COMP2521 24T3
Graphs (III)
Graph Problems

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cycle checking
connected components
hamiltonian paths/circuits
eulerian paths/circuits

Cycle Checking
Connected Components
Hamiltonian Path/Circuit
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Other Problems

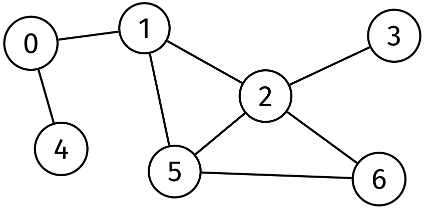
Graph Problems

Basic graph problems:

- Is there a cycle in the graph?
- How many connected components are there in the graph?
- Is there a path that passes through all vertices?
- Is there a path that passes through all edges?

Cycle Checking

A cycle is a path of length > 2
where the start vertex = end vertex
and no edge is used more than once



This graph has three distinct cycles:
1-2-5-1, 2-5-6-2, 1-2-6-5-1

(two cycles are distinct if they have different sets of edges)

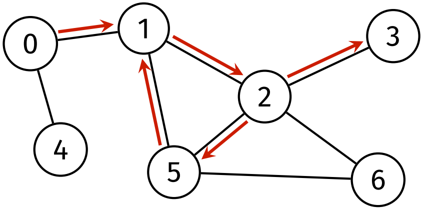
Cycle Checking

Attempt 1

How to check if a graph has a cycle?

Idea:

- Perform a DFS, starting from any vertex
- During the DFS, if the current vertex has an edge to an already-visited vertex, then there is a cycle



`tests/cycle1.txt`

Cycle Checking
Attempt 1
Attempt 2
Solution
Analysis
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Other Problems

Cycle Checking

Attempt 1

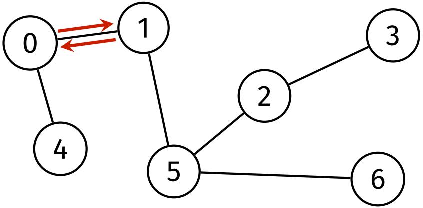
```
hasCycle(G):
    Input:  graph G
    Output: true if G has a cycle, false otherwise

    pick any vertex v in G
    create visited array, initialised to false
    return dfsHasCycle(G, v, visited)

dfsHasCycle(G, v, visited):
    visited[v] = true

    for each neighbour w of v in G:
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, visited):
            return true

    return false
```

Cycle Checking

Attempt 1

Problem:

- The algorithm does not check whether the neighbour w is the vertex that it just came from
- Therefore, it considers moving back and forth along a single edge to be a cycle (e.g., 0-1-0)



tests/cycle2.txt

Cycle Checking

Attempt 2

Improved idea:

- Perform a DFS, starting from any vertex
- **Keep track of previous vertex during DFS**
- During the DFS, if the current vertex has an edge to an already-visited vertex **which is not the previous vertex**, then there is a cycle

Cycle Checking

Attempt 2

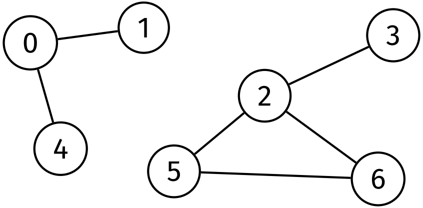
```
hasCycle(G):
    Input:  graph G
    Output: true if G has a cycle, false otherwise

    pick any vertex v in G
    create visited array, initialised to false
    return dfsHasCycle(G, v, v, visited)

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each neighbour w of v in G:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

Cycle Checking

Attempt 2

Problem:

- The algorithm only checks one connected component
  - The connected component that the initially chosen vertex belongs to



`tests/cycle3.txt`

Cycle Checking
Attempt 1
Attempt 2
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Connected Components

Hamiltonian Path/Circuit

Eulerian Path/Circuit

Other Problems

Working idea:

- Perform a DFS, starting from any vertex
- Keep track of previous vertex during DFS
- During the DFS, if the current vertex has an edge to an already-visited vertex which is not the previous vertex, then there is a cycle
- **After the DFS, if any vertex has not yet been visited, perform another DFS, this time starting from that vertex**
- **Repeat until all vertices have been visited**

Cycle Checking
Attempt 1
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Eulerian Path/Circuit

Other Problems

```
hasCycle(G):
    Input:  graph G
    Output: true if G has a cycle, false otherwise

    create visited array, initialised to false

    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, v, visited):
                return true

    return false

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each neighbour w of v in G:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

Cycle Checking
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Analysis for adjacency list representation:

- Algorithm is a slight modification of DFS
- A full DFS traversal is $O(V + E)$
- Thus, worst-case time complexity of cycle checking is $O(V + E)$

Cycle Checking
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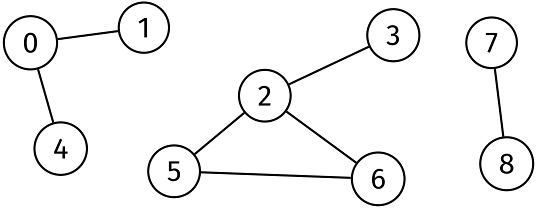
Connected Components

A connected component
is a maximally connected subgraph

For example, this graph has three connected components:



Connected Components

DEFINITIONS:

subgraph
a subset of vertices and edges of original graph

connected subgraph
there is a path between every pair of vertices in the subgraph

maximally connected subgraph
no way to include more edges/vertices from original graph into the subgraph
such that subgraph is still connected

Connected Components

Problems:

How many connected components are there?

Are two vertices in the same connected component?

Cycle Checking
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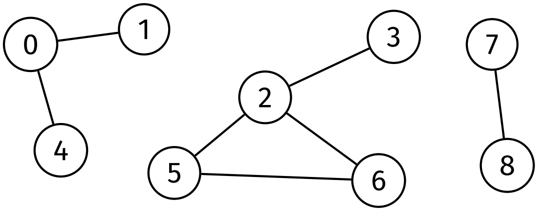
Connected Components

Goal:

- Compute an array which indicates which connected component each vertex is in
  - Let this array be called `componentOf`
  - `componentOf[`v`]` contains the component number of vertex v
- For example:



[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
0	0	1	1	0	1	1	2	2

Connected Components

Idea:

- Choose a vertex and perform a DFS starting at that vertex
  - During the DFS, assign all vertices visited to component 0
- After the DFS, if any vertex has not been assigned a component, perform a DFS starting at that vertex
  - During this DFS, assign all vertices visited to component 1
- Repeat until all vertices are assigned a component, increasing the component number each time

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```
components(G):
    Input:  graph G
    Output: componentOf array

    create componentOf array, initialised to -1

    compNo = 0
    for each vertex v in G:
        if componentOf[v] = -1:
            dfsComponents(G, v, componentOf, compNo)
            compNo = compNo + 1

    return componentOf

dfsComponents(G, v, componentOf, compNo):
    componentOf[v] = compNo
    for each neighbour w of v in G:
        if componentOf[w] = -1:
            dfsComponents(G, w, componentOf, compNo)
```

Cycle Checking
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Analysis for adjacency list representation:

- Algorithm performs a full DFS, which is $O(V + E)$

Connected Components

Suppose we frequently need to answer the following questions:

- How many connected components are there?
- Are v and w in the same connected component?
- Is there a path between v and w?

Note: The last two questions are actually equivalent in an undirected graph.

Connected Components

Solution:

- Cache the components array in the graph struct

```
struct graph {
    ...
    int nC;  // number of connected components
    int *cc; // componentOf array
};
```

Connected Components

This allows us to answer the questions very easily:

```
// How many connected components are there?
int numComponents(Graph g) {
    return g->nC;
}

// Are v and w in the same connected component?
bool inSameComponent(Graph g, Vertex v, Vertex w) {
    return g->cc[v] == g->cc[w];
}

// Is there a path between v and w?
bool hasPath(Graph g, Vertex v, Vertex w) {
    return g->cc[v] == g->cc[w];
}
```

Connected Components

However, this information needs to be maintained as the graph changes:

- Inserting an edge may reduce nC
  - If the endpoint vertices were in different components
- Removing an edge may increase nC
  - If the endpoint vertices were in the same component *and* there is no other path between them

Cycle Checking
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Hamiltonian Path and Circuit

A **Hamiltonian path** is
a path that includes each vertex exactly once

A **Hamiltonian circuit (or Hamiltonian cycle)** is
a cycle that includes each vertex exactly once
(a cycle that visits each vertex of a graph exactly once and returns to the starting vertex)

Cycle Checking
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Hamiltonian Path and Circuit

Graph needs to be connected (as expected)
If multiple components, then it's not possible

Hamiltonian Path and Circuit

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Named after
Irish mathematician, astronomer and physicist
Sir William Rowan Hamilton (1805-1865)

Cycle Checking
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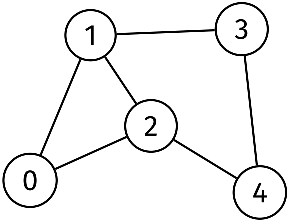
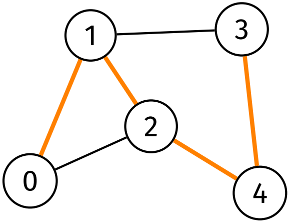
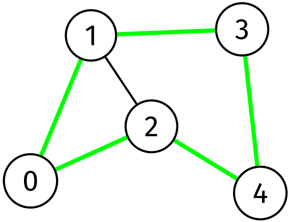
Hamiltonian Path and Circuit

Consider the following graph:





Hamiltonian path



Hamiltonian circuit

Hamiltonian Path

How to check if a graph has a Hamiltonian path?

Idea:

- Brute force
- Use DFS to check all possible paths
  - Recursive DFS is perfect, as it naturally allows backtracking
- Keep track of the number of vertices left to visit
- Stop when this number reaches 0

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Hamiltonian Path

```
hasHamiltonianPath(G):
    Input:  graph G
    Output: true if G has a Hamiltonian path
            false otherwise

    create visited array, initialised to false
    for each vertex v in G:
        if dfsHamiltonianPath(G, v, visited, #vertices(G)):
            return true

    return false
```

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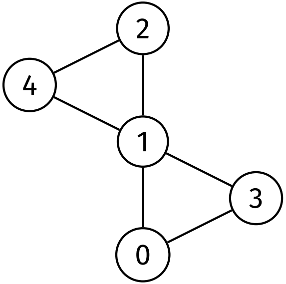
```
dfsHamiltonianPath(G, v, visited, numVerticesLeft):
    visited[v] = true
    numVerticesLeft = numVerticesLeft - 1

    if numVerticesLeft = 0:
        return true

    for each neighbour w of v in G:
        if visited[w] = false:
            if dfsHamiltonianPath(G, w, visited, numVerticesLeft):
                return true

    visited[v] = false
    return false
```

Hamiltonian Path

Why set `visited[`v`]` to false at the end of `dfsHamiltonianPath`?



allow the algorithm to explore other potential paths

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Hamiltonian Circuit

How to check if a graph has a Hamiltonian *circuit*?

- Similar approach as Hamiltonian path
- Don't need to try all starting vertices
- After a Hamiltonian path is found, check if the final vertex is adjacent to the starting vertex

Cycle Checking
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Hamiltonian Circuit

```
hasHamiltonianCircuit(G):
    Input:  graph G
    Output: true if G has a Hamiltonian circuit
            false otherwise

    if #vertices(G) < 3:
        return false

    create visited array, initialised to false
    return dfsHamiltonianCircuit(G, 0, visited, #vertices(G))

dfsHamiltonianCircuit(G, v, visited, numVerticesLeft):
    visited[v] = true
    numVerticesLeft = numVerticesLeft - 1

    if numVerticesLeft = 0 and adjacent(G, v, 0):
        return true

    for each neighbour w of v in G:
        if visited[w] = false:
            if dfsHamiltonianCircuit(G, w, visited, numVerticesLeft):
                return true

    visited[v] = false
    return false
```

Cycle Checking
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Analysis:

- Worst-case time complexity: $O(V!)$
- There are at most $V!$ paths to check (all possible permutations of the V vertices)
- There is no known polynomial time algorithm, so the Hamiltonian path problem is NP-hard (Non-deterministic Polynomial-time Hard)
- no easy way

Cycle Checking
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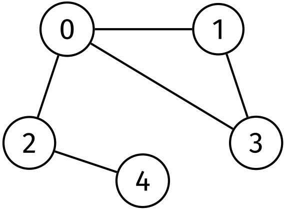
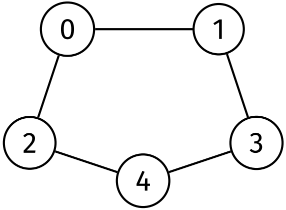
Eulerian Path and Circuit

An **Eulerian path** is
a path that visits each edge exactly once

An **Eulerian circuit** is
an Eulerian path that starts and ends at the same vertex



Eulerian path:
4-2-0-1-3-0



Eulerian circuit:
4-2-0-1-3-4

Eulerian Path and Circuit

Cycle Checking
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Other Problems

One-stroke problem: whether it is possible to draw a graph or figure in a single stroke without lifting the pen and without retracing any edge.

Cycle Checking
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Other Problems

Problem is named after
Swiss mathematician, physicist, astronomer, logician and engineer
Leonhard Euler (1707-1783)

Cycle Checking
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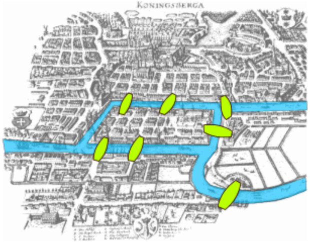
Eulerian Path and Circuit

Background

Problem was introduced by Euler while trying to solve the Seven Bridges of Konigsberg problem in 1736.



Is there a way to cross all the bridges exactly once on a walk through the town?

Cycle Checking
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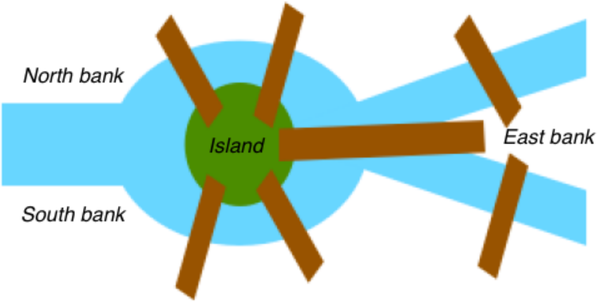
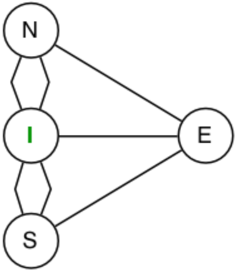
Other Problems

This is a graph problem:
vertices represent pieces of land
edges represent bridges



Bridges as schematic



Bridges as graph

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Eulerian Path and Circuit

How to check if a graph has an Eulerian path or circuit?

Can use the following theorems:

A graph has an Eulerian path if and only if
exactly zero or two vertices have odd degree,
and all vertices with non-zero degree belong to the same connected component

A graph has an Eulerian circuit if and only if
every vertex has even degree,
and all vertices with non-zero degree belong to the same connected component

Cycle Checking
Connected Components
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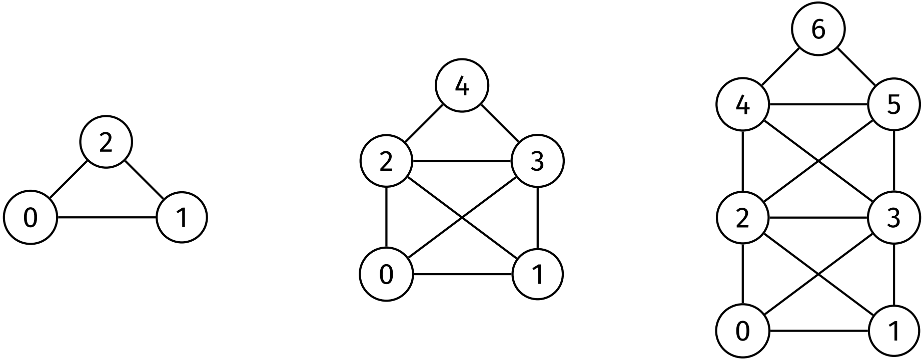
Which of these graphs have an Eulerian path? How about an Eulerian circuit?



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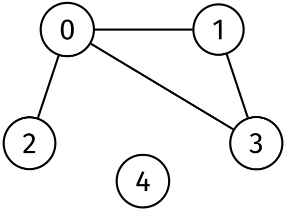
Eulerian Path and Circuit

Why
"all vertices with non-zero degree belong to the same connected component"?



Eulerian Path

```
hasEulerianPath(G):
    Input:   graph G
    Output:  true if G has an Eulerian path
             false otherwise

    numOddDegree = 0
    for each vertex v in G:
        if degree(G, v) is odd:
            numOddDegree = numOddDegree + 1

    return (numOddDegree = 0 or numOddDegree = 2) and
           eulerConnected(G)
```

Eulerian Path

```
eulerConnected(G):
    Input:  graph G
    Output: true if all vertices in G with non-zero degree
            belong to the same connected component
            false otherwise

    create visited array, initialised to false

    for each vertex v in G:
        if degree(G, v) > 0:
            dfsRec(G, v, visited)
            break

    for each vertex v in G:
        if degree(G, v) > 0 and visited[v] = false:
            return false

    return true
```

Eulerian Circuit

Cycle Checking
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Other Problems

```
hasEulerianCircuit(G):
    Input:  graph G
    Output: true if G has an Eulerian circuit
            false otherwise

    for each vertex v in G:
        if degree(G, v) is odd:
            return false

    return eulerConnected(G)
```

Cycle Checking
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Analysis for adjacency list representation:

- Finding degree of every vertex is $O(V + E)$
- Checking connectivity requires a DFS which is $O(V + E)$
- Therefore, worst-case time complexity is $O(V + E)$

So unlike the Hamiltonian path problem, the Eulerian path problem can be solved in polynomial time.

Cycle Checking
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Other Problems

Many graph problems are intractable – that is,
there is no known “efficient” algorithm to solve them.

In this context, “efficient” usually means polynomial time.

A tractable problem is one that is known to have a polynomial-time solution.

Other Graph Problems

Tractable and Intractable

tractable

what is the shortest path between two vertices?

intractable

how about the *longest* path?

Other Graph Problems

Tractable and Intractable

tractable	intractable
what is the shortest path between two vertices?	how about the *longest* path?
does a graph contain a clique?	what is the *largest* clique?

Other Graph Problems

Tractable and Intractable

tractable	intractable
what is the shortest path between two vertices?	how about the *longest* path?
does a graph contain a clique?	what is the *largest* clique?
given two colors, is it possible to colour every vertex in a graph such that no two adjacent vertices are the same colour?	what about *three* colours?

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Other Graph Problems

Tractable and Intractable

tractable	intractable
what is the shortest path between two vertices?	how about the *longest* path?
does a graph contain a clique?	what is the *largest* clique?
given two colors, is it possible to colour every vertex in a graph such that no two adjacent vertices are the same colour?	what about *three* colours?
does a graph contain an Eulerian path?	how about a Hamiltonian path?

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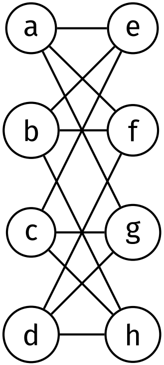
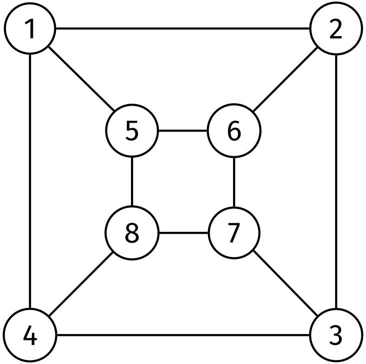
Other Graph Problems

Bonus Round!





Graph isomorphism:
Can we make two given graphs identical by renaming vertices?

Feedback

https://forms.office.com/r/zEqxUXvmLR