COMP2521 25T2
Graphs (I)
Introduction to Graphs

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graph fundamentals
graph adt
graph representations

Graph Fundamentals

Motivation

Up to this point, we've seen a few collection types...

lists: a *linear* sequence of items
each node is connected to its next node

trees: a *branched* hierarchy of items
each node is connected to its child node(s)

what if we want something more general?
each node is connected to arbitrarily many nodes

Motivation

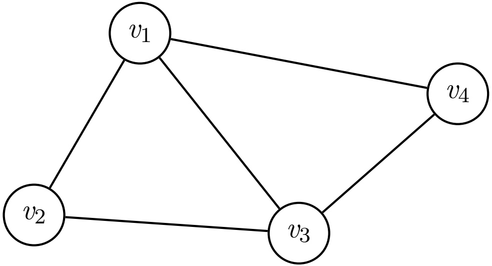
Many applications need to model relationships between items.

... on a map: cities, connected by roads
... on the Web: pages, connected by hyperlinks
... in a game: states, connected by legal moves
... in a social network: people, connected by friendships
... in scheduling: tasks, connected by constraints
... in circuits: components, connected by traces
... in networking: computers, connected by cables
... in programs: functions, connected by calls
... etc. etc. etc.

Graphs

A graph is a data structure consisting of:

- A set of vertices V
  - Also called nodes
- A set of edges E between pairs of vertices



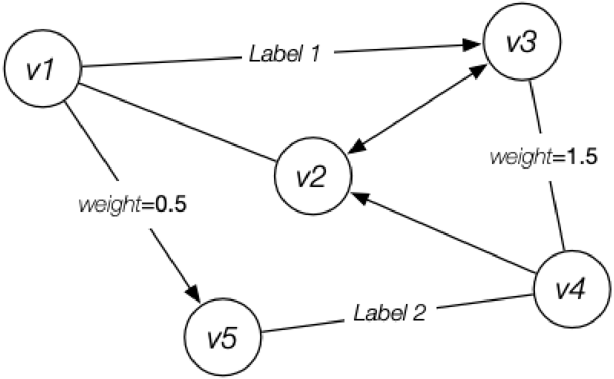
$$V = \{v_1, v_2, v_3, v_4\}$$

$$E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_3, v_4)\}$$

Vertices are distinguished by a unique identifier.

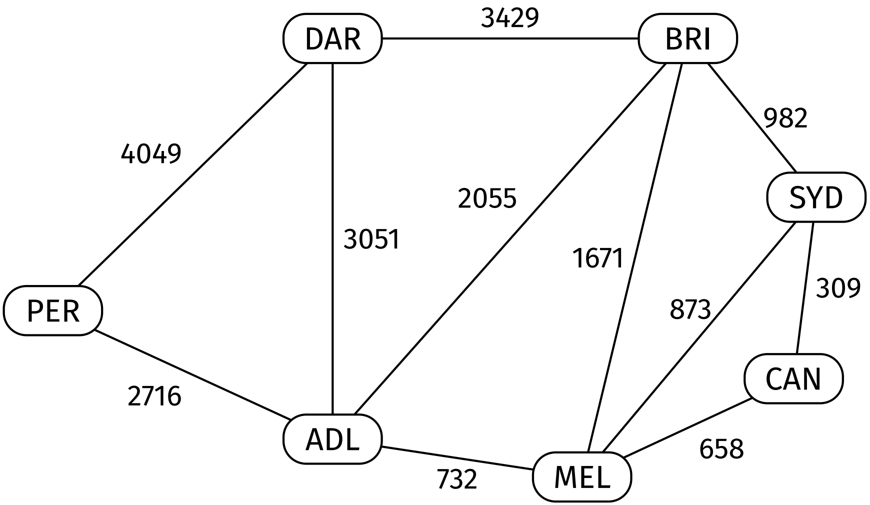
- In this course, usually an integer between 0 and $|V| - 1$

Edges may be (optionally) directed, weighted and/or labelled.



Example

Example: Australian cities and roads



Questions we could answer with a graph:

- Is there a way to get from A to B?
- What is the *best* way to get from A to B?
- In general, what vertices can we reach from A?
- Is there a path that lets me visit all vertices?
- Can we form a tree linking all vertices?
- Are two graphs “equivalent”?

Graph problems are generally more complex to solve than linked list problems:

- Items are not ordered
- Graphs may contain cycles
- Concrete representation is more complex

Graphs
Types of Graphs

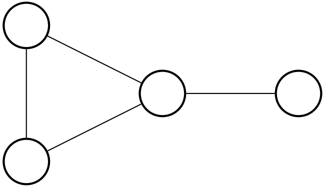
Graphs can be a combination of these types:

undirected	or	directed
unweighted	or	weighted
without loops	or	with loops
non-multigraph	or	multigraph

... and others ...

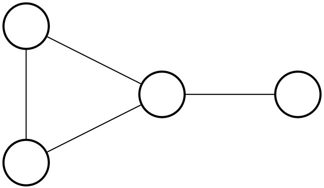
Undirected Graphs

In an undirected graph, edges do not have direction.
For example, Facebook friends.

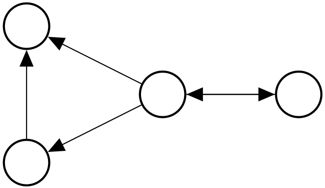


Directed Graphs

In a directed graph or digraph, each edge has a direction.
For example, road maps, Twitter follows.



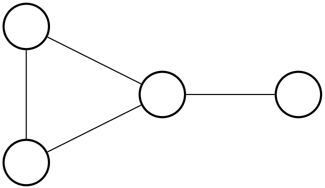
undirected graph



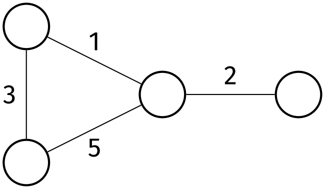
directed graph

Weighted Graphs

In a weighted graph, each edge has an associated weight.
For example, road maps, networks.



unweighted graph

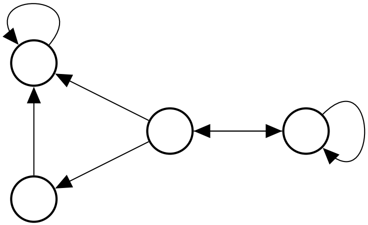


weighted graph

Graphs with Loops

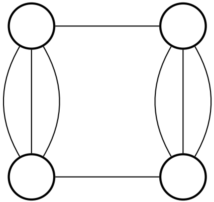
A loop is an edge from a vertex to itself.

Depending on the context,
a graph may or may not be able to have loops.



Multigraphs

In a multigraph,
multiple edges are allowed between two vertices.
For example, call graphs, maps.



Multigraphs will not be considered in this course.

Simple Graphs

A simple graph is an undirected graph
with no loops and no multiple edges.

For now, we will only consider simple graphs.

Simple Graphs

Question:

For a simple graph with V vertices,
what is the *maximum* possible number of edges?

Simple Graphs

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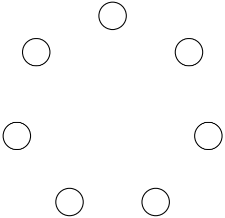
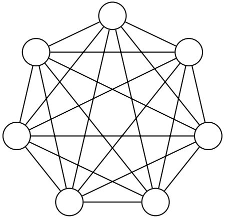
For a simple graph with V vertices,
what is the *maximum* possible number of edges?



$E = 0$



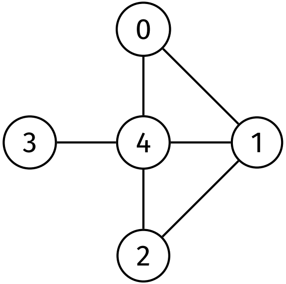
$E = V(V-1)/2$

Note on notation:
The number of vertices $|V|$ and the number of edges $|E|$ are normally written as V and E for simplicity.

Graph Terminology (I)

Two vertices v and w are adjacent if an edge $e := (v, w)$ connects them; we say e is incident on v and w.

The degree of a vertex v ($\deg(v)$) is the number of edges incident on v.

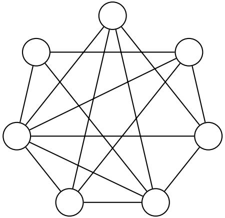


$$
\begin{aligned}
\deg(0) &= 2\\
\deg(1) &= 3\\
\deg(2) &= 2\\
\deg(3) &= 1\\
\deg(4) &= 4
\end{aligned}
$$

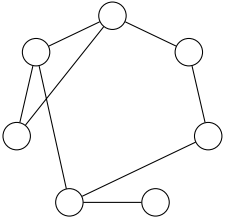
Graph Terminology (II)

The ratio E:V can vary considerably.

If E is closer to V^2, the graph is dense.
If E is closer to V, the graph is sparse.



dense graph



sparse graph

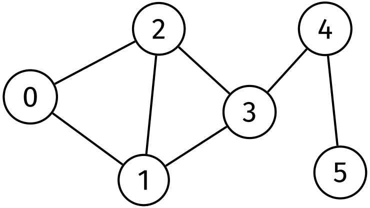
Knowing whether a graph is dense or sparse will affect our choice of representation and algorithms.

Graph Terminology (III)

A path is
a sequence of vertices where
each vertex has a edge to the next in the
sequence

A path is simple
if it has no repeating vertices

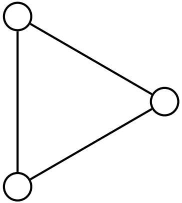
A cycle is a path where
only the first and last vertices are the same
0-1-2-0, 1-2-3-1, 0-1-3-2-0



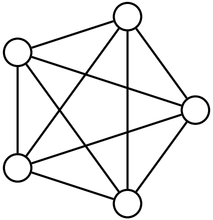
Graph Terminology (IV)

A complete graph is a graph where
every vertex is connected to every other vertex via an edge.

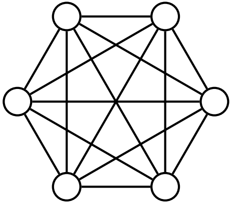
In a complete graph, $E = \frac{1}{2}V(V-1)$.



K_3



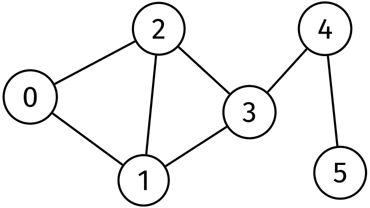
K_5



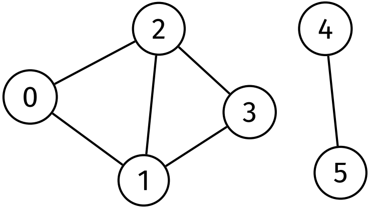
K_6

Graph Terminology (V)

A connected graph is a graph where
there is a path from every vertex to every other vertex.



Connected graph

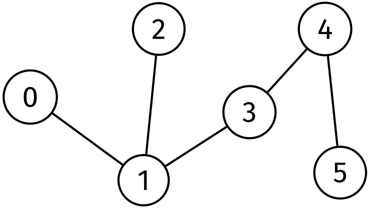


Disconnected graph

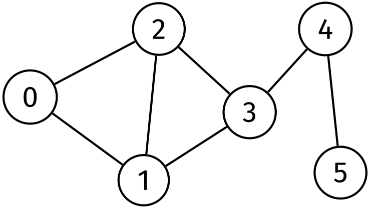
Graph Terminology (VI)

A tree is a connected graph with no cycles.

A tree has exactly one path between each pair of vertices.



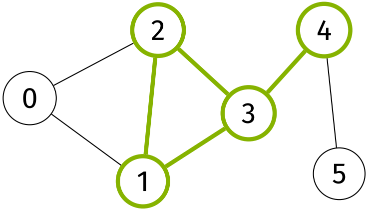
Tree

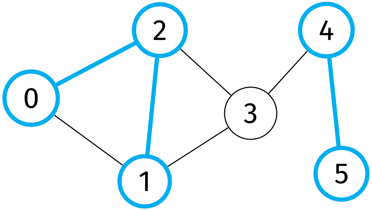


Not a tree

Graph Terminology (VII)

A subgraph of a graph G
is a graph that contains a subset of the vertices of G
and a subset of the edges between these vertices.

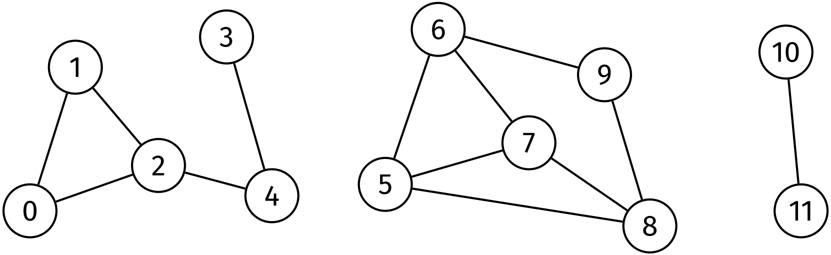




Graph Terminology (VIII)

A connected component is a maximally connected subgraph.

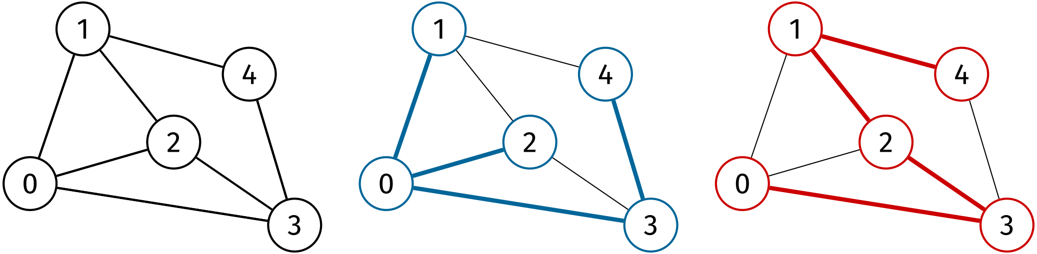
A connected graph has one connected component — the graph itself.
A disconnected graph has two or more connected components.



Graph Terminology (IX)

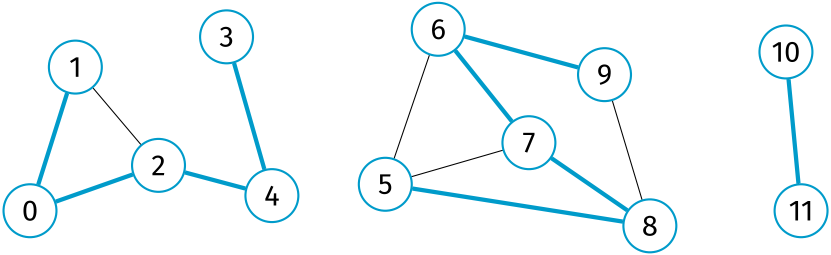
A spanning tree of a graph G
is a subgraph that contains all the vertices of G
and is a single tree.

Spanning trees only exist for connected graphs.



Graph Terminology (X)

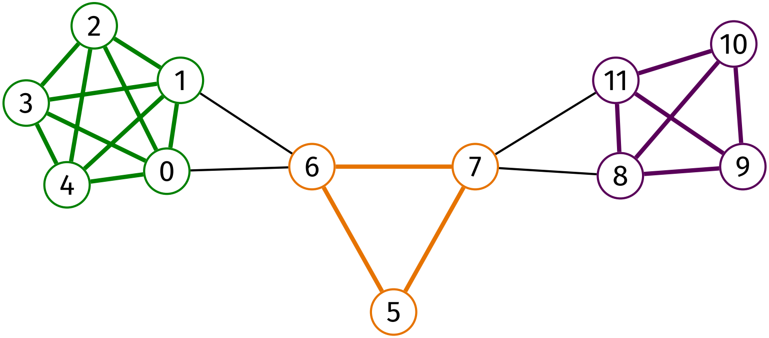
A spanning forest of a graph G
is a subgraph that contains all the vertices of G
and contains one tree for each connected component.



Graph Terminology (XI)

A clique is a complete subgraph.

A clique is non-trivial if it has 3 or more vertices.



Graph ADT

Graph ADT

What do we need to represent?
What operations do we need to support?

Graph ADT

What do we need to represent?

A set of vertices $V := \{v_1, \cdots, v_n\}$

A set of edges $E := \{(v, w) \mid v, w \in V\}$

What operations do we need to support?

create/destroy graph

add/remove edges

get #vertices, #edges

check if an edge exists

create/destroy
create a graph
free memory allocated to graph

query
get number of vertices
get number of edges
check if an edge exists

update
add edge
remove edge

We will extend this ADT with more complex operations later.

A Graph ADT

"Graph.h" - Operations to Create/Destroy

```
typedef struct graph *Graph;

// vertices denoted by integers 0..V-1
typedef int Vertex;

/** Creates a new graph with nV vertices */
Graph GraphNew(int nV);

/** Frees all memory allocated to a graph */
void GraphFree(Graph g);
```

A Graph ADT

"Graph.h" - Operations to Query

```
/** Returns the number of vertices in a graph */
int GraphNumVertices(Graph g);

/** Returns the number of edges in a graph */
int GraphNumEdges(Graph g);

/** Returns true if there is an edge between the given vertices
    and false otherwise */
bool GraphIsAdjacent(Graph g, Vertex v, Vertex w);
```

A Graph ADT

"Graph.h" - Operations to Update

```
/** Inserts an edge into a graph */
void GraphInsertEdge(Graph g, Vertex v, Vertex w);

/** Removes an edge from a graph */
void GraphRemoveEdge(Graph g, Vertex v, Vertex w);
```

Graph Representations

Graph Representations

3 main graph representations:

Adjacency Matrix
Edges defined by presence value in $V \times V$ matrix

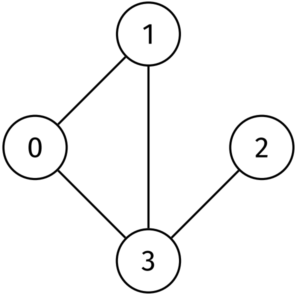
Adjacency List
Edges defined by entries in array of V lists

Array of Edges
Explicit representation of edges as (v, w) pairs

We'll consider these representations for *unweighted*, *undirected* graphs.

Adjacency Matrix

A $V \times V$ matrix
Each cell represents a pair of vertices,
with a 1 indicating an edge between them



	[0]	[1]	[2]	[3]
[0]	0	1	0	1
[1]	1	0	0	1
[2]	0	0	0	1
[3]	1	1	1	0

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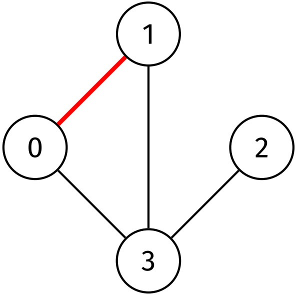

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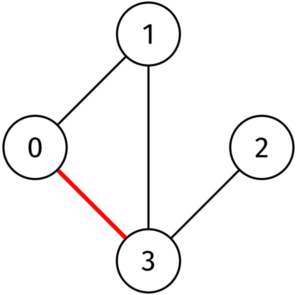

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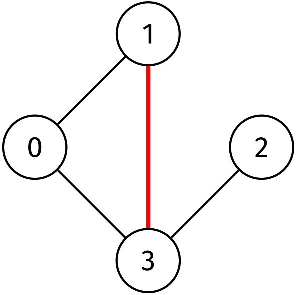

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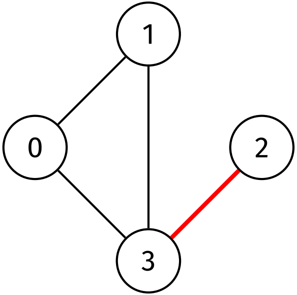
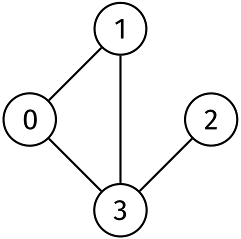


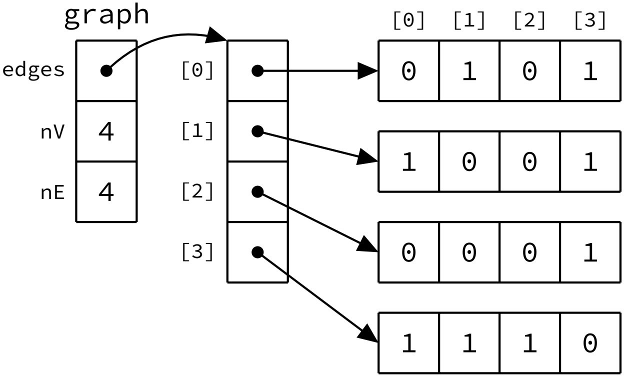
	[0]	[1]	[2]	[3]
[0]	0	1	0	1
[1]	1	0	0	1
[2]	0	0	0	1
[3]	1	1	1	0

Adjacency Matrix

Implementation in C

```c
struct graph {
    int nV;
    int nE;
    bool **edges;
};
```





Adjacency Matrix

Advantages and Disadvantages

Advantages

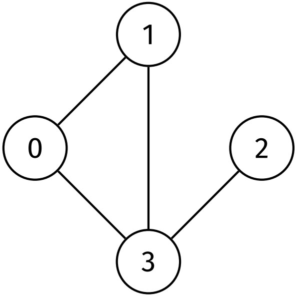
Efficient
edge insertion/deletion
and adjacency check ($O(1)$)

Disadvantages

Huge memory usage ($O(V^2)$)
sparse graph $\Rightarrow$ wasted space!
undirected graph $\Rightarrow$ wasted space!

Adjacency List

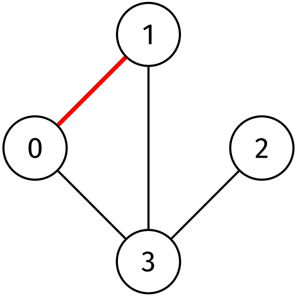
Array of V lists
List at index v contains the neighbours of vertex v



[0]	1, 3
[1]	0, 3
[2]	3
[3]	0, 1, 2

Adjacency List

Array of V lists
List at index v contains the neighbours of vertex v



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Adjacency List

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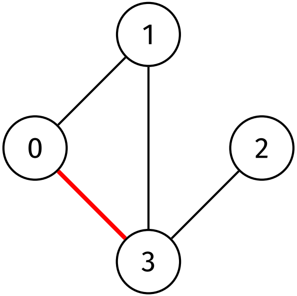

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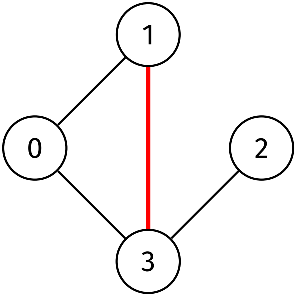
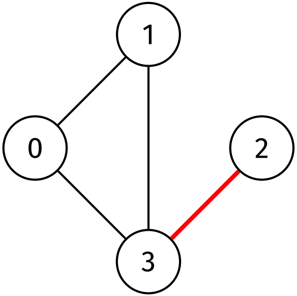


[0]	1, 3
[1]	0, 3
[2]	3
[3]	0, 1, 2

Adjacency List

Array of V lists
List at index v contains the neighbours of vertex v



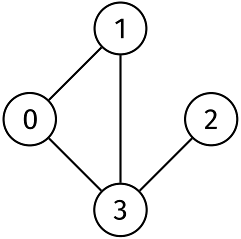
[0]	1, 3
[1]	0, 3
[2]	3
[3]	0, 1, 2

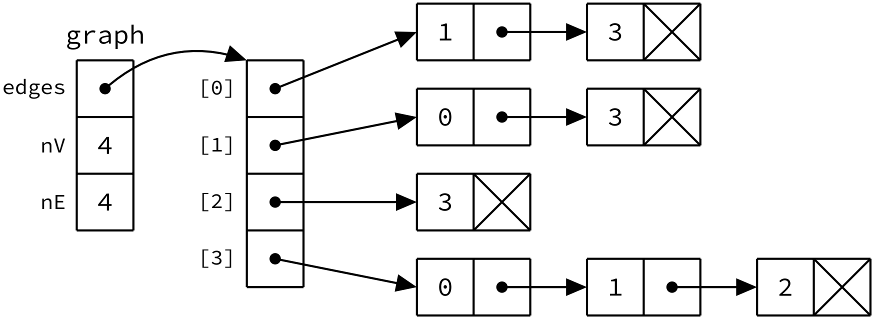
Adjacency List

Implementation in C


```c
struct graph {
    int nV;
    int nE;
    struct adjNode **edges;
};

struct adjNode {
    Vertex v;
    struct adjNode *next;
};
```





Adjacency List

Advantages and Disadvantages

Advantages

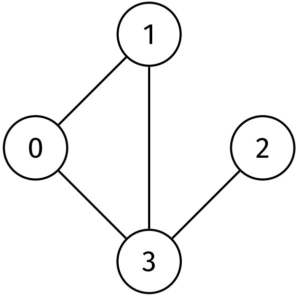
Space-efficient for sparse graphs
$O(V + E)$ memory usage

Disadvantages

Inefficient edge insertion/deletion ($O(V)$)
(matters less for sparse graphs)

Array of Edges

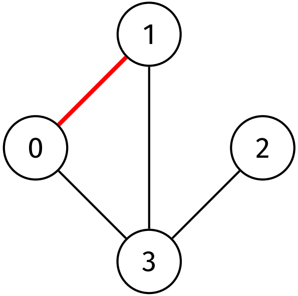
Explicit array of edges (pairs of vertices)



[0]	(0,1)
[1]	(0,3)
[2]	(1,3)
[3]	(2,3)

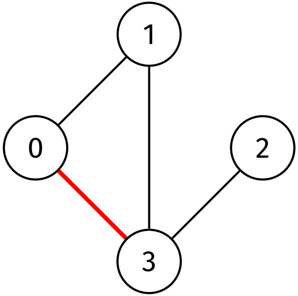
Array of Edges

Explicit array of edges (pairs of vertices)



[0]	(0,1)
[1]	(0,3)
[2]	(1,3)
[3]	(2,3)

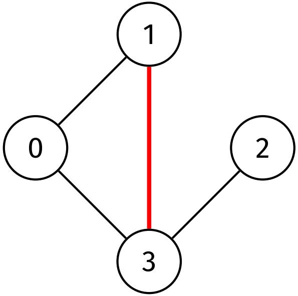
Explicit array of edges (pairs of vertices)



[0]	(0,1)
[1]	(0,3)
[2]	(1,3)
[3]	(2,3)

Array of Edges

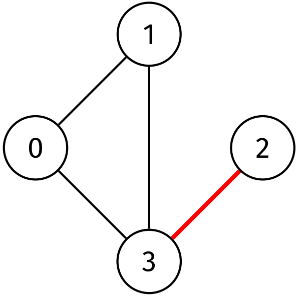
Explicit array of edges (pairs of vertices)



[0]	(0,1)
[1]	(0,3)
[2]	(1,3)
[3]	(2,3)

Array of Edges

Explicit array of edges (pairs of vertices)



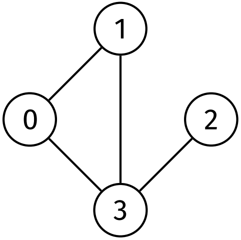
[0]	(0,1)
[1]	(0,3)
[2]	(1,3)
[3]	(2,3)

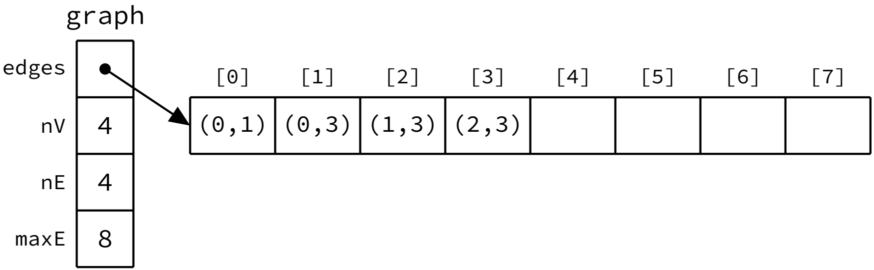
Array of Edges

Implementation in C


```c
struct graph {
    int nV;
    int nE;
    int maxE;
    struct edge *edges;
};

struct edge {
    Vertex v;
    Vertex w;
};
```





Array of Edges

Advantages and Disadvantages

Advantages

Very space-efficient for sparse graphs where $E < V$

Disadvantages

Inefficient edge insertion/deletion ($O(E)$)

Summary of Graph Representations

	Adjacency Matrix	Adjacency List	Array of Edges
Space usage	$O(V^2)$	$O(V+E)$	$O(E)$
Create	$O(V^2)$	$O(V)$	$O(1)$
Destroy	$O(V)$	$O(V+E)$	$O(1)$
Insert edge	$O(1)$	$O(V)$	$O(E)$
Remove edge	$O(1)$	$O(V)$	$O(E)$
Is adjacent	$O(1)$	$O(V)$	$O(E)$*
Degree	$O(V)$	$O(V)$	$O(E)$*

* Can be $O(\log E)$ if the array is ordered
and both directions of each edge are stored in an undirected graph