

Some Announcements

More material will continue to appear on the website.

- Tutorial 1 is up now, etc.

Assignment 0 will appear soon.

- Some parts will cover future material; more on that later.

Some students have asked about pre-requisites, co-requisites etc.

- We don't think it helps to constrain your choices.

Some students have asked about Haskell learning resources.

- There are various linked on the website.
- e.g. chapters 1-7 of “Learn You a Haskell for Great Good!”
- Helpful? Unhelpful? Broken link? Tell us on the forum.

More Announcements

Two of our tutors have set up a discord chat and begun reading/viewing some extra material.

- Totally unofficial.
- Find out more on the forum.

We've talked a lot about **induction**.

- By the end of this lecture we'll link it back to program syntax.

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We've talked a lot about **induction**.

- By the end of this lecture we'll link it back to program syntax.
- In the remainder of the course:
 - Parsing
 - Semantics
 - Type systems
 - Compilers and type checkers

Formalisation

To talk about languages in a mathematically precise way, we need to **formalise** them.

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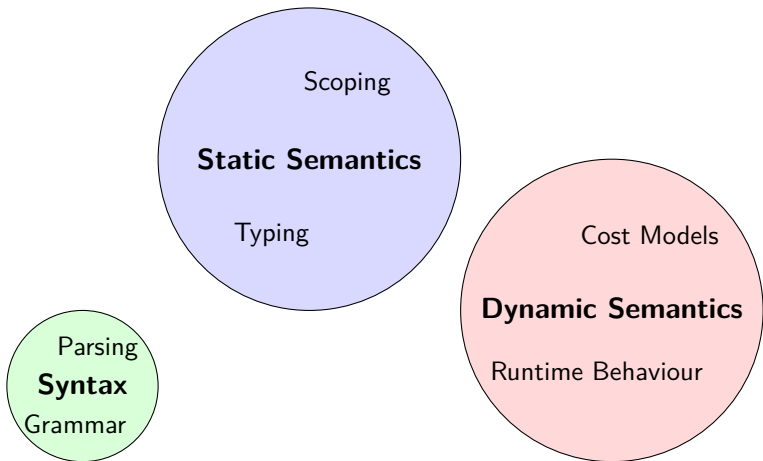
Formalisation

Formalisation is the process of giving a language a formal, **mathematical description**.

Typically, we describe the language in **another language**, called the *meta-language*. For implementations, it may be a programming language such as **Haskell**. For formalisations it is usually a minimal logic called a *meta-logic*.

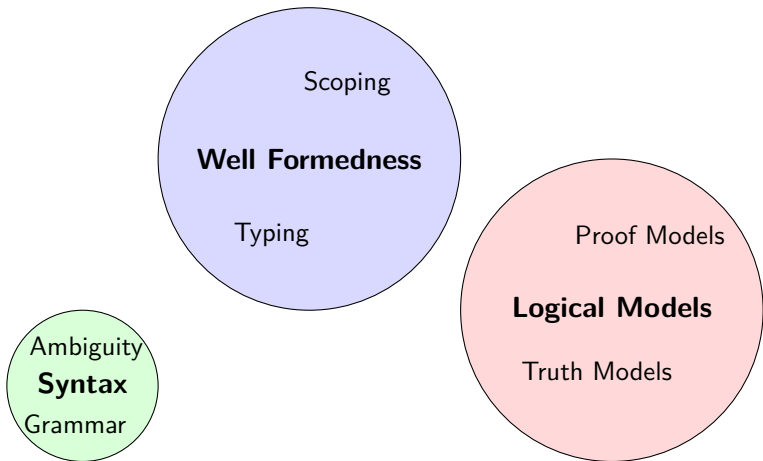
Learning from History

What sort of meta logic should we use? There are a number of things to formalise:



Learning from History

Logicians in the early 20th century had much the same desire to formalise *logics*.



Learning from History

In this course, we will use a meta-logic based on *Natural Deduction* and inductive inference rules, originally invented for formalising logics by Gerhard Gentzen in the mid 1930s.

Der Kalkül des natürlichen Schließens.

$$\frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \& \mathcal{B}}$$

$$\frac{\mathcal{A} \& \mathcal{B}}{\mathcal{A}}$$

$$\frac{\mathcal{A} \& \mathcal{B}}{\mathcal{B}}$$

Learning from History

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These days Gentzen-style rules are frequently used by programming language researchers, especially to describe type systems.

Judgements

A *judgement* is a statement asserting a certain property for an object.

Example (Informal Judgements)

- $3 + 4 \times 5$ is a valid arithmetic expression.
- The string *madam* is a palindrome.
- The string *snooze* is a palindrome
 \implies Judgements do not have to hold.

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 - The string *madam* is a palindrome.
 - The string *snooze* is a palindrome
- ⇒ Judgements do not have to hold.

Unary Judgements

Formally, we denote the judgement that a property **A** holds for an object *s* by writing *s* **A**.

Typically, *s* is a **string** when describing syntax, and *s* is a **term** when describing semantics.

Proving Judgements

We define how a judgement may be **proven** by providing a set of *inference rules*.

Inference Rules

An inference rule is written as:

$$\frac{J_1 \quad J_2 \quad \dots \quad J_n}{J}$$

This states that in order to prove judgement J (the *conclusion*), it suffices to prove all judgements J_1 through to J_n (the *premises*).

Rules with no premises are called *axioms*. Their conclusions **always hold**.

Examples

Example (Natural Numbers)

$$n \text{ Nat}$$

$$\frac{}{0 \text{ Nat}} N_1$$

0 is a natural number

$$\frac{n \text{ Nat}}{(S n) \text{ Nat}} N_2$$

if n is a natural number,
then the successor of n
is a natural number.

What terms are in the set $\{n \mid n \text{ Nat}\}$?

Examples

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if n is a natural number,
then the successor of n
is a natural number.

What terms are in the set $\{n \mid n \text{ Nat}\}$?

$$\{0, (S 0), (S (S 0)), (S (S (S 0))), \dots\}$$

Examples

Example (Even and Odd Numbers)

$$\begin{array}{ccc}
 \boxed{n \text{ Even}} & \boxed{n \text{ Odd}} & \\
 \frac{}{0 \text{ Even}} E_1 & \frac{n \text{ Even}}{(S (S n)) \text{ Even}} E_2 & \frac{n \text{ Even}}{(S n) \text{ Odd}} O_1
 \end{array}$$

The Proof Video Game

To show that a judgement $s \mathbf{A}$ holds:

- ① Find a rule whose conclusion matches $s \mathbf{A}$.
- ② The preconditions of the applied rules become new **proof obligations**.
- ③ Rinse and repeat until all obligations are proven up to axioms.

Examples

Example (Even and Odd Numbers)

$$\begin{array}{ccc}
 \frac{}{0 \text{ Even}} E_1 & \frac{\boxed{n \text{ Even}}}{(S (S n)) \text{ Even}} E_2 & \frac{\boxed{n \text{ Odd}}}{(S n) \text{ Odd}} O_1
 \end{array}$$

$$\frac{}{(S (S (S (S (S 0)))) \text{ Odd}}$$

Examples

Example (Even and Odd Numbers)

$$\begin{array}{ccc}
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 \end{array}$$

$$\frac{\overline{(S (S (S (S 0)))) \text{ Even}}}{(S (S (S (S (S 0)))) \text{ Odd}} O_1$$

Examples

Example (Even and Odd Numbers)

$$\frac{}{0 \text{ Even}} E_1 \qquad \frac{\boxed{n \text{ Even}}}{(S (S n)) \text{ Even}} E_2 \qquad \frac{\boxed{n \text{ Odd}}}{(S n) \text{ Odd}} O_1$$

$$\frac{\frac{(S (S 0)) \text{ Even}}{(S (S (S (S 0)))) \text{ Even}} E_2}{(S (S (S (S (S 0)))) \text{ Odd}} O_1$$

Examples

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Defining Languages

Example (Bracket Matching Language)

$$\mathbf{M} ::= \varepsilon \mid \mathbf{M}\mathbf{M} \mid (\mathbf{M})$$

Examples of strings: ε , $()$, $(())$, $()()$, $(())()$, ...

Three rules:

Axiom The empty string is in **M**

Juxtaposition Any two strings in **M** can be concatenated to give a new string in **M**

Nesting Any string in **M** can be surrounded by parentheses, giving a new string in **M**

With Rules

The Language M

$$\begin{array}{c}
 \boxed{s \ M} \\
 \\
 \frac{}{\varepsilon \ M} M_E \qquad \frac{s \ M}{(s) \ M} M_N \qquad \frac{s_1 \ M \quad s_2 \ M}{s_1 s_2 \ M} M_J
 \end{array}$$

$() (()) \ M$

With Rules

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The Language M

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$$\frac{\frac{\frac{}{\varepsilon \ M} M_E}{() \ M} M_N \quad \frac{\frac{\frac{}{\varepsilon \ M} M_E}{() \ M} M_N}{(()) \ M} M_N}{() (()) \ M} M_J$$

Getting Stuck

If we had started with rule M_N instead, we would have gotten stuck:

$$\frac{\frac{???}{) (() \mathbf{M}}}{() (()) \mathbf{M}} M_N$$

Takeaway

Getting stuck does **not** mean what you're trying to prove is false!

Admissibility

An additional rule is considered *admissible* to a language if admitting it (adding it to the list of defining rules) does not change the language.

For instance, consider the following rule:

$$\frac{s \mathbf{M}}{((s)) \mathbf{M}}$$

Would adding this rule to \mathbf{M} change the set of strings in \mathbf{M} ?

Derivability

Is this rule *admissible* to **M**?

$$\frac{s \mathbf{M}}{((s)) \mathbf{M}}$$

Yes, because we could always use rule M_N twice instead. Rules that are compositions of existing rules are called *derivable*:

$$\frac{\frac{s \mathbf{M}}{(s) \mathbf{M}} M_N}{((s)) \mathbf{M}} M_N$$

We can prove *rules* as well as *axioms*, by deriving the *conclusion* of the rule while taking the *premises* as local axioms.

Derivability

Is this rule derivable?

$$\frac{s \ M}{(s) s \ M}$$

Derivability

Is this rule derivable?

$$\frac{s \mathbf{M}}{(s) s \mathbf{M}}$$

We can derive it like so:

$$\frac{\frac{\overline{s \mathbf{M}}}{(s) \mathbf{M}} M_N \quad \frac{\overline{s \mathbf{M}}}{s \mathbf{M}} M_J}{(s) s \mathbf{M}}$$

Derivability

Is this rule derivable?

$$\frac{(s) \quad \mathbf{M}}{s \quad \mathbf{M}} \mathbf{Q}$$

Derivability

Is this rule admissible? If so, is it derivable?

$$\frac{()s M}{s M}$$

Derivability

Is this rule admissible? If so, is it derivable?

$$\frac{()s M}{s M}$$

- It is **admissible**, as it doesn't let us prove any new judgements about **M**.
- It is **not derivable**, as it is not made up of the composition of existing rules.
- We will see how to prove these sorts of rules are admissible later on.

Hypothetical Derivations

We can write a rule in a horizontal format as well:

$$\frac{A}{B} \text{ is the same as } A \vdash B$$

This allows us to neatly make **rules** premises of other rules, called *hypothetical derivations*:

Example

$$\frac{A \vdash B}{C}$$

Read as: *If assuming A we can derive B, then we can derive C.*

Specifying Logic

With hypotheticals we can specify logic, which was the original purpose of natural deduction. Let $A \text{ True}$ be the judgement that the proposition A is true.

Example (And and Implies)

$$\frac{A \text{ True} \quad B \text{ True}}{A \wedge B \text{ True}} \wedge_I \quad \frac{A \wedge B \text{ True}}{A \text{ True}} \wedge_{E1} \quad \frac{A \wedge B \text{ True}}{B \text{ True}} \wedge_{E2}$$

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 \\
 \frac{A \text{ True} \vdash B \text{ True}}{A \Rightarrow B \text{ True}} \Rightarrow_I \quad \frac{A \Rightarrow B \text{ True} \quad A \text{ True}}{B \text{ True}} \Rightarrow_E
 \end{array}$$

Specifying Logic, Continued

Example (Or, True, False and Not)

$$\begin{array}{c}
 \frac{A \text{ True}}{A \vee B \text{ True}} \vee_{I1} \quad \frac{B \text{ True}}{A \vee B \text{ True}} \vee_{I2} \\
 \\
 \frac{A \text{ True} \vdash C \text{ True} \quad B \text{ True} \vdash C \text{ True} \quad A \vee B \text{ True}}{C \text{ True}} \vee_E
 \end{array}$$

Specifying Logic, Continued

Example (Or, True, False and Not)

$$\begin{array}{c}
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 \\
 \frac{A \text{ True} \vdash C \text{ True} \quad B \text{ True} \vdash C \text{ True} \quad A \vee B \text{ True}}{C \text{ True}} \vee_E \\
 \\
 \frac{}{\top \text{ True}} \top_I \quad \frac{\perp \text{ True}}{A \text{ True}} \perp_E \\
 \\
 \frac{A \text{ True} \vdash \perp \text{ True}}{\neg A \text{ True}} \neg_I \quad \frac{\neg A \text{ True} \quad A \text{ True}}{B \text{ True}} \neg_E
 \end{array}$$

Minimal Definitions

$$\begin{array}{c}
 \boxed{s \mathbf{M}} \\
 \frac{}{\varepsilon \mathbf{M}} M_E \qquad \frac{s \mathbf{M}}{(s) \mathbf{M}} M_N \qquad \frac{s_1 \mathbf{M} \quad s_2 \mathbf{M}}{s_1 s_2 \mathbf{M}} M_J
 \end{array}$$

The above rules are the **smallest set of rules** to define every string in \mathbf{M} .

Therefore

If we know that a string satisfies $s \mathbf{M}$, it must have been through a (finite) derivation using these rules.

This is called an **inductive definition** of \mathbf{M} .

Rule Induction

Suppose we want to show that a property $P(s)$ of strings s holds for any string $s \mathbf{M}$. We will use *rule induction*.

If we show that

$$\frac{}{\varepsilon \mathbf{M}} M_E$$

$P(\varepsilon)$ holds, and

$$\frac{s \mathbf{M}}{(s) \mathbf{M}} M_N$$

$P(s)$ implies $P((s))$, and

$$\frac{s_1 \mathbf{M} \quad s_2 \mathbf{M}}{s_1 s_2 \mathbf{M}} M_J$$

$P(s_1)$ and $P(s_2)$ implies $P(s_1 s_2)$

Then we have shown $P(s)$ for all $s \mathbf{M}$.

These assumptions are called *inductive hypotheses*.

Rule Induction

Example (Counting Pairs)

Let $op(s)$ denote the number of opening parentheses in s , and $cl(s)$ denote the number of closing parentheses. We shall prove that

$$s \mathbf{M} \implies op(s) = cl(s)$$

by doing rule induction on $s \mathbf{M}$.

Rule Induction

Example (Counting Prens)

$$\frac{}{\varepsilon \mathbf{M}} M_E$$

Base Case: $op(\varepsilon) = 0 = cl(\varepsilon)$

Rule Induction

Example (Counting Prens)

$$\frac{}{\varepsilon \mathbf{M}} M_E$$

$$\frac{s \mathbf{M}}{(s) \mathbf{M}} M_N$$

Base Case: $op(\varepsilon) = 0 = cl(\varepsilon)$

Inductive Case: Assuming I.H:

$$op(s) = cl(s)$$

$$op((s)) = op(s) + 1 = cl(s) + 1 = cl((s))$$

Rule Induction

Example (Counting Prens)

$$\frac{}{\varepsilon \text{ M}} M_E$$

Base Case: $op(\varepsilon) = 0 = cl(\varepsilon)$

$$\frac{s \text{ M}}{(s) \text{ M}} M_N$$

Inductive Case: Assuming I.H:

$$op(s) = cl(s)$$

$$op((s)) = op(s) + 1 = cl(s) + 1 = cl((s))$$

$$\frac{s_1 \text{ M} \quad s_2 \text{ M}}{s_1 s_2 \text{ M}} M_J$$

Inductive Case: Assuming I.Hs:

$$op(s_1) = cl(s_1) \text{ and } op(s_2) = cl(s_2)$$

$$op(s_1 s_2) = op(s_1) + op(s_2) = cl(s_1 s_2)$$

Rule Induction in General

Rule Induction Method

Given a set of rules R , we may prove a property P **inductively** for all judgements that can be inferred with R by showing, for each rule of the form

$$\frac{J_1 \quad J_2 \quad \dots \quad J_n}{J}$$

that if P holds for each of $J_1 \dots J_n$, then P holds for J .

Therefore, axioms are the **base cases** of the induction, all other rules form **inductive cases**, and the premises of each rule give rise to **inductive hypotheses**.

Structural Induction

Conventional *structural induction* such as that on natural numbers, which we have encountered before, is a *special case* of rule induction.

Natural Number Induction

To show a property $P(n)$ for all $n \in \mathbb{N}$, it suffices to:

$\frac{}{0 \text{ Nat}}$ Show that $P(0)$ holds, and

$\frac{n \text{ Nat}}{(S n) \text{ Nat}}$ Assuming $P(n)$, show $P(n + 1)$.

Another Example

Recall our definition of even numbers:

$$\boxed{n \text{ Even}}$$

$$\frac{}{0 \text{ Even}} E_1 \qquad \frac{n \text{ Even}}{(S (S n)) \text{ Even}} E_2$$

We could define odd numbers differently:

$$\boxed{n \text{ Odd}'}$$

$$\frac{}{(S 0) \text{ Odd}'} O'_1 \qquad \frac{n \text{ Odd}'}{(S (S n)) \text{ Odd}'} O'_2$$

Let's prove the original **Odd** rule, but for **Odd'** (to “whiteboard”):

$$\frac{n \text{ Even}}{(S n) \text{ Odd}'}$$

Arithmetic

Example (Arithmetic Expression)

Arith ::= i | **Arith** × **Arith** | **Arith** + **Arith** | (**Arith**) ($i \in \mathbb{Z}$)

Arithmetic

Example (Arithmetic Expression)

$\mathbf{Arith} ::= i \mid \mathbf{Arith} \times \mathbf{Arith} \mid \mathbf{Arith} + \mathbf{Arith} \mid (\mathbf{Arith}) \quad (i \in \mathbb{Z})$

$$\frac{i \in \mathbb{Z}}{i \mathbf{Arith}} L \quad \frac{a \mathbf{Arith} \quad b \mathbf{Arith}}{a \times b \mathbf{Arith}} P \quad \frac{a \mathbf{Arith} \quad b \mathbf{Arith}}{a + b \mathbf{Arith}} S \quad \frac{a \mathbf{Arith}}{(a) \mathbf{Arith}}$$

We can infer $1 + 2 \times 3 \mathbf{Arith}$ in two different ways.

Ambiguity

Arith is *ambiguous*, which means that there are multiple ways to derive the same judgement.

For syntax, this is a **big problem**, as different interpretations of syntax can lead to semantic inconsistency:

$$\begin{array}{c}
 \frac{1 \in \mathbb{Z}}{1 \text{ Arith}} \\
 \frac{2 \in \mathbb{Z} \quad 3 \in \mathbb{Z}}{2 \text{ Arith} \quad 3 \text{ Arith}} \\
 \hline
 \frac{1 \text{ Arith} \quad 2 \times 3 \text{ Arith}}{1 + 2 \times 3 \text{ Arith}}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{1 \in \mathbb{Z} \quad 2 \in \mathbb{Z}}{1 \text{ Arith} \quad 2 \text{ Arith}} \\
 \frac{1 \text{ Arith} \quad 2 \text{ Arith}}{1 + 2 \text{ Arith}} \\
 \frac{1 + 2 \text{ Arith} \quad 3 \in \mathbb{Z}}{1 + 2 \times 3 \text{ Arith}}
 \end{array}$$

Second Attempt

We want to specify **Arith** in such a way that enforces **order of operations**.

Here we will use **multiple judgements**:

Example (Arithmetic Expression)

Atom ::= $i \mid (\text{SExp}) \quad (i \in \mathbb{Z})$

PExp ::= **Atom** \mid **PExp** \times **PExp**

SExp ::= **PExp** \mid **SExp** $+$ **SExp**

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$$\begin{array}{c}
 \frac{i \in \mathbb{Z}}{i \text{ Atom}} \quad \frac{a \text{ SExp}}{(a) \text{ Atom}} \quad \frac{e \text{ Atom}}{e \text{ PExp}} \quad \frac{e \text{ PExp}}{e \text{ SExp}} \\
 \\
 \frac{a \text{ PExp} \quad b \text{ PExp}}{a \times b \text{ PExp}} \quad \frac{a \text{ SExp} \quad b \text{ SExp}}{a + b \text{ SExp}}
 \end{array}$$

Consider: Is there still any ambiguity here?

More ambiguity

$$\begin{array}{c}
 \frac{1 \in \mathbb{Z}}{1 \text{ Atom}} \\
 \frac{2 \in \mathbb{Z}}{2 \text{ Atom}} \\
 \frac{3 \in \mathbb{Z}}{3 \text{ Atom}} \\
 \frac{1 \text{ Atom} \quad 2 \text{ Atom} \quad 3 \text{ Atom}}{1 \times 2 \times 3 \text{ PExp}} \\
 \frac{1 \text{ PExp} \quad 2 \text{ PExp} \quad 3 \text{ PExp}}{1 \times 2 \times 3 \text{ PExp}}
 \end{array}
 \qquad
 \begin{array}{c}
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 \frac{2 \in \mathbb{Z}}{2 \text{ Atom}} \\
 \frac{1 \text{ Atom} \quad 2 \text{ Atom}}{1 \times 2 \text{ PExp}} \\
 \frac{1 \times 2 \text{ PExp} \quad 3 \text{ Atom}}{1 \times 2 \times 3 \text{ PExp}}
 \end{array}$$

This ambiguity seems harmless, but it would not be harmless for some other operations. Which ones?

More ambiguity

$$\begin{array}{c}
 \frac{1 \in \mathbb{Z}}{1 \text{ Atom}} \\
 \frac{1 \text{ Atom}}{1 \text{ PExp}} \\
 \frac{2 \in \mathbb{Z}}{2 \text{ Atom}} \\
 \frac{2 \text{ Atom}}{2 \text{ PExp}} \\
 \frac{3 \in \mathbb{Z}}{3 \text{ Atom}} \\
 \frac{3 \text{ Atom}}{3 \text{ PExp}} \\
 \frac{2 \text{ PExp} \quad 3 \text{ PExp}}{2 \times 3 \text{ PExp}} \\
 \frac{1 \text{ PExp} \quad 2 \times 3 \text{ PExp}}{1 \times 2 \times 3 \text{ PExp}}
 \end{array}
 \qquad
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 \frac{1 \text{ Atom}}{1 \text{ PExp}} \\
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 \frac{2 \text{ Atom}}{2 \text{ PExp}} \\
 \frac{1 \text{ PExp} \quad 2 \text{ PExp}}{1 \times 2 \text{ PExp}} \\
 \frac{3 \in \mathbb{Z}}{3 \text{ Atom}} \\
 \frac{3 \text{ Atom}}{3 \text{ PExp}} \\
 \frac{1 \times 2 \text{ PExp} \quad 3 \text{ PExp}}{1 \times 2 \times 3 \text{ PExp}}
 \end{array}$$

This ambiguity seems harmless, but it would not be harmless for some other operations. Which ones? Operators that are not *associative*.

We have to specify the *associativity* of operators. How?

Associativities

Operators have various *associativity* constraints:

Associative

All associativities are equal.

Left-Associative

$$A \odot B \odot C = (A \odot B) \odot C$$

Right-Associative

$$A \odot B \odot C = A \odot (B \odot C)$$

Try to think of some examples!

Enforcing associativity

We force the grammar to accept a smaller set of expressions on **one** side of the operator only. Show how this works on the “whiteboard”.

Example (Arithmetic Expression)

$$\begin{aligned} \text{Atom} &::= i \mid (\text{SExp}) \quad (i \in \mathbb{Z}) \\ \text{PExp} &::= \text{Atom} \mid \text{Atom} \times \text{PExp} \\ \text{SExp} &::= \text{PExp} \mid \text{PExp} + \text{SExp} \end{aligned}$$

Enforcing associativity

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Example (Arithmetic Expression)

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SExp ::= **PExp** \mid **PExp** $+$ **SExp**

$$\begin{array}{c}
 \frac{i \in \mathbb{Z}}{i \text{ **Atom**}} \quad \frac{a \text{ **SExp**}}{(a) \text{ **Atom**}} \quad \frac{e \text{ **Atom**}}{e \text{ **PExp**}} \quad \frac{e \text{ **PExp**}}{e \text{ **SExp**}} \\
 \frac{a \text{ **Atom**}}{a \times b \text{ **PExp**}} \quad \frac{b \text{ **PExp**}}{a + b \text{ **SExp**}} \quad \frac{a \text{ **PExp**}}{a + b \text{ **SExp**}} \quad \frac{b \text{ **SExp**}}{a + b \text{ **SExp**}}
 \end{array}$$

Here we made multiplication and addition **right** associative. How would we do **left**?

Bring Back Parentheses

The Parenthetical Language

$$\begin{array}{c}
 \boxed{s \ M} \\
 \frac{\quad}{\varepsilon \ M} M_E \qquad \frac{s \ M}{(s) \ M} M_N \qquad \frac{s_1 \ M \quad s_2 \ M}{s_1 s_2 \ M} M_J
 \end{array}$$

Is this language ambiguous? to “whiteboard”

Ambiguity in Parentheses

Not only is it ambiguous, it is **infinitely so**. Strings like $()()()$ could be split at two different locations by rule M_J , but if we use ε , then even the string $()$ is ambiguous:

$$\frac{\frac{\varepsilon \mathbf{M}^{M_E}}{() \mathbf{M}} M_N}{() \mathbf{M}} M_J \qquad \frac{\frac{\varepsilon \mathbf{M}^{M_E}}{() \mathbf{M}} M_N \quad \frac{\varepsilon \mathbf{M}^{M_E}}{() \mathbf{M}} M_N}{() \mathbf{M}} M_J$$

$$\frac{\frac{\varepsilon \mathbf{M}^{M_E}}{() \mathbf{M}} M_J \quad \frac{\frac{\varepsilon \mathbf{M}^{M_E}}{() \mathbf{M}} M_N \quad \frac{\varepsilon \mathbf{M}^{M_E}}{() \mathbf{M}} M_N}{() \mathbf{M}} M_J}{() \mathbf{M}} M_J$$

We will eliminate the ambiguity by once again splitting **M** into two judgements, **N** and **L**.

The crucial observation is that terms in **M** are a **list (L)** of terms nested within parentheses (**N**).

Example (Unambiguous Parentheses)

$$\begin{array}{ccc}
 \boxed{s \ L} & & \boxed{s \ N} \\
 \\
 \frac{}{\varepsilon \ L} L_E & \quad \quad & \frac{s \ L}{(s) \ N} N_N & \quad \quad & \frac{s_1 \ N \quad s_2 \ L}{s_1 s_2 \ L} L_J
 \end{array}$$

Proving Equivalence

Now we shall prove $\mathbf{M} = \mathbf{L}$. There are two cases, each dispatched with rule induction:

$$\frac{s \mathbf{M}}{s \mathbf{L}} \quad \frac{s \mathbf{L}}{s \mathbf{M}}$$

The first case requires proving a *lemma*. The second requires *simultaneous induction*.

These proofs will be carried out on the “board”.