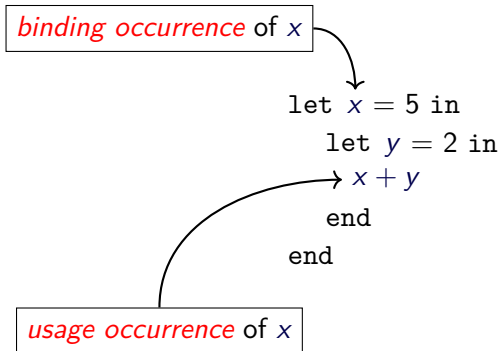
# Scope

*binding occurrence* of  $x$

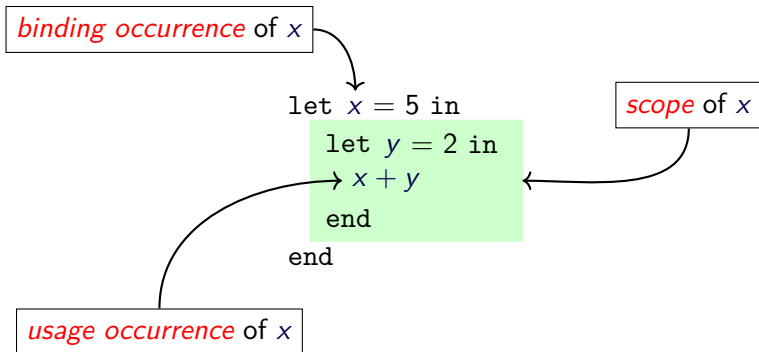


```
let x = 5 in
  let y = 2 in
    x + y
  end
end
```

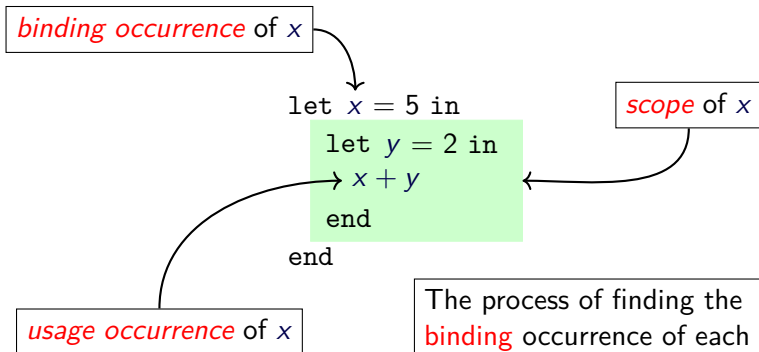
# Scope



# Scope



# Scope



The process of finding the *binding* occurrence of each used variable is called *scope resolution*. Usually this is done *statically*. If no binding can be found, an *out of scope* error is raised.

# Shadowing

What does this program evaluate to?

```
let x = 5 in
  let x = 2 in
    x + x
  end
+ x end
```

# Shadowing

What does this program evaluate to?

```
let x = 5 in
  let x = 2 in
    x + x
  end
+ x end
```

x is *shadowed* here

This program results in 9.

## $\alpha$ -equivalence

What is the difference between these two programs?

```
let x = 5 in
  let x = 2 in
    x + x
  end
end
```

```
let a = 5 in
  let y = 2 in
    y + y
  end
end
```

## $\alpha$ -equivalence

What is the difference between these two programs?

```
let x = 5 in
  let x = 2 in
    x + x
  end
end

let a = 5 in
  let y = 2 in
    y + y
  end
end
```

They are **semantically** identical, but differ in the choice of **bound variable names**. Such expressions are called  **$\alpha$ -equivalent**.

We write  $e_1 \equiv_\alpha e_2$  if  $e_1$  is  $\alpha$ -equivalent to  $e_2$ . The relation  $\equiv_\alpha$  is an **equivalence relation**. That is, it is **reflexive**, **transitive** and **symmetric**.

The process of consistently renaming variables that preserves  $\alpha$ -equivalence is called  **$\alpha$ -renaming**.

# Substitution

A variable  $x$  is *free* in an expression  $e$  if  $x$  occurs in  $e$  but is not bound in  $e$ .

## Example (Free Variables)

The variable  $x$  is free in  $x + 1$ , but not in `let  $x = 3$  in  $x + 1$  end.`

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## Example (Free Variables)

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A *substitution*, written  $e[x := t]$  (or  $e[t/x]$  in some other courses), is the replacement of all *free* occurrences of  $x$  in  $e$  with the term  $t$ .

## Example (Simple Substitution)

$(5 \times x + 7)[x := y \times 4]$  is the same as  $(5 \times (y \times 4) + 7)$ .

## Problems with substitution

Consider these two  $\alpha$ -equivalent expressions.

```
let y = 5 in y × x + 7 end
```

and

```
let z = 5 in z × x + 7 end
```

What happens if you apply the substitution  $[x := y \times 3]$  to both expressions?

## Problems with substitution

Consider these two  $\alpha$ -equivalent expressions.

```
let y = 5 in y × x + 7 end
```

and

```
let z = 5 in z × x + 7 end
```

What happens if you apply the substitution  $[x := y \times 3]$  to both expressions? You get two **non- $\alpha$ -equivalent** expressions!

```
let y = 5 in y × (y × 3) + 7 end
```

and

```
let z = 5 in z × (y × 3) + 7 end
```

This problem is called **capture**.

# Variable Capture

*Capture* can occur for a substitution  $e[x := t]$  when a bound variable in  $e$  clashes with a free variable occurring in  $t$ .

## Fortunately

It is **always possible** to avoid capture.

- **$\alpha$ -rename** the offending bound variable to an unused name, or
- If you have access to the free variable's definition, renaming the free variable, or
- Use a **different abstract syntax representation** that makes capture impossible (More on this later).

## Abstract Syntax for Variables

We shall extend our AST and parsing relation to include a definition for `let` and variables.

### Let Syntax

$$\begin{array}{c}
 x \text{ Ident} \\
 \hline
 x \text{ Atom} \longleftrightarrow (\text{Var } x) \text{ AST} \\
 \\
 \frac{x \text{ Ident} \quad e_1 \text{ SExp} \longleftrightarrow a_1 \text{ AST} \quad e_2 \text{ SExp} \longleftrightarrow a_2 \text{ AST}}{\text{let } x = e_1 \text{ in } e_2 \text{ end Atom} \longleftrightarrow (\text{Let } x \ a_1 \ a_2) \text{ AST}}
 \end{array}$$

## First Order Abstract Syntax

Consider the following two pieces of abstract syntax:

```
(Let "x" (Num 5) (Plus (Num 4) (Var "x")))
```

```
(Let "y" (Num 5) (Plus (Num 4) (Var "y")))
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This demonstrates some problems with our abstract syntax approach.

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This demonstrates some problems with our abstract syntax approach.

- 1 Substitution capture is a problem.
- 2  $\alpha$ -equivalent expressions are **not equal**. Determining if an expression is  $\alpha$ -equivalent requires us to search for a consistent  $\alpha$ -renaming of variables.

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This demonstrates some problems with our abstract syntax approach.

- 1 Substitution capture is a problem.
- 2  $\alpha$ -equivalent expressions are **not equal**. Determining if an expression is  $\alpha$ -equivalent requires us to search for a consistent  $\alpha$ -renaming of variables.
- 3 Computing whether names are in scope or might be captured requires handling sets of names of variables (e.g. free-variable sets), which Avoiding capture or scope errors requires computing free-variable sets.

# de Bruijn Indices

One popular alternative is *de Bruijn indices*.

## Key Idea

- 1 Remove the identifiers from binding expressions like Let.
- 2 Replace the identifier in a Var with a number indicating how many binders we must skip in order to find the binder for that variable.

```
(Let "a" (Num 5)
 (Let "y" (Num 2)
  (Plus (Var "a") (Var "y")))))
```

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(Let "a" (Num 5)	(Let (Num 5)
(Let "y" (Num 2)	(Let (Num 2)
(Plus (Var "a") (Var "y"))))	(Plus (Var <b>1</b> ) (Var <b>0</b> )))))

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(Let "a" (Num 5)
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```

```
(Let (Num 5)
 (Let (Num 2)
  (Plus (Var 1) (Var 0))))
```



# Debruijnification

## Algorithm

To convert *first order abstract syntax* with explicit variable names to de Bruijn indices, we keep a *stack* of variable names in scope. Within the scope of a `Let` we push the new name. When we encounter a variable, the de Bruijn index is the **first position** of its name in the stack.

```
namedToB :: [String] → Named_Term → DB_Term
```

```
namedToB names (NLet nm e1 e2) =
```

```
  DLet (namedToB names e1) (namedToB (nm : names) e2)
```

```
namedToB names (NVar nm) =
```

```
  case (Data.List.findIndex (== nm) names) of
```

```
    Nothing → error (nm ++ " out of scope")
```

```
    Just i → DVar i
```

## de Bruijn Advantages

Converting to de Bruijn form naturally eliminates **shadowing**. It is always possible to name every variable that is in scope.

This gives us one possible algorithm for  $\alpha$ -equivalence of named terms: convert to de Bruijn representation then check equality.

It's also possible, but harder, to have de Bruijn indices going in the other direction (from the bottom of the stack, upwards).

## de Bruijn Substitution

Substitution can be made **capture avoiding** without considering free-variable sets.

$$\begin{aligned}(\text{Num } i)[n := t] &= (\text{Num } i) \\(\text{Plus } a \ b)[n := t] &= (\text{Plus } a[n := t] \ b[n := t]) \\(\text{Times } a \ b)[n := t] &= (\text{Times } a[n := t] \ b[n := t])\end{aligned}$$

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 (\text{Let } e_1 \ e_2)[n := t] &= (\text{Let } e_1[n := t] \ e_2[n + 1 := t_{\uparrow 0}])
 \end{aligned}$$

Where  $e_{\uparrow n}$  is an **up-shifting** operation defined as follows:

$$\begin{aligned}
 (\text{Num } i)_{\uparrow n} &= (\text{Num } i) \\
 (\text{Plus } a \ b)_{\uparrow n} &= (\text{Plus } a_{\uparrow n} \ b_{\uparrow n}) \\
 (\text{Times } a \ b)_{\uparrow n} &= (\text{Times } a_{\uparrow n} \ b_{\uparrow n}) \\
 (\text{Var } m)_{\uparrow n} &= \begin{cases} (\text{Var } (m + 1)) & \text{if } m \geq n \\ (\text{Var } m) & \text{otherwise} \end{cases} \\
 (\text{Let } e_1 \ e_2)_{\uparrow n} &= (\text{Let } e_{1\uparrow n} \ e_{2\uparrow n+1})
 \end{aligned}$$

## Examining de Bruijn indices

How do de Bruijn indices stack up against explicit names?

- 1 Substitution capture **solved**.

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- 1 Substitution capture **solved**.
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- 3 Defining substitution machinery is a bit involved.
- 4 It is still possible to make **malformed syntax** – indices that overflow the stack, for example.
- 5 Naming the correct variable changes from a **rare** issue to a **standard** one.

Can we do better by changing the **term language**?

## Higher Order Terms

We shall change our term language to include **built-in** notions of variables and binding.

$$\begin{array}{ll}
 t ::= \text{Symbol} & \text{(symbols)} \\
 | x & \text{(variables)} \\
 | t_1 t_2 & \text{(application)} \\
 | x. t & \text{(binding or abstraction)}
 \end{array}$$

As in Haskell, we shall say that application is **left-associative**, so

$$(\text{Plus} (\text{Num } 3) (\text{Num } 4)) = ((\text{Plus} (\text{Num } 3)) (\text{Num } 4))$$

Now the **binding** and **usage** occurrences of variables are distinguished from regular symbols in our term language. Let's see what this lets us do...

## Representing Let

$$\frac{a_1 \text{ AST} \quad a_2 \text{ AST}}{(\text{Let } a_1 (x. a_2)) \text{ AST}}$$

We no longer need a rule for variables, because they're baked into the structure of terms.

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How would we represent this AST in **Haskell**?

```
data AST = Num Int
          | Plus AST AST
          | Times AST AST
          | Let AST ???
```

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How would we represent this AST in **Haskell**?

```
data AST = Num Int
          | Plus AST AST
          | Times AST AST
          | Let AST (AST → AST)
```

So `let x = 3 in x + 2 end` becomes, in Haskell:

```
(Let (Num 3) (λx → Plus x (Num 2)))
```

## Substitution

We can now define substitution across **all** terms **in the meta-logic**:

$$\begin{aligned} \text{Symbol}[x := e] &= \text{Symbol} \\ y[x := e] &= \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} \\ (t_1 \ t_2)[x := e] &= t_1[x := e] \ t_2[x := e] \\ (y. \ t)[x := e] &= \begin{cases} (y. \ t) & \text{if } x = y \\ (y. \ t[x := e]) & \text{if } y \notin \text{FV}(e) \\ \text{undefined} & \text{otherwise} \end{cases} \end{aligned}$$

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Where  $\text{FV}(\cdot)$  is the set of all **free variables** in a term:

$$\begin{aligned}
 \text{FV}(\text{Symbol}) &= \emptyset \\
 \text{FV}(x) &= \{x\} \\
 \text{FV}(t_1 \ t_2) &= \text{FV}(t_1) \cup \text{FV}(t_2) \\
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 \text{FV}(x. \ t) &= \text{FV}(t) \setminus \{x\}
 \end{aligned}$$

## Cheating Outrageously

Substitution **capture** is still a problem in HOAS. But it is **not our problem**. Because substitution is defined in the **meta-language**, it's the job of the implementors of the meta-language (if any) to deal with capture issues.

- When **Haskell** is our meta-language, it's the job of the GHC developers.
- When we are doing proofs in our **meta-logic**, there is no implementation, so we can just say that we assume  $\alpha$ -equivalent terms to be equal, and therefore assume that variables are always renamed to avoid capture.

So, we have solved the problem by making it someone else's problem. **Outrageous cheating!**

## Evaluating All Approaches

	HOAS		FOAS	
	Proofs	Haskell	Strings	de Bruijn
Capture	Cheat	Cheat	Problem	Solved
$\alpha$ -equivalence	Cheat	Cheat	Problem	Solved
Generic subst.	Solved	Solved	Problem	Problem
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- In **embedded languages** and in pen and paper **proofs**, HOAS is very common.

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- In **embedded languages** and in pen and paper **proofs**, HOAS is very common.
- In conventional language implementations and machine-checked formalisations, de Bruijn indices are more popular.
- In your assignments, strings will be used 😊

## More Approaches

There are yet more approaches to naming in the research literature.

- Locally nameless: use de Bruijn indices for local variables, but a name-carrying representation for free variables.
- Nominative sets: build on permutations of name sets rather than updates of name sets, to get a theory with nicer properties.