COMP2521 25T2

Graphs (VII)

Minimum Spanning Trees

Sim Mautner

cs2521@cse.unsw.edu.au

minimum spanning trees
kruskal's algorithm
prim's algorithm

Minimum Spanning Trees

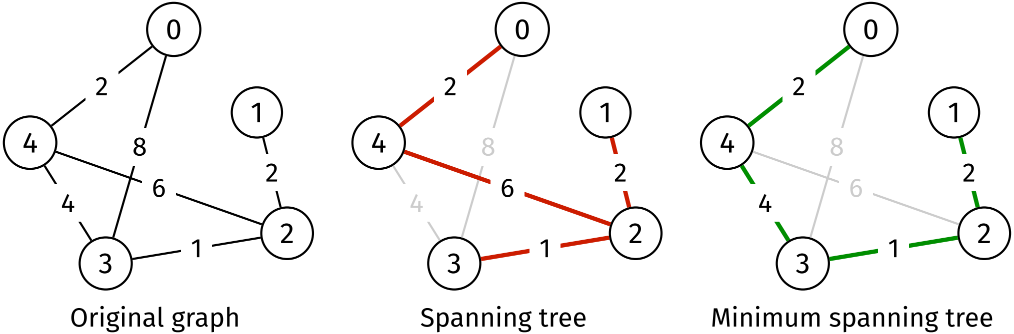
A spanning tree of an undirected graph G
is a subgraph of G that contains all vertices of G,
that is connected and contains no cycles

A minimum spanning tree of an undirected weighted graph G
is a spanning tree of G that has
minimum total edge weight among all spanning trees of G

Applications:
Electrical grids, networks
Any situation where we want to connect nodes as cheaply as possible

Minimum Spanning Trees

Example



Minimum Spanning Tree Algorithms

Basic minimum spanning tree algorithms:

- Kruskal's algorithm
- Prim's algorithm

Kruskal's Algorithm

Invented by
American mathematician, statistician, computer scientist
Joseph Kruskal in 1956

Kruskal's Algorithm

Algorithm:

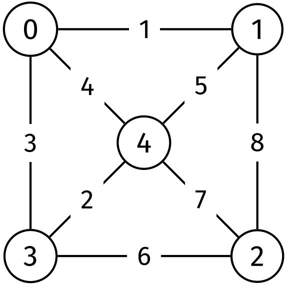
1. Start with an empty graph
   - With same vertices as original graph
2. Consider edges in increasing weight order
   - Add edge if it does not form a cycle in the MST
3. Repeat until $V - 1$ edges have been added

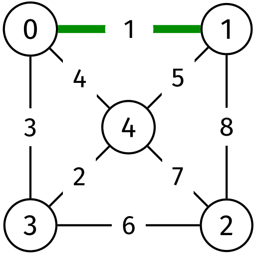
Critical operations:

- Iterating over edges in weight order
- Checking if adding an edge would form a cycle

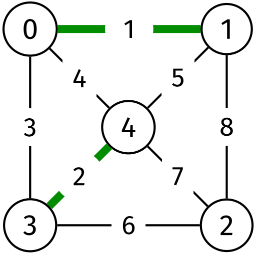
Kruskal's Algorithm
Example

Run Kruskal's algorithm on this graph:

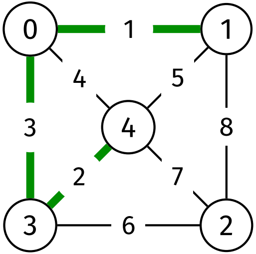




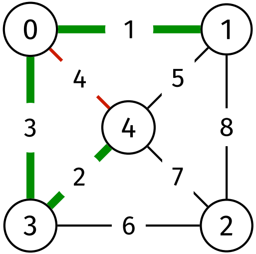
Add 0-1



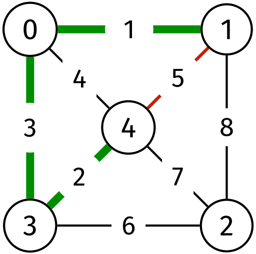
Add 3-4



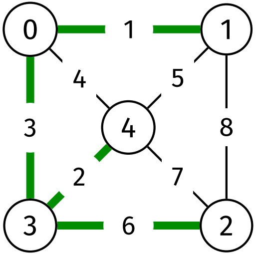
Add 0-3



Don't add 0-4



Don't add 1-4

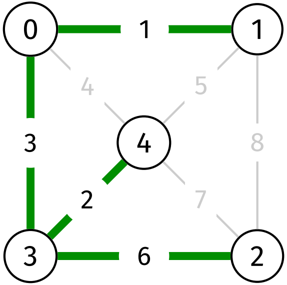


Add 2-3

Kruskal's Algorithm

Example

MST:



Kruskal's Algorithm

Pseudocode (Version 1)


```
kruskalMst(G):
    Input:  graph G with V vertices
    Output: minimum spanning tree of G

    mst = empty graph with V vertices

    sortedEdges = sort edges of G by weight
    for each edge e in sortedEdges:
        add e to mst
        if mst has a cycle:
            remove e from mst

        if mst has V - 1 edges:
            return mst
```

Kruskal's Algorithm

Pseudocode (Version 2)


```
kruskalMst(G):
    Input:  graph G with V vertices
    Output: minimum spanning tree of G

    mst = empty graph with V vertices

    sortedEdges = sort edges of G by weight
    for each edge (v, w, weight) in sortedEdges:
        if there is no path between v and w in mst:
            add edge (v, w, weight) to mst

        if mst has V - 1 edges:
            return mst
```

Proof by exchange argument.

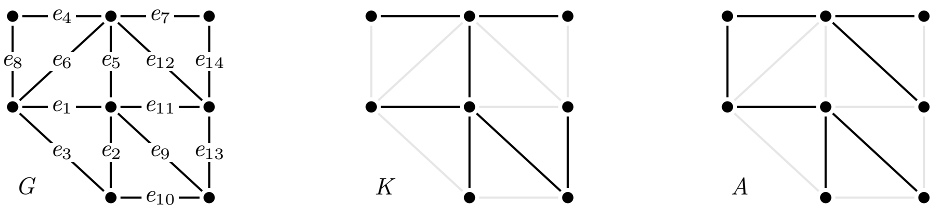
Idea:

- Suppose there exists another algorithm A which makes a different set of choices
  - In this case, chooses a different set of edges for the MST
- Identify *one* choice made by A which is not made by our algorithm
- Show that by exchanging that choice with one of the choices made by our algorithm, the solution does not become worse or less optimal
  - In this case, the "solution" is the spanning tree produced
  - In this case, an "optimal" solution is a spanning tree that costs as little as possible

Sort the edges of G in increasing order.

Let K be the set of edges selected by Kruskal's algorithm.
Let A be the set of edges selected by a different algorithm.

edges of G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A	e_1	e_2		e_4			e_7	e_8	e_9	$\cdots$

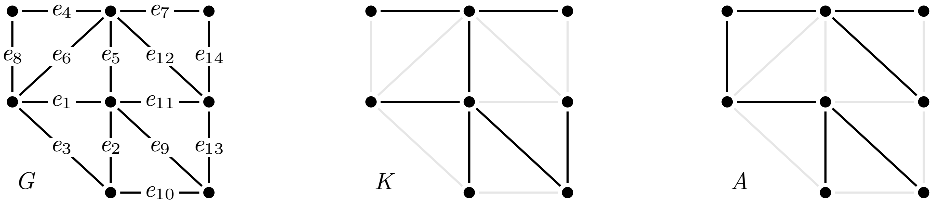


Kruskal's Algorithm

Analysis - Correctness

Consider the first edge that is chosen by K but *not* by A.

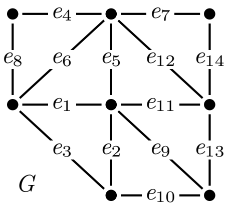
edges of G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A	e_1	e_2		e_4			e_7	e_8	e_9	$\cdots$

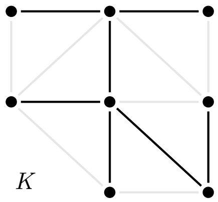


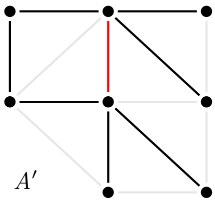
Consider the first edge that is chosen by K but *not* by A.

Add this edge to a copy of A (call it A'). This forms a cycle in A'.

edges of G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A'	e_1	e_2		e_4	e_5		e_7	e_8	e_9	$\cdots$





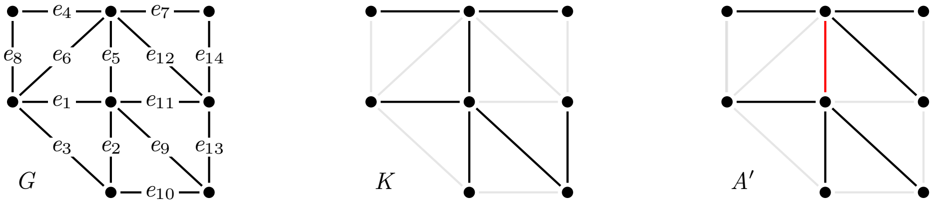


Consider the first edge that is chosen by K but *not* by A.

Add this edge to a copy of A (call it A'). This forms a cycle in A'.

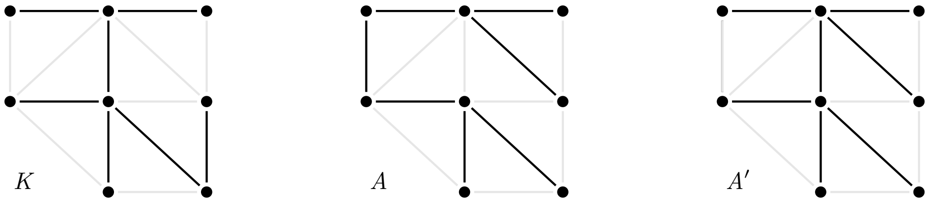
Now find the highest-weight edge in this cycle and *remove* it from A'.

edges of G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A'	e_1	e_2		e_4	e_5		e_7	e_8	e_9	$\cdots$



Kruskal's Algorithm

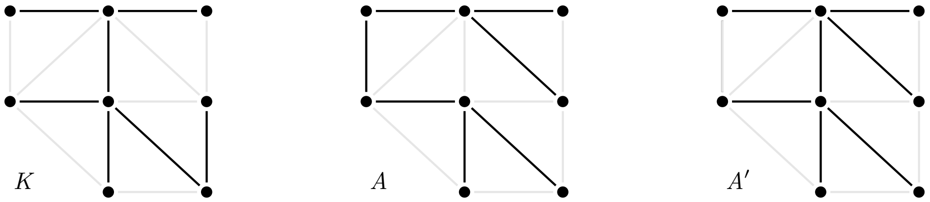
Analysis - Correctness



Now A' is once again a spanning tree,
but it is more similar to K than A and it costs no more than A.

Kruskal's Algorithm

Analysis - Correctness



Now A' is once again a spanning tree,
but it is more similar to K than A and it costs no more than A.

Repeat until A' is identical to K. Each time we perform an exchange, the spanning tree does not increase in cost.

Therefore, K is an optimal spanning tree (MST).

Kruskal's Algorithm

Analysis - Time complexity

Analysis:

- Sorting edges is $O(E \cdot \log E)$
- Main loop has at most E iterations
- Different ways to check if adding an edge would form a cycle
  - Cycle/path checking is $O(V)$ in the worst case (adjacency list)
    $\Rightarrow$ overall cost $= O(E \cdot \log E + E \cdot V) = O(E \cdot V)$
  - Using union-find data structure is close to $O(1)$ in the worst case
    $\Rightarrow$ overall cost $= O(E \cdot \log E + E) = O(E \cdot \log E) = O(E \cdot \log V)$

Prim's Algorithm

Developed by Vojtěch Jarník in 1930
and rediscovered by Robert C. Prim in 1957

Vojtěch Jarník

Robert C. Prim

Prim's Algorithm

Algorithm:

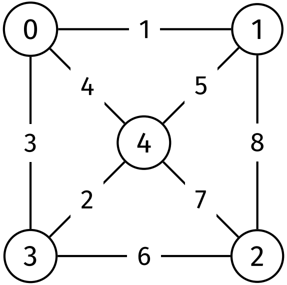
1. Start with an empty graph
2. Start from any vertex, add it to the MST
3. Choose cheapest edge s–t such that:
   - s has been added to the MST, and
   - t has not been added to the MST

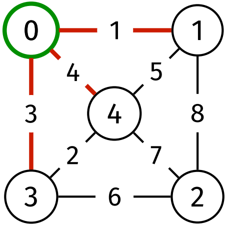
   and add this edge and the vertex t to the MST
4. Repeat previous step until $V - 1$ edges have been added
   - Or until all vertices have been added

Critical operations:

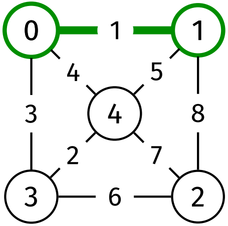
- Finding the cheapest edge s–t such that
  s has been added to the MST and t has not been added to the MST

Run Prim's algorithm on this graph (starting at 0):

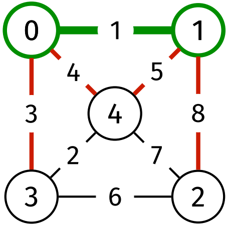




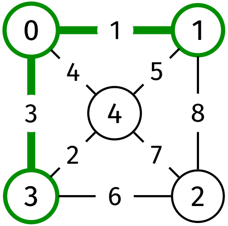
Start of step 1



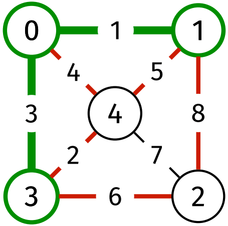
End of step 1



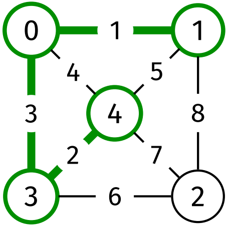
Start of step 2



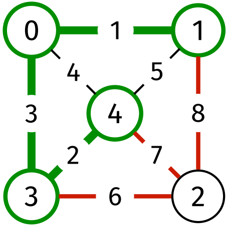
End of step 2



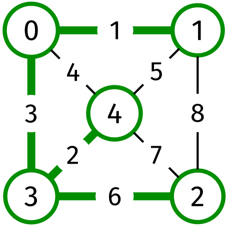
Start of step 3



End of step 3



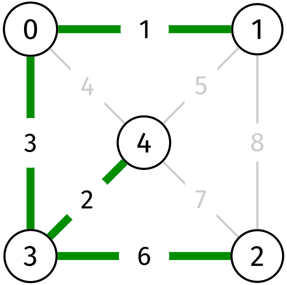
Start of step 4



End of step 4

Prim's Algorithm
Example

MST:



Prim's Algorithm
Pseudocode

```
primMst(G):
    Input:  graph G with V vertices
    Output: minimum spanning tree of G

    mst = empty graph with V vertices
    usedV = {0}
    unusedE = edges of G
    while |usedV| < V:
        find cheapest edge e (s, t, weight) in unusedE such that
                s ∈ usedV and t ∉ usedV

        add e to mst
        add t to usedV
        remove e from unusedE

    return mst
```

Analysis:

- Algorithm considers at most E edges $\Rightarrow O(E)$
- Loop has V iterations
- In each iteration, finding the minimum-weighted edge:
  - With set of edges is $O(E)$
    $\Rightarrow$ overall cost $= O(E + V \cdot E) = O(V \cdot E)$
  - With Fibonacci heap is $O(\log E) = O(\log V)$
    $\Rightarrow$ overall cost $= O(E + V \cdot \log V)$

Comparison

Kruskal's algorithm vs Prim's algorithm

Kruskal's algorithm...

- is $O(E \cdot \log V)$
- uses array-based data structures
- performs better on sparse graphs

Prim's algorithm...

- is $O(E + V \cdot \log V)$
- uses complex linked data structures
  - in its most efficient implementation (Fibonacci heap)
- performs better on dense graphs

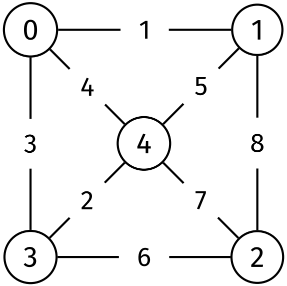
Other MST Algorithms

- Boruvka's algorithm
  - Oldest MST algorithm
  - Start with V separate components
  - Join components using min cost links
  - Continue until only a single component
  - Worst-case time complexity: $O(E \cdot \log V)$
- Karger, Klein and Tarjan
  - Based on Boruvka's algorithm, but non-deterministic
  - Randomly selects subset of edges to consider
  - Time complexity: $O(E)$ on average

Appendix

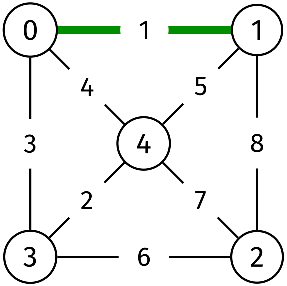
Kruskal’s Algorithm Example

Original graph



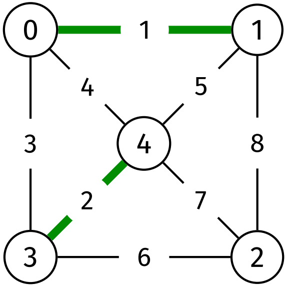
Kruskal's Algorithm Example

Adding 0-1 would not create a cycle



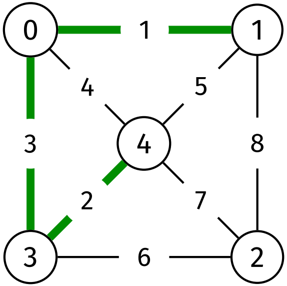
Kruskal's Algorithm Example

Adding 3-4 would not create a cycle



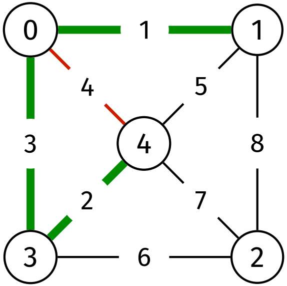
Kruskal's Algorithm Example

Adding 0-3 would not create a cycle



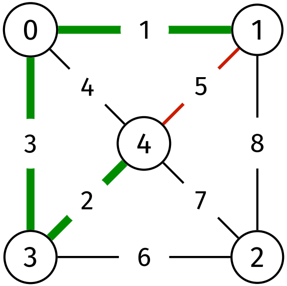
Kruskal's Algorithm Example

Adding 0-4 would create a cycle



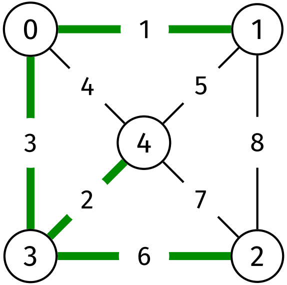
Kruskal's Algorithm Example

Adding 1-4 would create a cycle



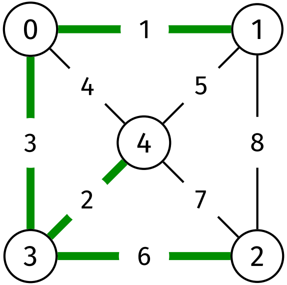
Kruskal's Algorithm Example

Adding 2-3 would not create a cycle



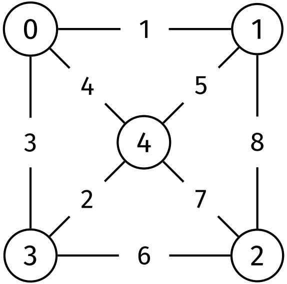
Kruskal's Algorithm Example

Done - MST has 4 edges



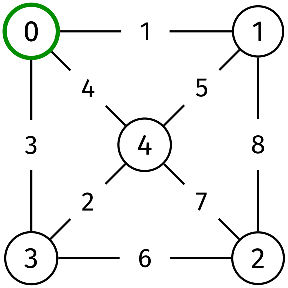
Prim's Algorithm Example

Original graph



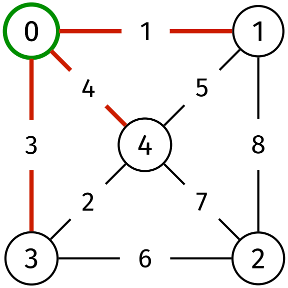
Prim's Algorithm Example

Start at vertex 0



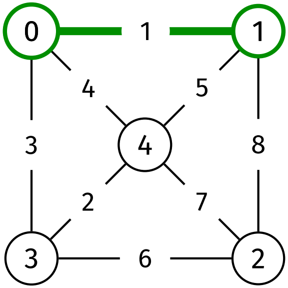
Prim's Algorithm Example

Choose cheapest edge out of these (in red)



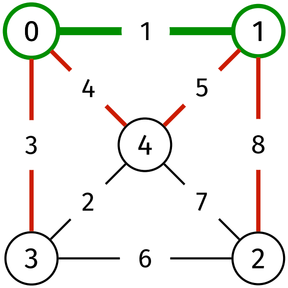
Prim's Algorithm Example

Add 0-1 to MST



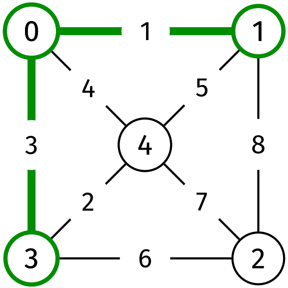
Prim's Algorithm Example

Choose cheapest edge out of these (in red)



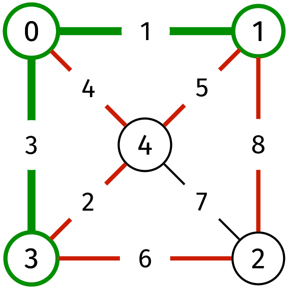
Prim's Algorithm Example

Add 0-3 to MST



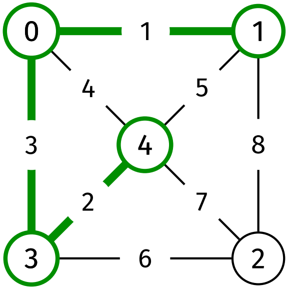
Prim’s Algorithm Example

Choose cheapest edge out of these (in red)



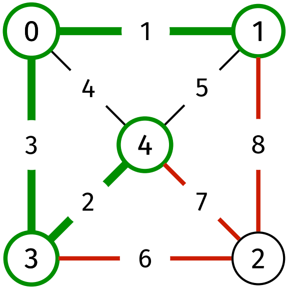
Prim's Algorithm Example

Add 3-4 to MST



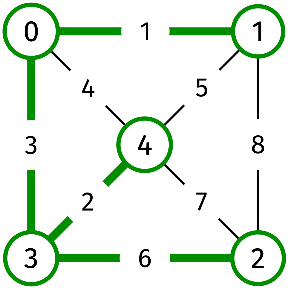
Prim's Algorithm Example

Choose cheapest edge out of these (in red)



Prim's Algorithm Example

Add 3-2 to MST



Prim's Algorithm Example

Done - MST has 4 edges