COMP2521 25T3
Graphs (III)
Graph Problems

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cycle checking
connected components
hamiltonian paths/circuits
eulerian paths/circuits

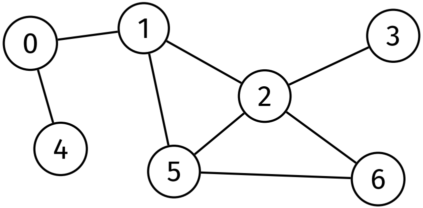
Graph Problems

Basic graph problems:

- Is there a cycle?
- How many connected components are there?
- Is there a simple path/cycle that passes through all vertices?
- Is there a path/cycle that passes through each edge exactly once?

Cycle Checking

A cycle is a path of length > 2
where the start vertex = end vertex
and no edge is used more than once



This graph has three distinct cycles:
1-2-5-1, 2-5-6-2, 1-2-6-5-1

(two cycles are distinct if they have different sets of edges)

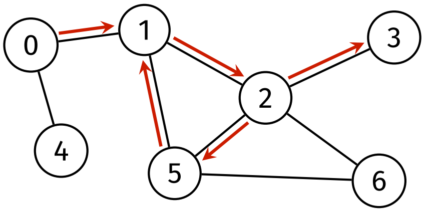
Cycle Checking

Attempt 1

How to check if a graph has a cycle?

Idea:

- Perform a DFS, starting from any vertex
- During the DFS, if the current vertex has an edge to an already-visited vertex, then there is a cycle



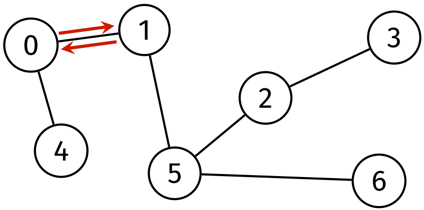
`tests/cycle1.txt`

Cycle Checking
Attempt 1

```
hasCycle(G):
    Input:  graph G
    Output: true if G has a cycle, false otherwise

    pick any vertex v in G
    create visited array, initialised to false
    return dfsHasCycle(G, v, visited)

dfsHasCycle(G, v, visited):
    visited[v] = true

    for each neighbour w of v in G:
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, visited):
            return true

    return false
```

Problem:

- The algorithm does not check whether the neighbour w is the vertex that it just came from
- Therefore, it considers moving back and forth along a single edge to be a cycle (e.g., 0-1-0)



`tests/cycle2.txt`

Cycle Checking

Attempt 2

Improved idea:

- Perform a DFS, starting from any vertex
- **Keep track of previous vertex during DFS**
- During the DFS, if the current vertex has an edge to an already-visited vertex **which is not the previous vertex**, then there is a cycle

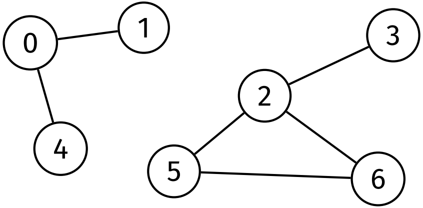
Cycle Checking

Attempt 2

```
hasCycle(G):
    Input:  graph G
    Output: true if G has a cycle, false otherwise

    pick any vertex v in G
    create visited array, initialised to false
    return dfsHasCycle(G, v, v, visited)

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each neighbour w of v in G:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

Problem:

- The algorithm only checks one connected component
  - The connected component that the initially chosen vertex belongs to



`tests/cycle3.txt`

Working idea:

- Perform a DFS, starting from any vertex
- Keep track of previous vertex during DFS
- During the DFS, if the current vertex has an edge to an already-visited vertex which is not the previous vertex, then there is a cycle
- **After the DFS, if any vertex has not yet been visited, perform another DFS, this time starting from that vertex**
- **Repeat until all vertices have been visited**

Cycle Checking

Working Solution

```
hasCycle(G):
    Input:  graph G
    Output: true if G has a cycle, false otherwise

    create visited array, initialised to false

    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, v, visited):
                return true

    return false

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each neighbour w of v in G:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

Cycle Checking

Analysis

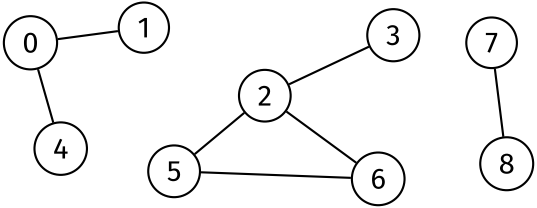
Analysis for adjacency list representation:

- Algorithm is a slight modification of DFS
- A full DFS traversal is $O(V + E)$
- Thus, worst-case time complexity of cycle checking is $O(V + E)$

Connected Components

A connected component
is a maximally connected subgraph

For example, this graph has three connected components:



Connected Components

DEFINITIONS:

subgraph
a subset of vertices and edges of original graph

connected subgraph
there is a path between every pair of vertices in the subgraph

maximally connected subgraph
no way to include more edges/vertices from original graph into the subgraph
such that subgraph is still connected

Connected Components

Problems:

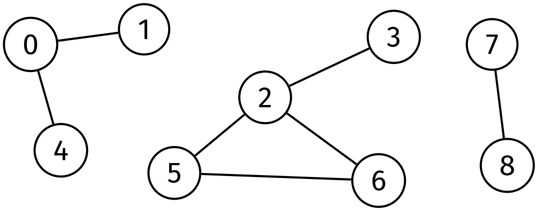
How many connected components are there?

Are two vertices in the same connected component?

Connected Components

Goal:

- Compute an array which indicates which connected component each vertex is in
  - Let this array be called `componentOf`
  - `componentOf[`v`]` contains the component number of vertex v
- For example:



[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
0	0	1	1	0	1	1	2	2

Connected Components

Idea:

- Choose a vertex and perform a DFS starting at that vertex
  - During the DFS, assign all vertices visited to component 0
- After the DFS, if any vertex has not been assigned a component, perform a DFS starting at that vertex
  - During this DFS, assign all vertices visited to component 1
- Repeat until all vertices are assigned a component, increasing the component number each time

```
components(G):
    Input:  graph G
    Output: componentOf array

    create componentOf array, initialised to -1

    compNo = 0
    for each vertex v in G:
        if componentOf[v] = -1:
            dfsComponents(G, v, componentOf, compNo)
            compNo = compNo + 1

    return componentOf

dfsComponents(G, v, componentOf, compNo):
    componentOf[v] = compNo
    for each neighbour w of v in G:
        if componentOf[w] = -1:
            dfsComponents(G, w, componentOf, compNo)
```

Connected Components

Analysis

Analysis for adjacency list representation:

- Algorithm performs a full DFS, which is $O(V + E)$

Connected Components

Suppose we frequently need to answer the following questions:

- How many connected components are there?
- Are v and w in the same connected component?
- Is there a path between v and w?

Note: The last two questions are actually equivalent in an undirected graph.

Connected Components

Solution:

- Cache the components array in the graph struct

```
struct graph {
    ...
    int nC;  // number of connected components
    int *cc; // componentOf array
};
```

Connected Components

This allows us to answer the questions very easily:

```
// How many connected components are there?
int numComponents(Graph g) {
    return g->nC;
}

// Are v and w in the same connected component?
bool inSameComponent(Graph g, Vertex v, Vertex w) {
    return g->cc[v] == g->cc[w];
}

// Is there a path between v and w?
bool hasPath(Graph g, Vertex v, Vertex w) {
    return g->cc[v] == g->cc[w];
}
```

Connected Components

However, this information needs to be maintained as the graph changes:

- Inserting an edge may reduce nC
  - If the endpoint vertices were in different components
- Removing an edge may increase nC
  - If the endpoint vertices were in the same component *and* there is no other path between them

Hamiltonian Path and Circuit

A Hamiltonian path is
a path that includes each vertex exactly once

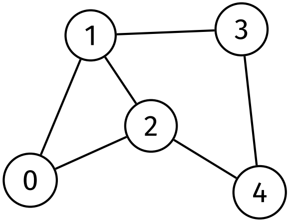
A Hamiltonian circuit is
a cycle that includes each vertex exactly once

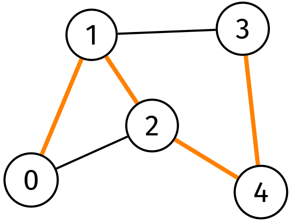
Hamiltonian Path and Circuit

Named after
Irish mathematician, astronomer and physicist
Sir William Rowan Hamilton (1805-1865)

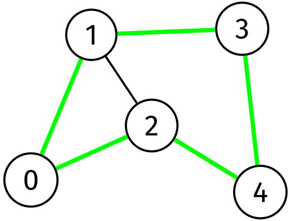
Hamiltonian Path and Circuit

Consider the following graph:





Hamiltonian path



Hamiltonian circuit

Hamiltonian Path

How to check if a graph has a Hamiltonian path?

Idea:

- Brute force
- Use DFS to check all possible paths
    - Recursive DFS is perfect, as it naturally allows backtracking
- Keep track of the number of vertices left to visit
- Stop when this number reaches 0

Hamiltonian Path

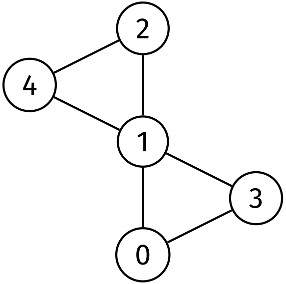
```
hasHamiltonianPath(G):
    Input:  graph G
    Output: true if G has a Hamiltonian path
            false otherwise

    create visited array, initialised to false
    for each vertex v in G:
        if dfsHamiltonianPath(G, v, visited, #vertices(G)):
            return true

    return false
```

Hamiltonian Path

```
dfsHamiltonianPath(G, v, visited, numVerticesLeft):
    visited[v] = true
    numVerticesLeft = numVerticesLeft - 1

    if numVerticesLeft = 0:
        return true

    for each neighbour w of v in G:
        if visited[w] = false:
            if dfsHamiltonianPath(G, w, visited, numVerticesLeft):
                return true

    visited[v] = false
    return false
```

Hamiltonian Path

Why set `visited[`v`]` to false at the end of `dfsHamiltonianPath`?



Hamiltonian Circuit

How to check if a graph has a Hamiltonian *circuit*?

- Similar approach as Hamiltonian path
- Don't need to try all starting vertices
- After a Hamiltonian path is found, check if the final vertex is adjacent to the starting vertex

Hamiltonian Circuit

```
hasHamiltonianCircuit(G):
    Input:  graph G
    Output: true if G has a Hamiltonian circuit
            false otherwise

    if #vertices(G) < 3:
        return false

    create visited array, initialised to false
    return dfsHamiltonianCircuit(G, 0, visited, #vertices(G))

dfsHamiltonianCircuit(G, v, visited, numVerticesLeft):
    visited[v] = true
    numVerticesLeft = numVerticesLeft - 1

    if numVerticesLeft = 0 and adjacent(G, v, 0):
        return true

    for each neighbour w of v in G:
        if visited[w] = false:
            if dfsHamiltonianCircuit(G, w, visited, numVerticesLeft):
                return true

    visited[v] = false
    return false
```

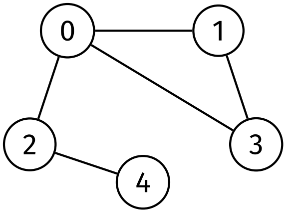
Analysis:

- Worst-case time complexity: $O(V!)$
- There are at most $V!$ paths to check ($\approx \sqrt{2\pi V}(V/e)^V$ by Stirling's approximation)
- There is no known polynomial time algorithm, so the Hamiltonian path problem is NP-hard

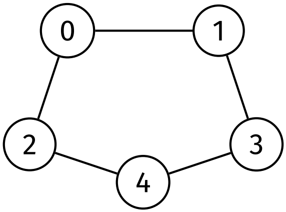
Eulerian Path and Circuit

An Eulerian path is
a path that visits each edge exactly once

An Eulerian circuit is
an Eulerian path that starts and ends at the same vertex



Eulerian path:
4-2-0-1-3-0



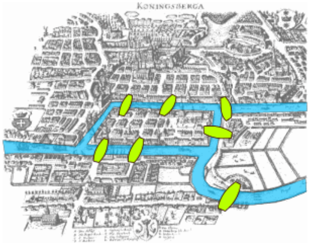
Eulerian circuit:
4-2-0-1-3-4

Problem is named after
Swiss mathematician, physicist, astronomer, logician and engineer
Leonhard Euler (1707-1783)

Eulerian Path and Circuit

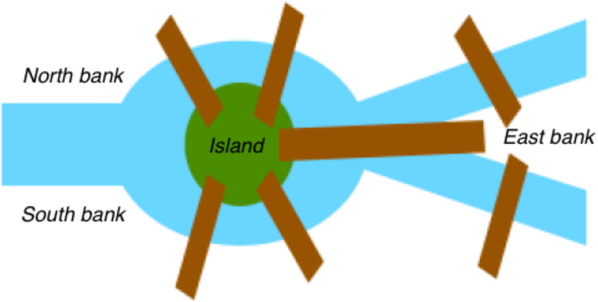
Background

Problem was introduced by Euler while trying to solve the Seven Bridges of Konigsberg problem in 1736.

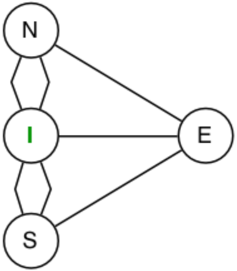


Is there a way to cross all the bridges exactly once on a walk through the town?

This is a graph problem:
vertices represent pieces of land
edges represent bridges



Bridges as schematic



Bridges as graph

Eulerian Path and Circuit

How to check if a graph has an Eulerian path or circuit?

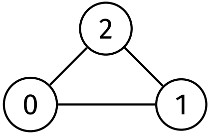
Can use the following theorems:

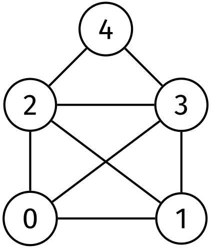
A graph has an Eulerian path if and only if
exactly zero or two vertices have odd degree,
and all vertices with non-zero degree belong to the same connected component

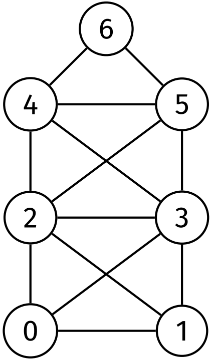
A graph has an Eulerian circuit if and only if
every vertex has even degree,
and all vertices with non-zero degree belong to the same connected component

Eulerian Path and Circuit

Which of these graphs have an Eulerian path? How about an Eulerian circuit?

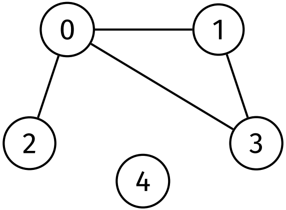






Eulerian Path and Circuit

Why
“all vertices with non-zero degree belong to the same connected component”?



Eulerian Path

```
hasEulerianPath(G):
    Input:   graph G
    Output:  true if G has an Eulerian path
             false otherwise

    numOddDegree = 0
    for each vertex v in G:
        if degree(G, v) is odd:
            numOddDegree = numOddDegree + 1

    return (numOddDegree = 0 or numOddDegree = 2) and
           eulerConnected(G)
```

```
eulerConnected(G):
    Input:  graph G
    Output: true if all vertices in G with non-zero degree
            belong to the same connected component
            false otherwise

    create visited array, initialised to false

    for each vertex v in G:
        if degree(G, v) > 0:
            dfsRec(G, v, visited)
            break

    for each vertex v in G:
        if degree(G, v) > 0 and visited[v] = false:
            return false

    return true
```

Eulerian Circuit

```
hasEulerianCircuit(G):
    Input:  graph G
    Output: true if G has an Eulerian circuit
            false otherwise

    for each vertex v in G:
        if degree(G, v) is odd:
            return false

    return eulerConnected(G)
```

Eulerian Path and Circuit

Analysis

Analysis for adjacency list representation:

- Finding degree of every vertex is $O(V + E)$
- Checking connectivity requires a DFS which is $O(V + E)$
- Therefore, worst-case time complexity is $O(V + E)$

So unlike the Hamiltonian path problem, the Eulerian path problem can be solved in polynomial time.

Other Graph Problems

Tractable and Intractable

Many graph problems are intractable – that is,
there is no known "efficient" algorithm to solve them.

In this context, "efficient" usually means polynomial time.

A tractable problem is one that is known to have a
polynomial-time solution.

Other Graph Problems

Tractable and Intractable

tractable	intractable
what is the shortest path between two vertices?	how about the *longest* path?

Other Graph Problems

Tractable and Intractable

tractable	intractable
what is the shortest path between two vertices?	how about the *longest* path?
does a graph contain a clique?	what is the *largest* clique?

Other Graph Problems

Tractable and Intractable

tractable	intractable
what is the shortest path between two vertices?	how about the *longest* path?
does a graph contain a clique?	what is the *largest* clique?
given two colors, is it possible to colour every vertex in a graph such that no two adjacent vertices are the same colour?	what about *three* colours?

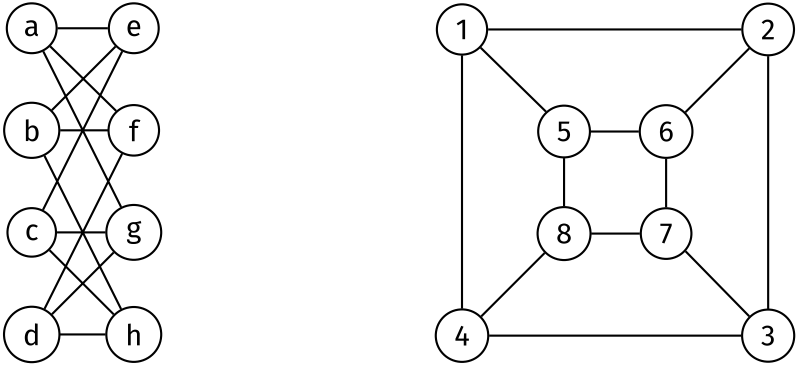
Other Graph Problems

Tractable and Intractable

tractable	intractable
what is the shortest path between two vertices?	how about the *longest* path?
does a graph contain a clique?	what is the *largest* clique?
given two colors, is it possible to colour every vertex in a graph such that no two adjacent vertices are the same colour?	what about *three* colours?
does a graph contain an Eulerian path?	how about a Hamiltonian path?

Other Graph Problems

Bonus Round!



Graph isomorphism:
Can we make two given graphs identical by renaming vertices?