COMP2521 25T2
Sorting Algorithms (III)
Divide-and-Conquer Sorting Algorithms

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merge sort
quick sort

Divide-and-Conquer Algorithms

divide-and-conquer algorithms
split a problem into two or more subproblems,
solve the subproblems recursively,
and then combine the results.

Merge Sort

Merge Sort

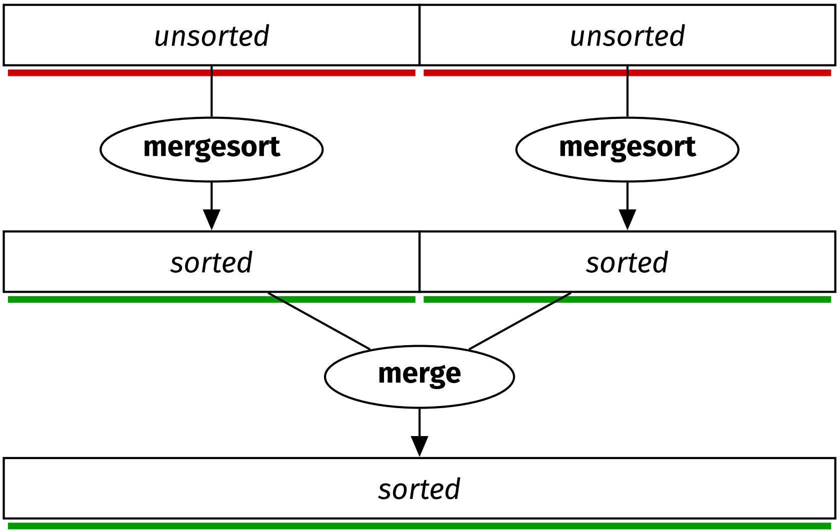
Invented by John von Neumann
in 1945

Merge Sort

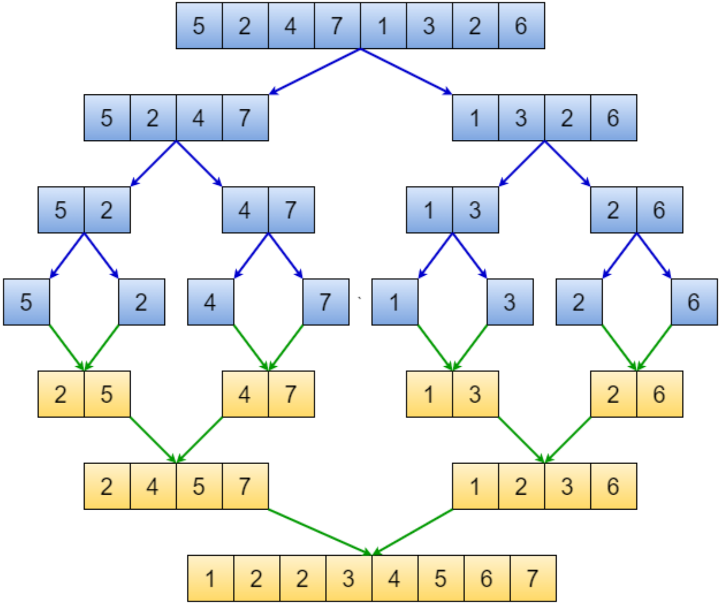
A divide-and-conquer sorting algorithm:

split the array into two roughly equal-sized parts
recursively sort each of the partitions
merge the two now-sorted partitions into a sorted array

Merge Sort


unsorted
unsorted
mergesort
mergesort
sorted
sorted
merge
sorted

Merge Sort


5 2 4 7 1 3 2 6
5 2 4 7
1 3 2 6
5 2
4 7
1 3
2 6
5
2
4
7
1
3
2
6
2 5
4 7
1 3
2 6
2 4 5 7
1 2 3 6
1 2 2 3 4 5 6 7

How do we split the array?

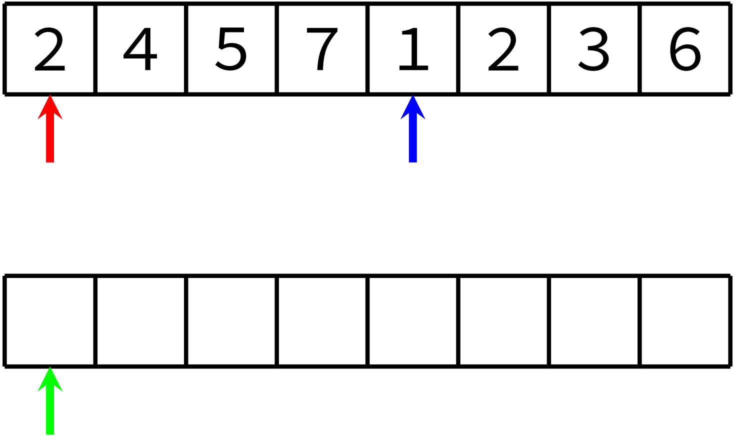
- We don't physically split the array
- We simply calculate the midpoint of the array
    - `mid = (lo + hi) / 2`
- Then recursively sort each half by passing in appropriate indices
    - Sort between indices `lo` and `mid`
    - Sort between indices `mid + 1` and `hi`
- This means the time complexity of splitting the array is $O(1)$

How do we merge two sorted subarrays?

- We merge the subarrays into a *temporary array*
- Keep track of the smallest element that has not been merged in each subarray
- Copy the smaller of the two elements into the temporary array
  - If the elements are equal, take from the left subarray
- Repeat until all elements have been merged
- Then copy from the temporary array back to the original array

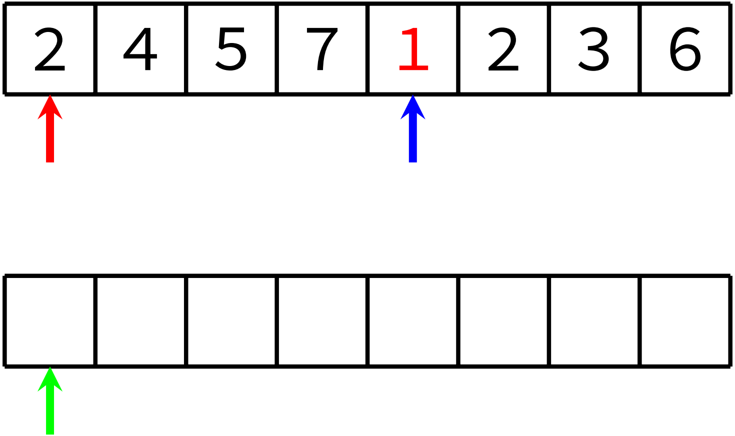
Merge Sort

Merging - Example 1



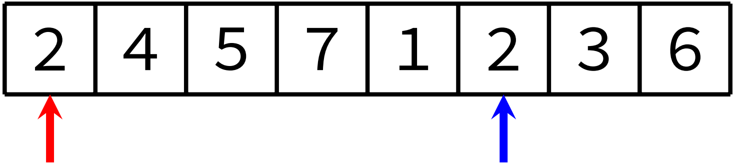
Merge Sort

Merging - Example 1

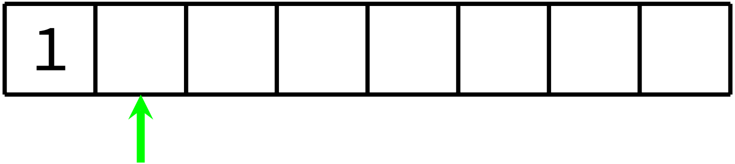


Merge Sort

Merging - Example 1

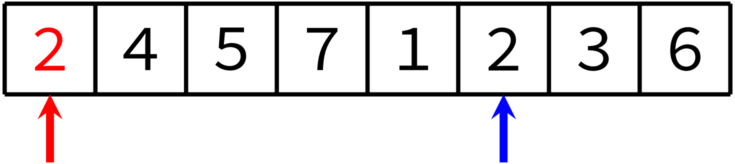


When items are equal, merge takes from the left subarray
(this ensures stability)



Merge Sort

Merging - Example 1



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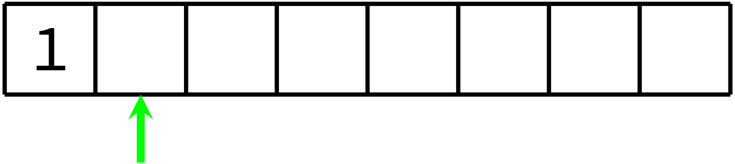

Merge Sort

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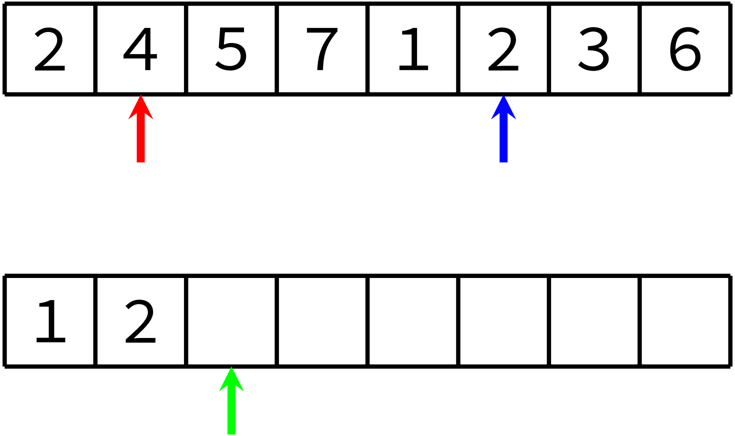
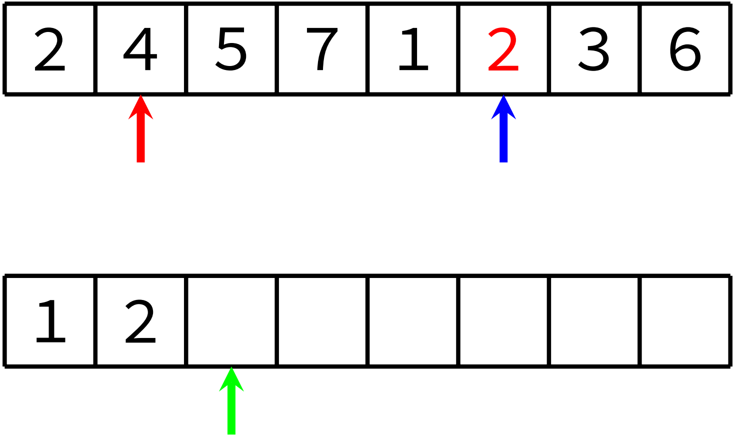
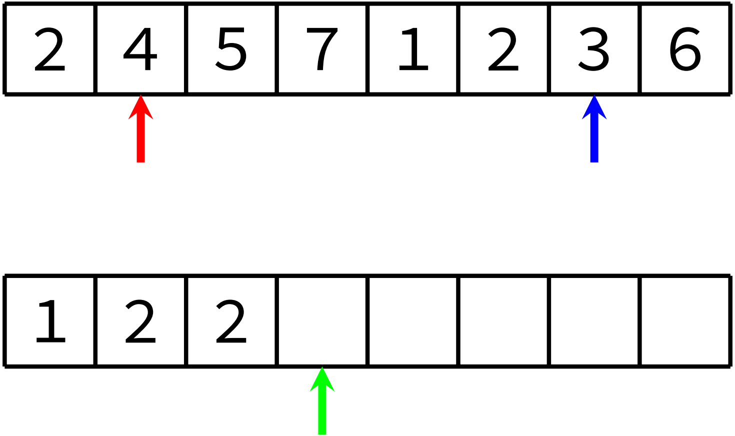


Merge Sort

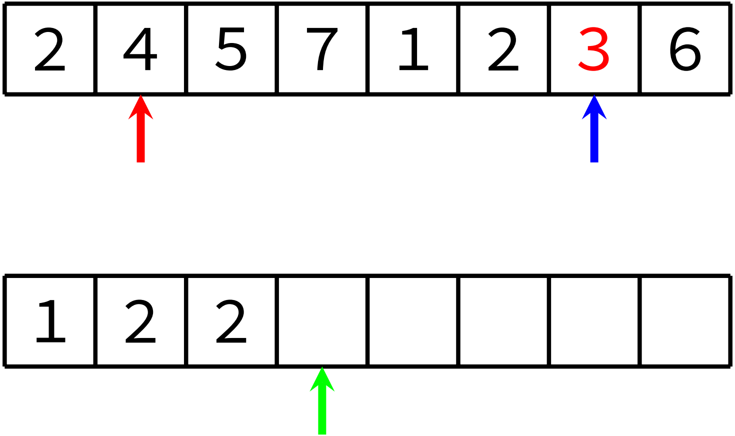
Merging - Example 1




2
4
5
7
1
2
3
6
1
2
2

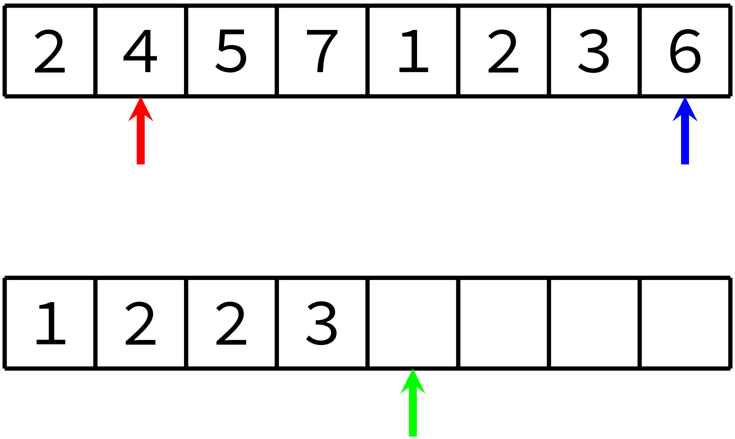
Merge Sort

Merging - Example 1



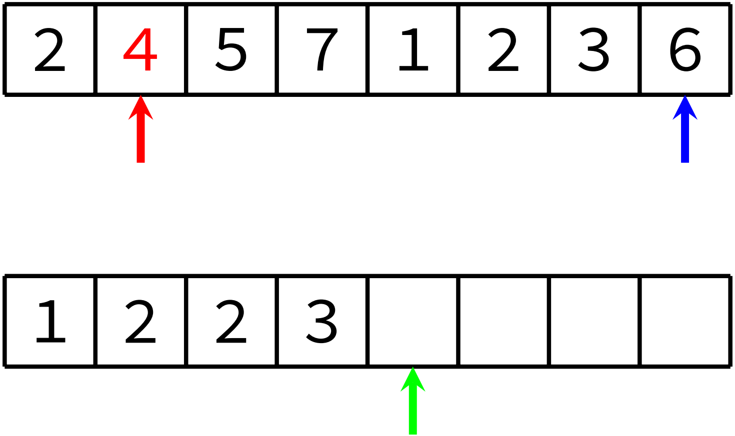
Merge Sort

Merging - Example 1



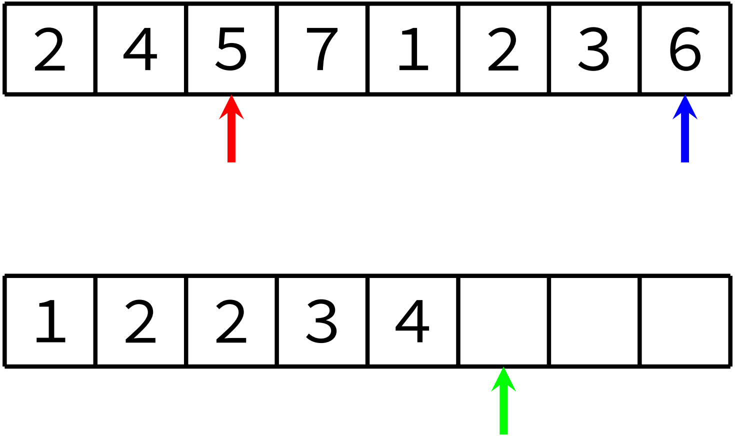
Merge Sort

Merging - Example 1



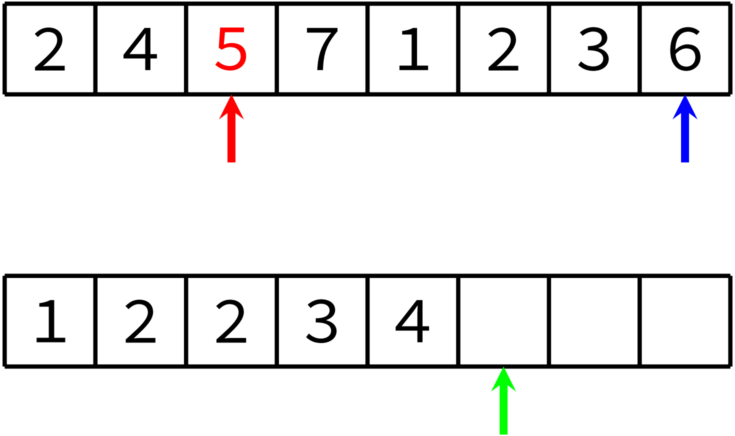
Merge Sort

Merging - Example 1



Merge Sort

Merging - Example 1



Merge Sort

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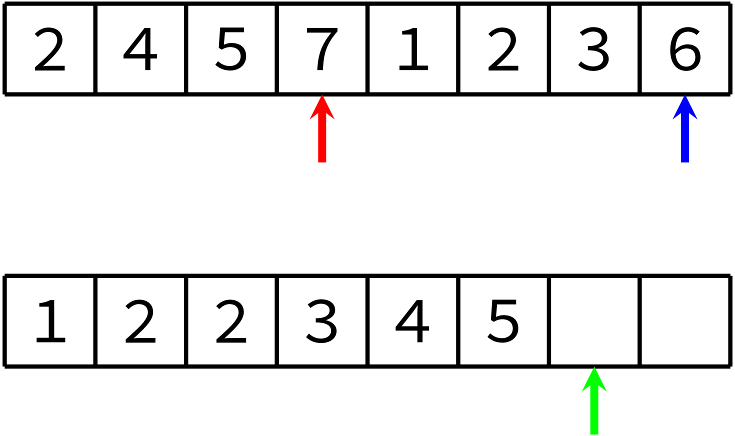
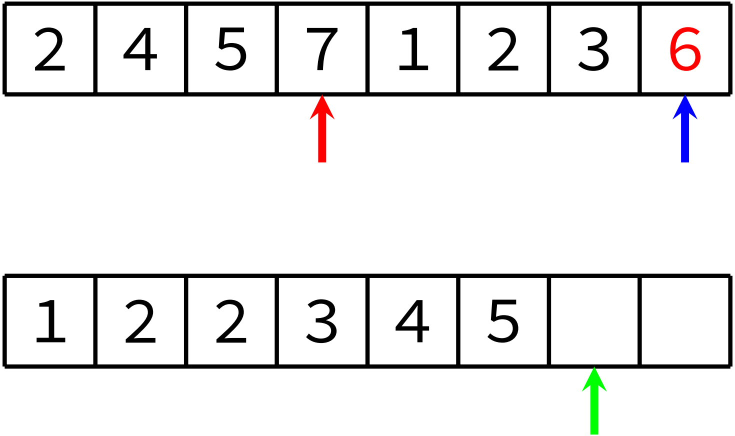


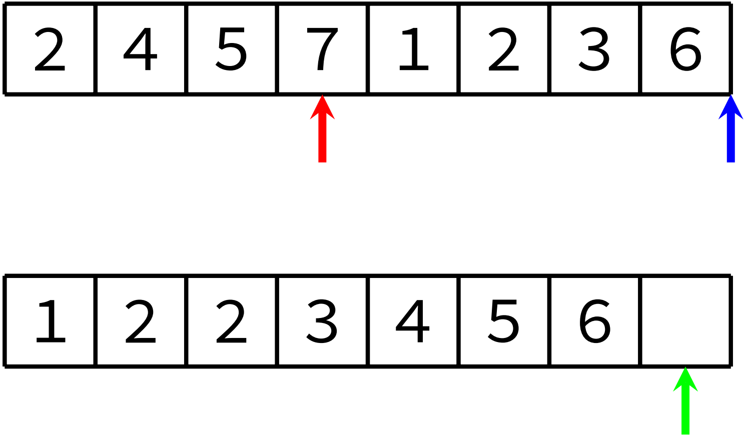
Merge Sort

Merging - Example 1



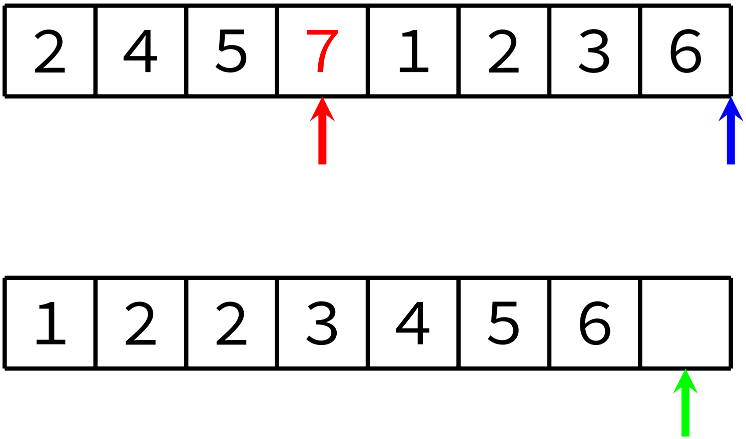
Merge Sort

Merging - Example 1



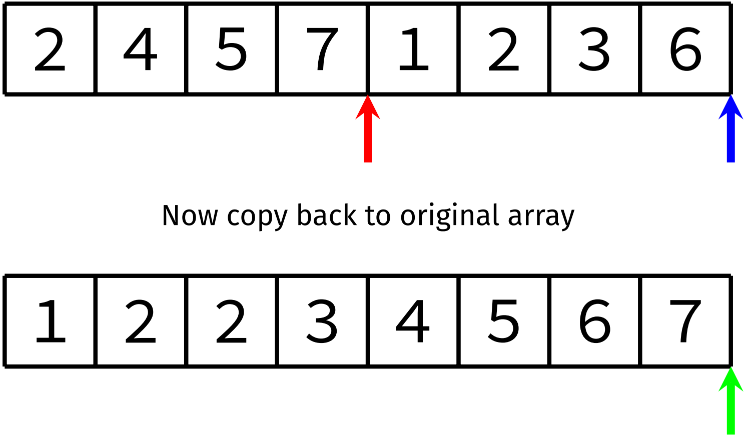
Merge Sort

Merging - Example 1



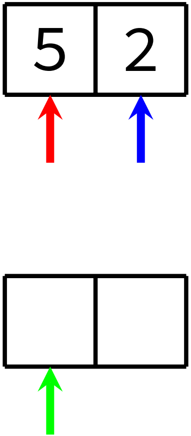
Merge Sort

Merging - Example 1



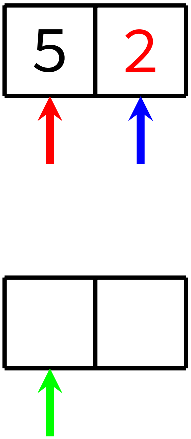
Merge Sort

Merging - Example 2



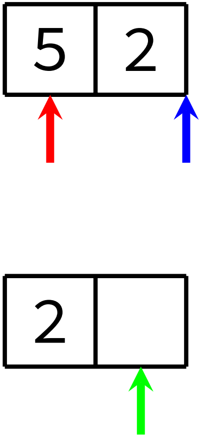
Merge Sort

Merging - Example 2



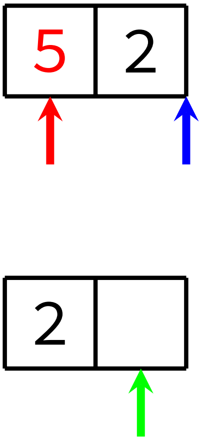
Merge Sort

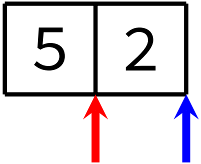
Merging - Example 2



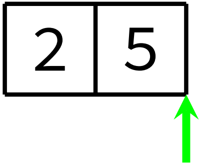
Merge Sort

Merging - Example 2





Now copy back to original array



- The time complexity of merging two sorted subarrays is $O(n)$, where n is the total number of elements in both subarrays
- Therefore:
  - Merging two subarrays of size 1 takes 2 “steps”
  - Merging two subarrays of size 2 takes 4 “steps”
  - Merging two subarrays of size 4 takes 8 “steps”
  - ...

Merge Sort

C Implementation: Sort

```
void mergeSort(Item items[], int lo, int hi) {
    if (lo >= hi) return;
    int mid = (lo + hi) / 2;
    mergeSort(items, lo, mid);
    mergeSort(items, mid + 1, hi);
    merge(items, lo, mid, hi);
}
```

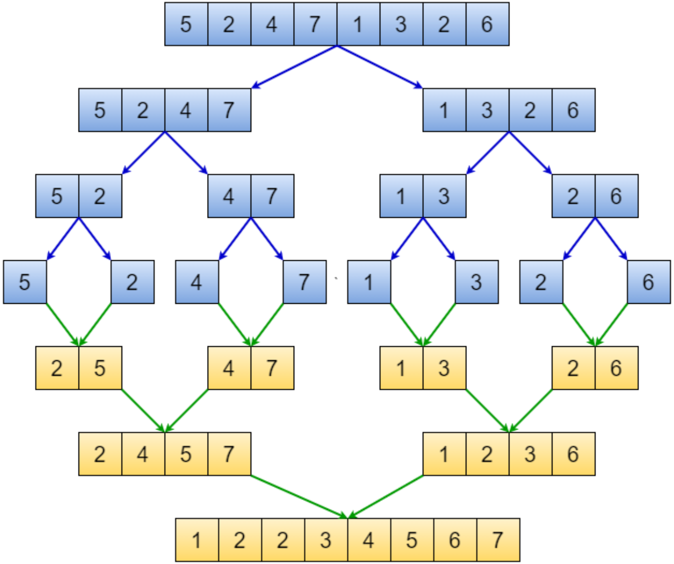
Merge Sort

C Implementation: Merge

```c
void merge(Item items[], int lo, int mid, int hi) {
    Item *tmp = malloc((hi - lo + 1) * sizeof(Item));
    int i = lo, j = mid + 1, k = 0;

    // Scan both segments, copying to `tmp'.
    while (i <= mid && j <= hi) {
        if (le(items[i], items[j])) {
            tmp[k++] = items[i++];
        } else {
            tmp[k++] = items[j++];
        }
    }

    // Copy items from unfinished segment.
    while (i <= mid) tmp[k++] = items[i++];
    while (j <= hi) tmp[k++] = items[j++];

    // Copy `tmp' back to main array.
    for (i = lo, k = 0; i <= hi; i++, k++) {
        items[i] = tmp[k];
    }

    free(tmp);
}
```

Merge Sort

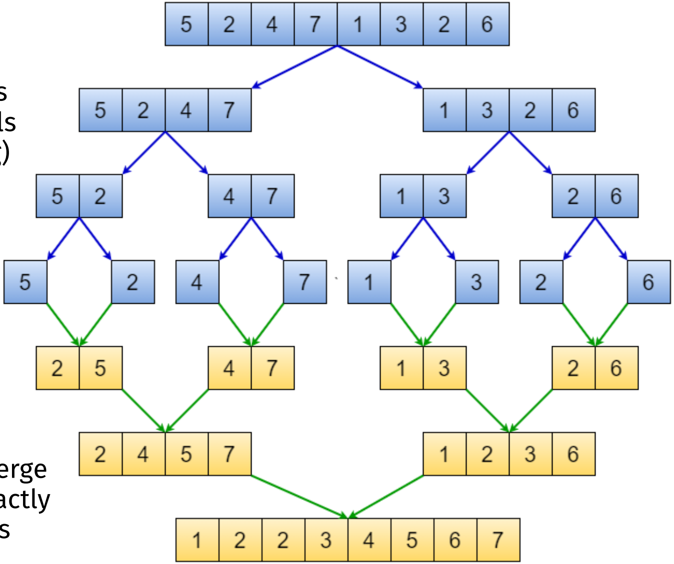
Analysis



Merge Sort
Analysis

Split

$n - 1$ splits
($\log_2 n$ levels of splitting)



$O(n)$

Merge

We have to merge n numbers exactly $\log_2 n$ times

$O(n \log n)$

Analysis:

- Merge sort splits the array into equal-sized partitions halving at each level $\Rightarrow \log_2 n$ levels
- The same operations happen at every recursive level
- Each 'level' requires $\leq n$ comparisons

Therefore:

- The time complexity of merge sort is $O(n \log n)$
  - Best-case, average-case, and worst-case time complexities are all the same

Note: Not required knowledge in COMP2521!

Let $T(n)$ be the time taken to sort n elements.

Splitting arrays into two halves takes constant time.
Merging two sorted arrays takes n steps.

So we have that:

$$T(n) = 2T(n/2) + n$$

Then the Master Theorem (see COMP3121) can be used to show that the time complexity is $O(n \log n)$.

Merge Sort

Properties

Stable

Due to taking from left subarray if items are equal during merge

Non-adaptive

$O(n \log n)$ best case, average case, worst case

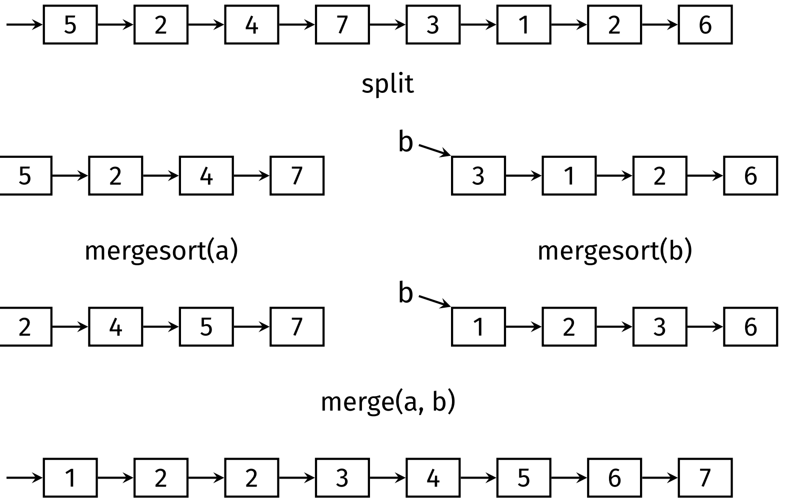
Not in-place

Merge uses a temporary array of size up to n

Note: Merge sort also uses $O(\log n)$ stack space

Merge Sort on Lists

It is possible to apply merge sort on linked lists.



Bottom-Up Merge Sort

An approach that works non-recursively!

- On each pass, our array contains sorted *runs* of length m.
- Initially, n sorted runs of length 1.
- The first pass merges adjacent elements into runs of length 2.
- The second pass merges adjacent elements into runs of length 4.
- Continue until we have a single sorted run of length n.

Can be used for *external* sorting;
e.g., sorting disk-file contents

Bottom-Up Merge Sort

Example

	[0]	[1]	[2]					.....								[15]
Original	A	S	O	R	T	I	N	G	E	X	E	M	P	L	A	R
After 1st pass *sorted slices of length 2*	A	S	O	R	I	T	G	N	E	X	E	M	L	P	A	R
After 2nd pass *sorted slices of length 4*	A	O	R	S	G	I	N	T	E	E	M	X	A	L	P	R
After 3rd pass *sorted slices of length 8*	A	G	I	N	O	R	S	T	A	E	E	L	M	P	R	X
After 4th pass *sorted slice of length 16*	A	A	E	E	G	I	L	M	N	O	P	R	R	S	T	X

Bottom-Up Merge Sort

C Implementation

```
void mergeSortBottomUp(Item items[], int lo, int hi) {
    for (int m = 1; m <= hi - lo; m *= 2) {
        for (int i = lo; i <= hi - m; i += 2 * m) {
            int end = min(i + 2 * m - 1, hi);
            merge(items, i, i + m - 1, end);
        }
    }
}
```

Quick Sort

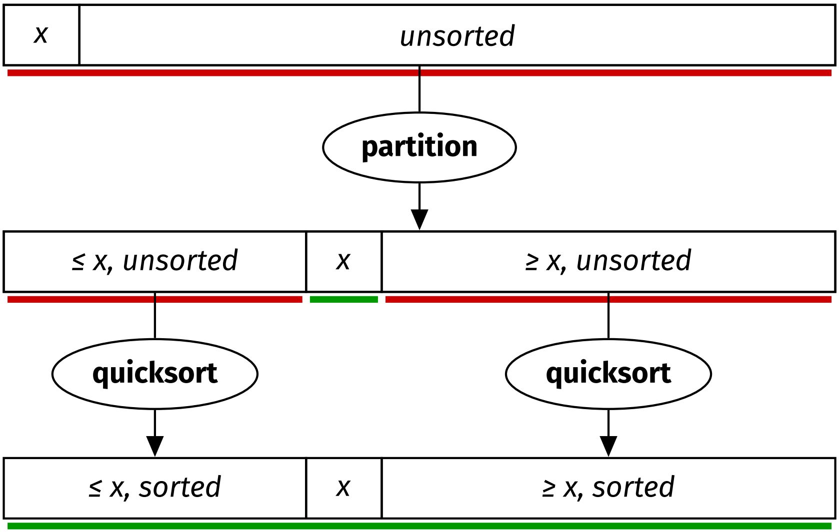
Quick Sort

Invented by Tony Hoare
in 1959

Method:

1. Choose an item to be a pivot
2. Rearrange (partition) the array so that
    - All elements to the left of the pivot are less than (or equal to) the pivot
    - All elements to the right of the pivot are greater than (or equal to) the pivot
3. Recursively sort each of the partitions

Quick Sort


x
unsorted
partition
≤ x, unsorted
x
≥ x, unsorted
quicksort
quicksort
≤ x, sorted
x
≥ x, sorted

Partitioning

How to partition an array?

- Assume the pivot is stored at index `lo`
- Create index `l` to start of array (`lo + 1`)
- Create index `r` to end of array (`hi`)
- Until `l` and `r` meet:
  - Increment `l` until `a[l]` is greater than pivot
  - Decrement `r` until `a[r]` is less than pivot
  - Swap items at indices `l` and `r`
- Swap the pivot with index `l` or `l - 1` (depending on the item at index `l`)

Partitioning

Example 1

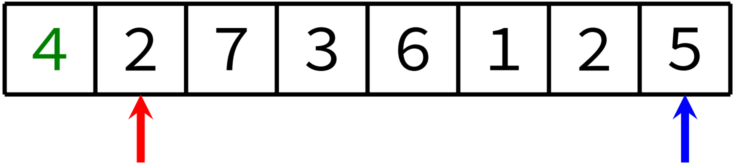
Pivot is 4

4	2	7	3	6	1	2	5

Partitioning

Example 1

Create left and right indices

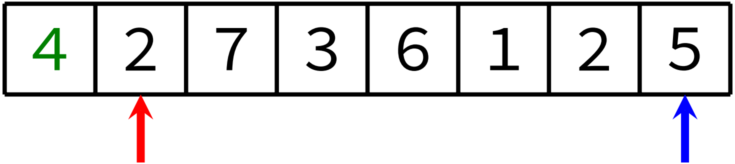


Partitioning

Example 1

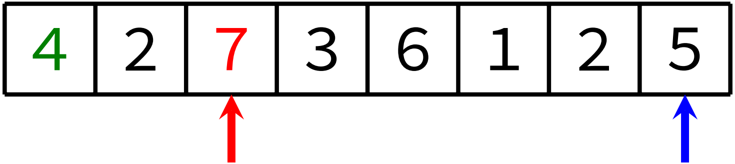
Until the indices meet:

Increment left index while element is $\leq$ pivot



Until the indices meet:

Increment left index while element is $\leq$ pivot

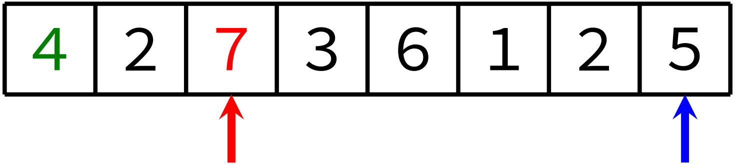


Partitioning

Example 1

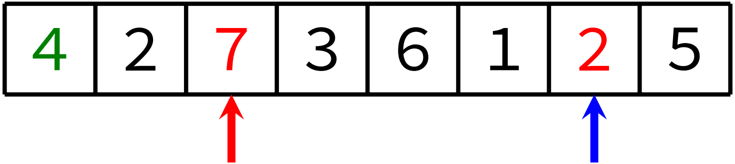
Until the indices meet:

Decrement right index while element is $\geq$ pivot



Until the indices meet:

Decrement right index while element is $\geq$ pivot

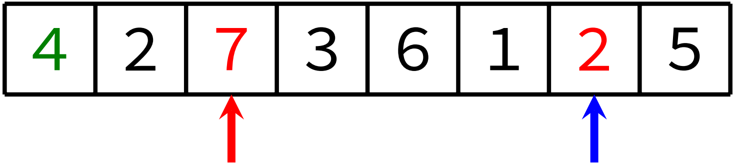


Partitioning

Example 1

Until the indices meet:

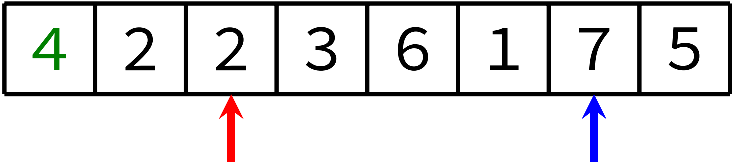
Swap the two elements



Partitioning
Example 1

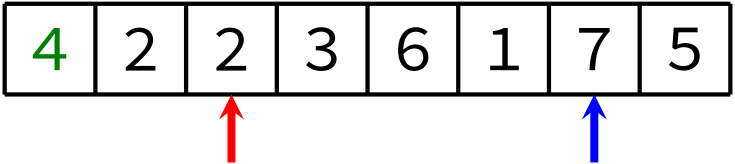
Until the indices meet:

Swap the two elements



Until the indices meet:

Increment left index while element is $\leq$ pivot



Partitioning

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Until the indices meet:

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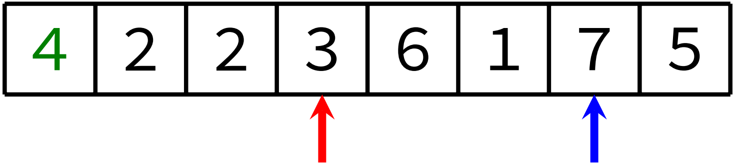

Partitioning

Example 1

Until the indices meet:

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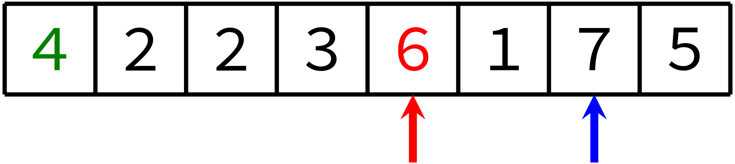
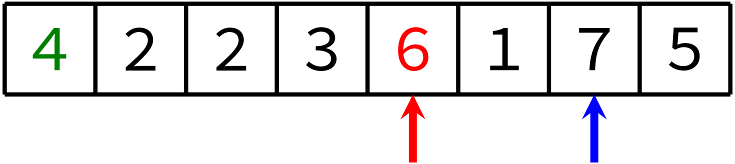


Partitioning

Example 1

Until the indices meet:

Decrement right index while element is $\geq$ pivot

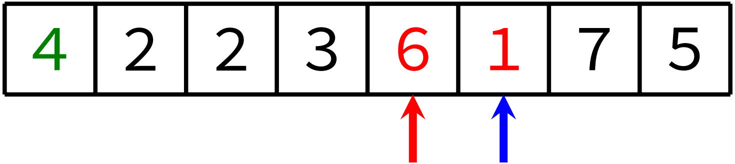


Partitioning

Example 1

Until the indices meet:

Decrement right index while element is $\geq$ pivot

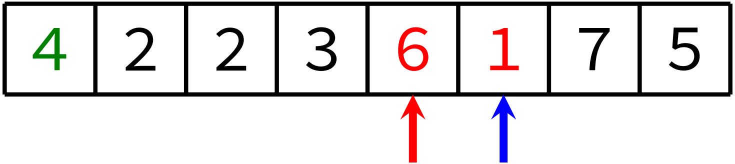


Partitioning

Example 1

Until the indices meet:

Swap the two elements



Partitioning
Example 1

Until the indices meet:

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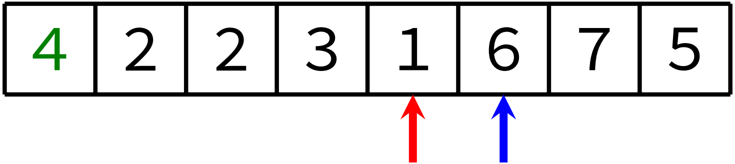

Partitioning

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Until the indices meet:

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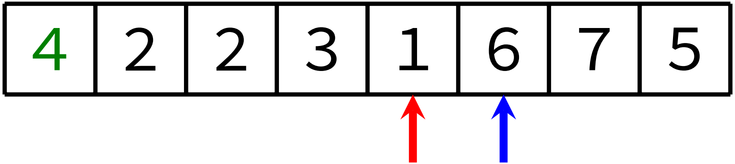

Partitioning

Example 1

Until the indices meet:

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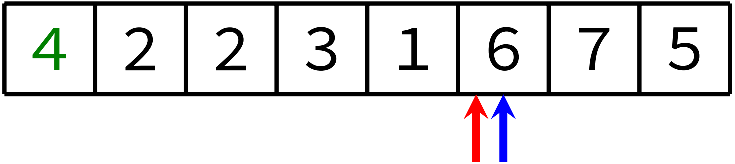
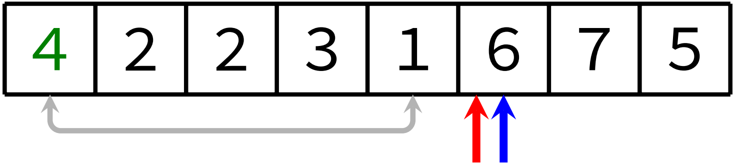


Partitioning

Example 1

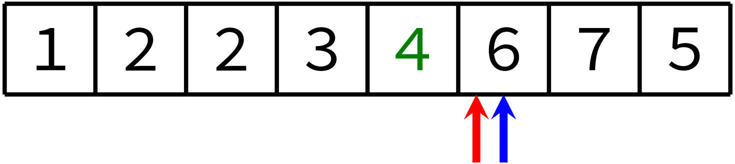
Swap the pivot into the middle (be careful!)



Partitioning

Example 1

Swap the pivot into the middle (be careful!)



Done

1	2	2	3	4	6	7	5

≤ 4 (under 1, 2, 2, 3) ≥ 4 (under 6, 7, 5)

Partitioning

Example 2

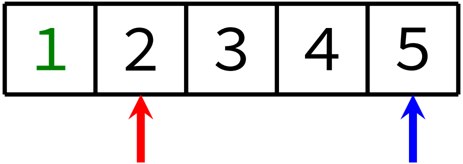
Pivot is 1

1	2	3	4	5

Partitioning

Example 2

Create left and right indices

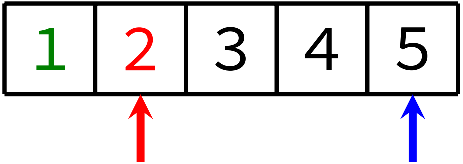


Partitioning

Example 2

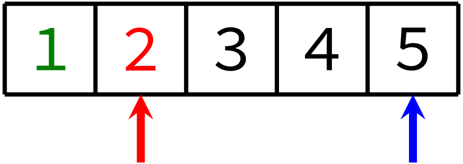
Until the indices meet:

Increment left index while element is $\leq$ pivot



Until the indices meet:

Decrement right index while element is $\geq$ pivot

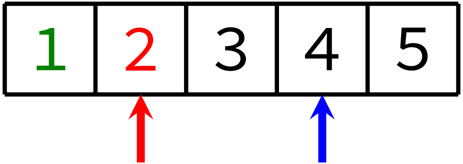


Partitioning

Example 2

Until the indices meet:

Decrement right index while element is $\geq$ pivot

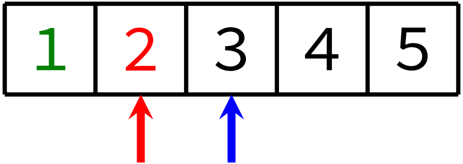


Partitioning

Example 2

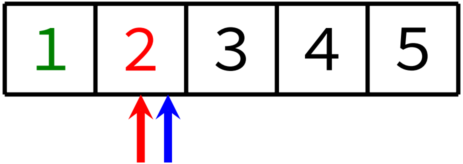
Until the indices meet:

Decrement right index while element is $\geq$ pivot



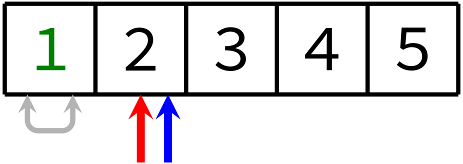
Until the indices meet:

Decrement right index while element is $\geq$ pivot



Partitioning
Example 2

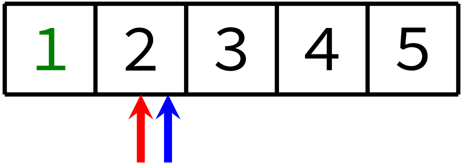
Swap the pivot into the middle (be careful!)



Partitioning

Example 2

Swap the pivot into the middle (be careful!)



Partitioning
Example 2

Done

1	2	3	4	5

≥ 1

Partitioning

Analysis

- Partitioning is $O(n)$, where n is the number of elements being partitioned
  - About n comparisons are performed, at most $\frac{n}{2}$ swaps are performed

Quick Sort

C Implementation: Sort

```
void naiveQuickSort(Item items[], int lo, int hi) {
    if (lo >= hi) return;
    int pivotIndex = partition(items, lo, hi);
    naiveQuickSort(items, lo, pivotIndex - 1);
    naiveQuickSort(items, pivotIndex + 1, hi);
}
```

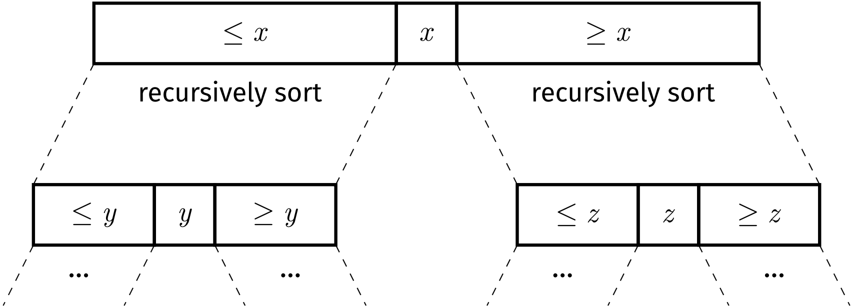
Quick Sort

C Implementation: Partition

```c
int partition(Item items[], int lo, int hi) {
    Item pivot = items[lo];

    int l = lo + 1;
    int r = hi;
    while (l < r) {
        while (l < r && le(items[l], pivot)) l++;
        while (l < r && ge(items[r], pivot)) r--;
        if (l == r) break;
        swap(items, l, r);
    }

    if (lt(pivot, items[l])) l--;
    swap(items, lo, l);
    return l;
}
```

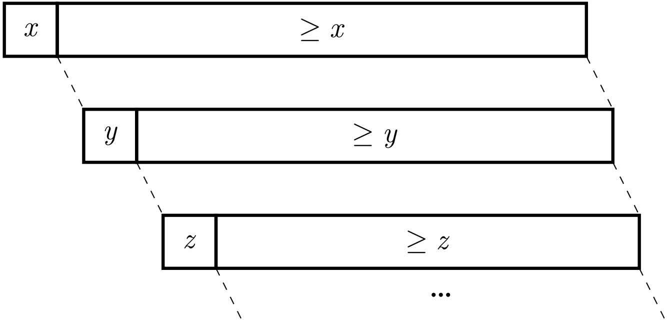
Best case: $O(n \log n)$

- Choice of pivot gives two equal-sized partitions
- Same happens at every recursive call
  - Resulting in $\log_2 n$ recursive levels
- Each "level" requires approximately n comparisons



Worst case: $O(n^2)$

- Always choose lowest/highest value for pivot
  - Resulting in partitions of size 0 and $n-1$
  - Resulting in n recursive levels
- Each "level" requires one less comparison than the level above



Average case: $O(n \log n)$

- If array is randomly ordered, chance of repeatedly choosing a bad pivot is very low
- Can also show empirically by generating random sequences and sorting them

Unstable
Due to long-range swaps

Non-adaptive
$O(n \log n)$ average case, sorted input does not improve this

In-place
Partitioning is done in-place
Stack depth is $O(n)$ worst-case, $O(\log n)$ average

Issues with Quick Sort

Choice of pivot can have a significant effect:

- Ideal pivot is the median value
- Always choosing largest/smallest $\Rightarrow$ worst case

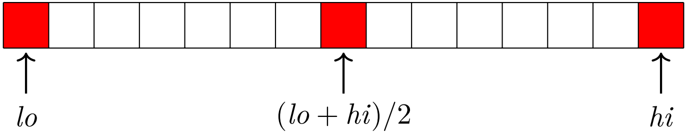
Therefore, always picking the first or last element as pivot is not a good idea:

- Existing order is a worst case
- Existing reverse order is a worst case
- Will result in partitions of size $n - 1$ and 0
- This pivot selection strategy is called naïve quick sort

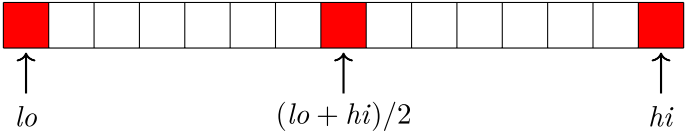
Quick Sort with Median-of-Three Partitioning

Take three values: left-most, middle, right-most.
Pick the median of these three values as our pivot.

Ordered data is no longer a worst-case scenario.
In general, doesn't eliminate the worst-case ...
... but makes it much less likely.



Quick Sort with Median-of-Three Partitioning

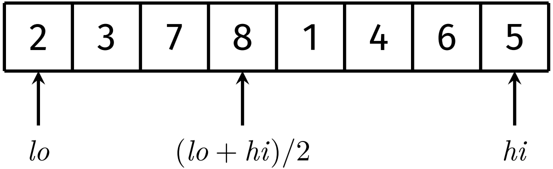


1. Sort a$[lo]$, a$[(lo+hi)/2]$, a$[hi]$, such that a$[(lo+hi)/2] \leq$ a$[lo] \leq$ a$[hi]$
2. Partition on a$[lo]$ to a$[hi]$

Quick Sort with Median-of-Three Partitioning

Example

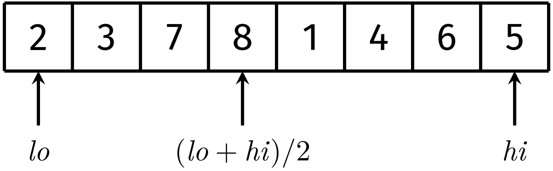
Which element is selected as the pivot?



Quick Sort with Median-of-Three Partitioning

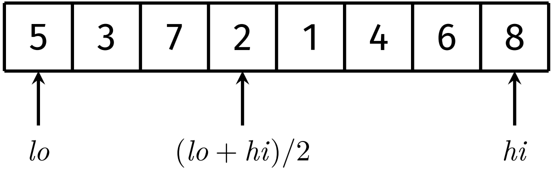
Example

Answer: 5



Quick Sort with Median-of-Three Partitioning

Example



Quick Sort with Median-of-Three Partitioning

C Implementation


```
void medianOfThreeQuickSort(Item items[], int lo, int hi) {
    if (lo >= hi) return;
    medianOfThree(items, lo, hi);
    int pivotIndex = partition(items, lo, hi);
    medianOfThreeQuickSort(items, lo, pivotIndex - 1);
    medianOfThreeQuickSort(items, pivotIndex + 1, hi);
}

void medianOfThree(Item a[], int lo, int hi) {
    int mid = (lo + hi) / 2;
    if (gt(a[mid], a[lo])) swap(a, mid, lo);
    if (gt(a[lo],  a[hi])) swap(a, lo,  hi);
    if (gt(a[mid], a[lo])) swap(a, mid, lo);
    // now, we have a[mid] <= a[lo] <= a[hi]
}
```

Quick Sort with Randomised Partitioning

Idea: Pick a random value for the pivot

This makes it *nearly* impossible to systematically generate inputs that would lead to $O(n^2)$ performance

Quick Sort with Randomised Partitioning

C Implementation

```
void randomisedQuickSort(Item items[], int lo, int hi) {
    if (lo >= hi) return;
    swap(items, lo, randint(lo, hi));
    int pivotIndex = partition(items, lo, hi);
    randomisedQuickSort(items, lo, pivotIndex - 1);
    randomisedQuickSort(items, pivotIndex + 1, hi);
}

int randint(int lo, int hi) {
    int i = rand() % (hi - lo + 1);
    return lo + i;
}
```

Note: `rand()` is a pseudo-random number generator provided by `<stdlib.h>`.
The generator should be initialised with `srand()`.

Insertion Sort Improvement

For small sequences (when $n < 5$, say),
quick sort is expensive
because of the recursion overhead.

Solution: Handle small partitions with insertion sort

Insertion Sort Improvement

C Implementation - Version 1


```
#define THRESHOLD 5

void quickSort(Item items[], int lo, int hi) {
    if (hi - lo < THRESHOLD) {
        insertionSort(items, lo, hi);
        return;
    }

    medianOfThree(items, lo, hi);
    int pivotIndex = partition(items, lo, hi);
    quickSort(items, lo, pivotIndex - 1);
    quickSort(items, pivotIndex + 1, hi);
}
```

Insertion Sort Improvement

C Implementation - Version 2

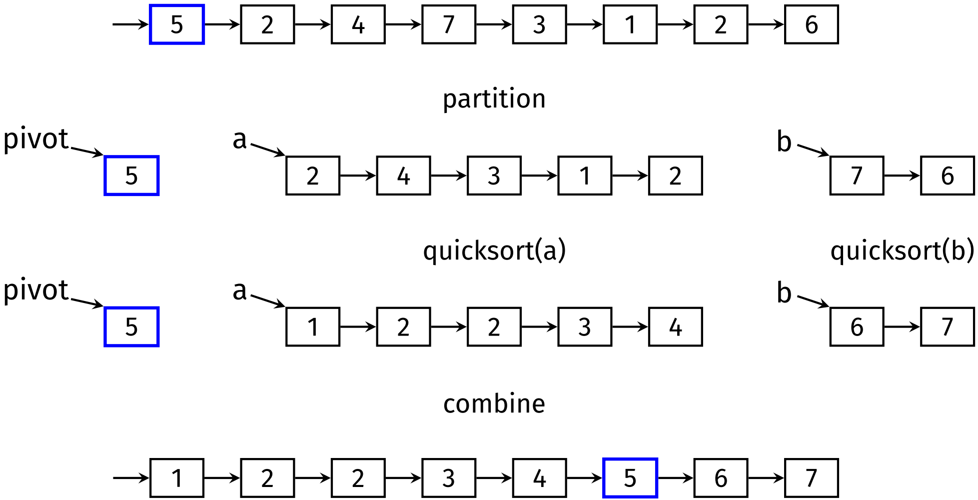

```c
#define THRESHOLD 5

void quickSort(Item items[], int lo, int hi) {
    doQuickSort(items, lo, hi);
    insertionSort(items, lo, hi);
}

void doQuickSort(Item items[], int lo, int hi) {
    if (hi - lo < THRESHOLD) return;

    medianOfThree(items, lo, hi);
    int pivotIndex = partition(items, lo, hi);
    doQuickSort(items, lo, pivotIndex - 1);
    doQuickSort(items, pivotIndex + 1, hi);
}
```

Quick Sort on Lists

It is possible to quick sort a linked list:

1. Pick first element as pivot
   - Alternatively, can use median-of-three or random pivot
2. Create two empty linked lists A and B
3. For each element in original list (excluding pivot):
   - If element is less than (or equal to) pivot, add it to A
   - If element is greater than pivot, add it to B
4. Recursively sort A and B
5. Form sorted linked list using sorted A, the pivot, and then sorted B

Quick Sort on Lists



Quick Sort vs Merge Sort

Design of modern CPUs mean,
for sorting arrays in RAM
quick sort *generally* outperforms merge sort.

Quick sort is more 'cache friendly':
good locality of access on arrays.

On the other hand, merge sort is
readily stable, readily parallel,
a good choice for sorting linked lists

Summary of Divide-and-Conquer Sorts

	Time complexity			Properties	
	Best	Average	Worst	Stable	Adaptive
Merge sort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	Yes	No
Quick sort	$O(n \log n)$	$O(n \log n)$	$O(n^2)$	No	No