COMP2521 24T3
AVL Trees

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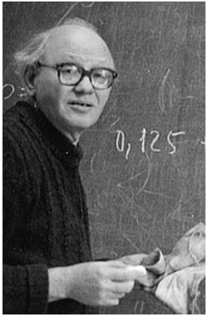
Invented by Georgy Adelson-Velsky and Evgenii Landis in 1962



AVL Trees

Approach:

- Keep tree height-balanced
- Repair balance as soon as imbalance occurs
  - During insertion or deletion
- Repairs are done locally, not by restructuring entire tree

Height of an AVL tree

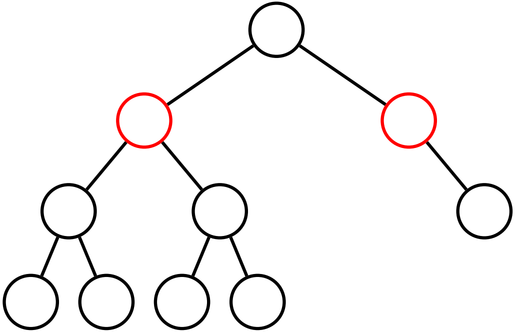
Since AVL trees are always height-balanced,
the height of an AVL tree is guaranteed to be at most
$\log_\phi (n+2) - 0.3277$ (where ϕ is the golden ratio)
$\approx 1.4404 \log_2(n+2) - 0.3277 = O(\log n)$

If you are interested in this:

- `https://people.csail.mit.edu/alinush/6.006-spring-2014/avl-height-proof.pdf`
- `https://www.youtube.com/watch?v=UJqkhRzBR9E`

One More Note:

AVL trees are not necessarily size-balanced.
For example, the following is a perfectly valid AVL tree:



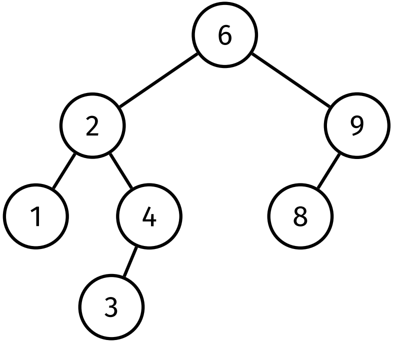
AVL Tree Insertion

Method:

- Insert item recursively
- Check balance at each node along the insertion path *in reverse*
  - i.e., from bottom to top
- Fix imbalances as they are found

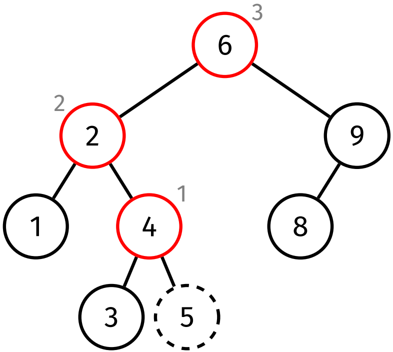
AVL Tree Insertion

Example: Insert 5 into this tree



AVL Tree Insertion

Example: Insert 5 into this tree



Balance must be checked at 4, then at 2, then at 6

AVL Tree Insertion

How to check balance along insertion path *in reverse*?

- Perform balance checking as a *postorder* operation in the insertion function
  - In other words - add balance checking code *below* recursive calls
  - Insert operation after recursion: perform operation during unwinding
  - In this case, the balance checking will be done from bottom to top (postorder)

AVL Tree Insertion

Outline of insertion process:

1. if the tree is empty:
   - return new node
2. insert recursively
3. check (and fix) balance
4. return root of updated tree

AVL Tree Insertion

Rebalancing

There are 4 rebalancing cases:

Left Left

Left Right

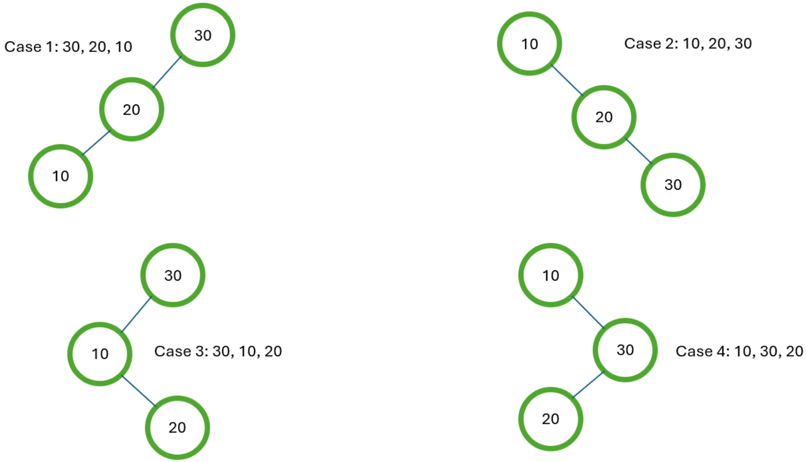
Right Left

Right Right

Examples

Considering three items: 10, 20, 30

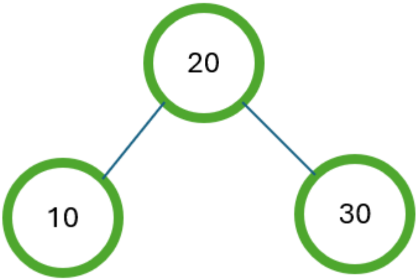
There are 6 insertion orders.



Examples

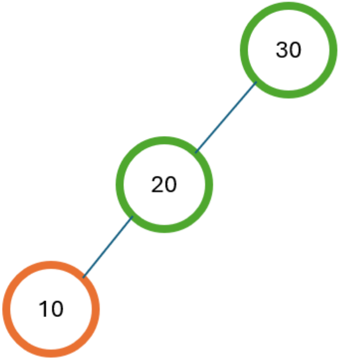
The rest 2 cases (balanced)

Best Case: 20, 30, 10 (20, 10, 30)



Examples
LL-imbalance

Case 1: 30, 20, 10

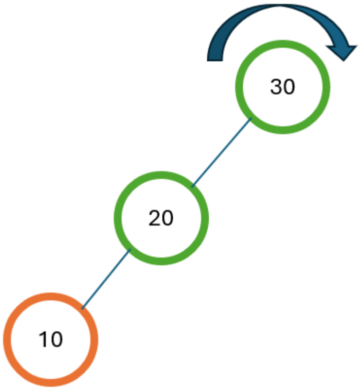


LL-imbalance

Examples

LL-rotation

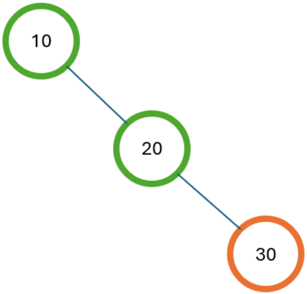
Case 1: 30, 20, 10



LL-rotation

Examples
RR-imbalance

Case 2: 10, 20, 30

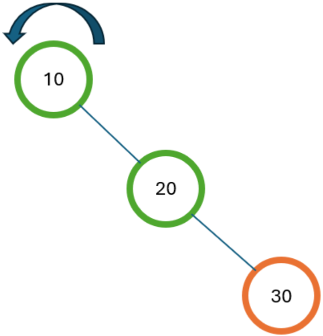


RR-imbalance

Examples

RR-rotation

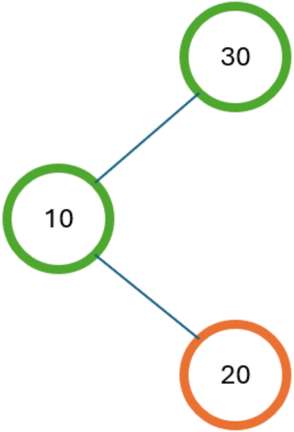
Case 2: 10, 20, 30



RR-rotation

Examples
LR-imbalance

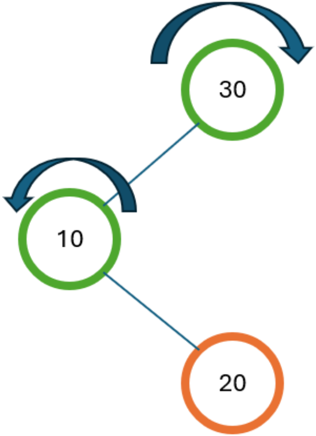
Case 3: 30, 10, 20



LR-imbalance

Examples
LR-rotation

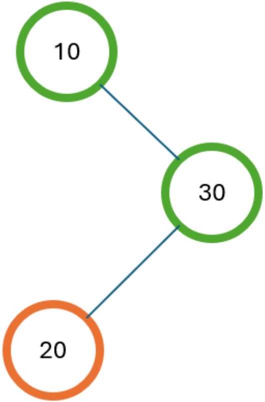
Case 3: 30, 10, 20



LR-rotation

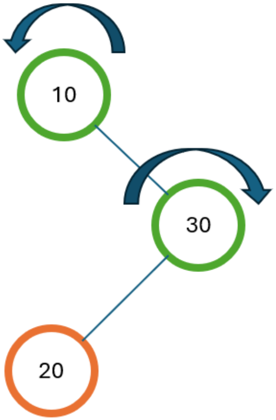
Examples
RL-imbalance

Case 4: 10, 30, 20



RL-imbalance

Case 4: 10, 30, 20



RL-rotation

```
avlInsert(t, v):
    Input:  AVL tree t, item v
    Output: t with v inserted

    if t is empty:
        return new node containing v
    else if v < t->item:
        t->left = avlInsert(t->left, v)
    else if v > t->item:
        t->right = avlInsert(t->right, v)
    else:
        return t

    return avlRebalance(t)
```

AVL Tree Insertion

Pseudocode

```
avlRebalance(t):
    Input:  possibly unbalanced tree t
    Output: balanced t

    bal = balance(t)
    if bal > 1:
        if balance(t->left) < 0:
            t->left = rotateLeft(t->left)//case 3
        t = rotateRight(t)
    else if bal < -1:
        if balance(t->right) > 0:
            t->right = rotateRight(t->right)//case 4
        t = rotateLeft(t)

    return t

balance(t):
    Input:  tree t
    Output: balance factor of t

    return height(t->left) - height(t->right)
```

AVL Tree Insertion

Rebalancing

There are 4 rebalancing cases:

Left Left

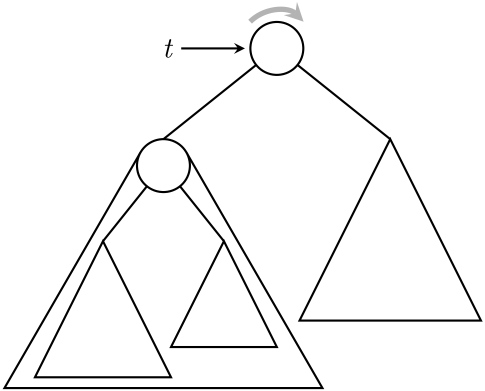
Left Right

Right Left

Right Right

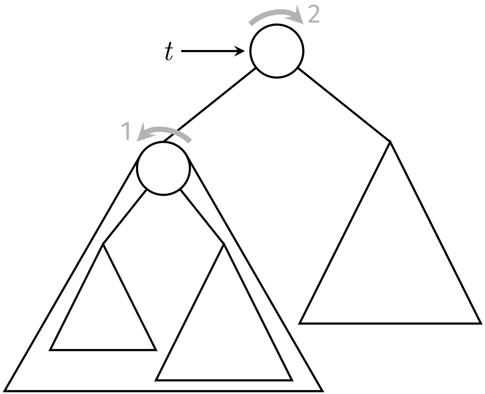
Left Left

```
bal = balance(t)
if bal > 1: (true)
    if balance(t->left) < 0: (false)
        t->left = rotateLeft(t->left)
    t = rotateRight(t)
else if bal < -1:
    if balance(t->right) > 0:
        t->right = rotateRight(t->right)
    t = rotateLeft(t)
```



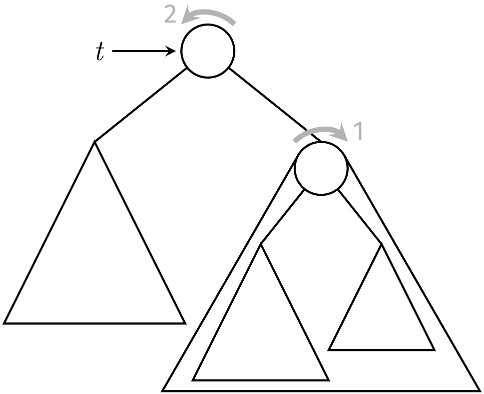
Left Right

```
bal = balance(t)
if bal > 1: (true)
    if balance(t->left) < 0: (true)
        t->left = rotateLeft(t->left)
    t = rotateRight(t)
else if bal < -1:
    if balance(t->right) > 0:
        t->right = rotateRight(t->right)
    t = rotateLeft(t)
```



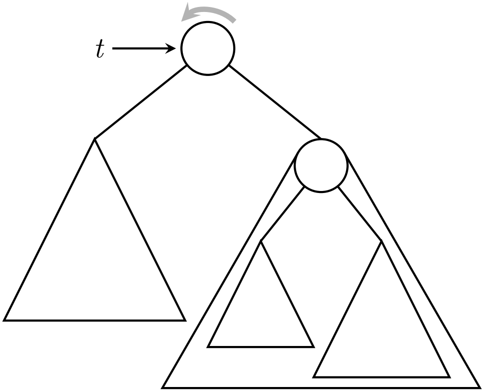
Right Left

```
bal = balance(t)
if bal > 1: (false)
    if balance(t->left) < 0:
        t->left = rotateLeft(t->left)
    t = rotateRight(t)
else if bal < -1: (true)
    if balance(t->right) > 0: (true)
        t->right = rotateRight(t->right)
    t = rotateLeft(t)
```



Right Right

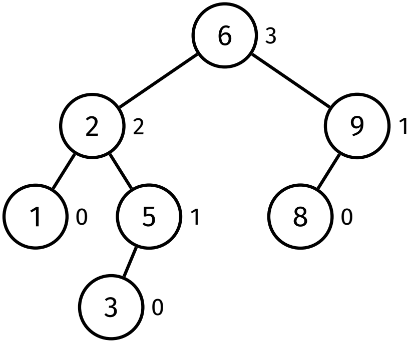
```
bal = balance(t)
if bal > 1: (false)
    if balance(t->left) < 0:
        t->left = rotateLeft(t->left)
    t = rotateRight(t)
else if bal < -1: (true)
    if balance(t->right) > 0: (false)
        t->right = rotateRight(t->right)
    t = rotateLeft(t)
```



AVL Tree Insertion

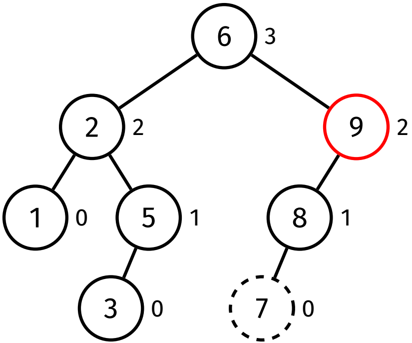
Rebalancing Example 1 - Left Left

Insert 7 into this tree:



AVL Tree Insertion

Rebalancing Example 1 - Left Left

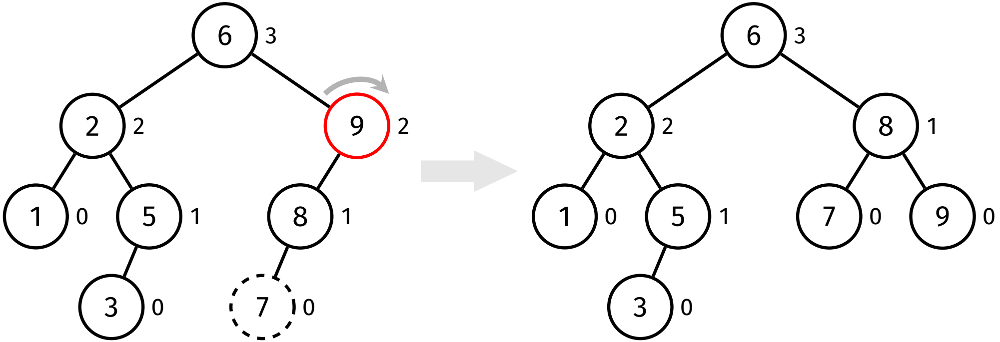


Check for balance at 8, then at 9, then at 6.

9 is unbalanced.

AVL Tree Insertion

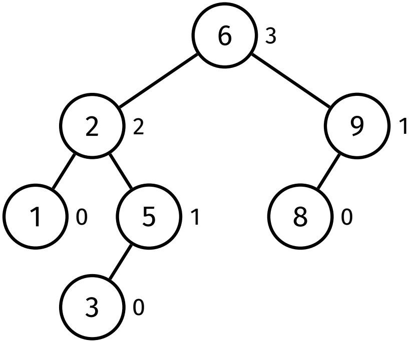
Rebalancing Example 1 - Left Left



AVL Tree Insertion

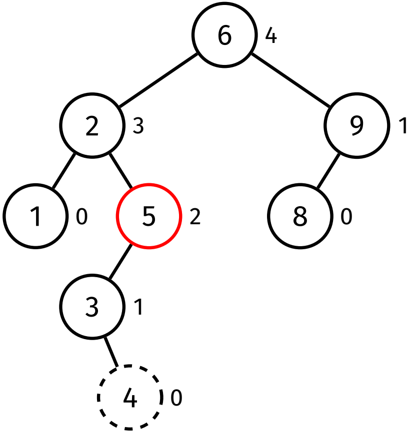
Rebalancing Example 2 - Left Right

Insert 4 into this tree:



AVL Tree Insertion

Rebalancing Example 2 - Left Right

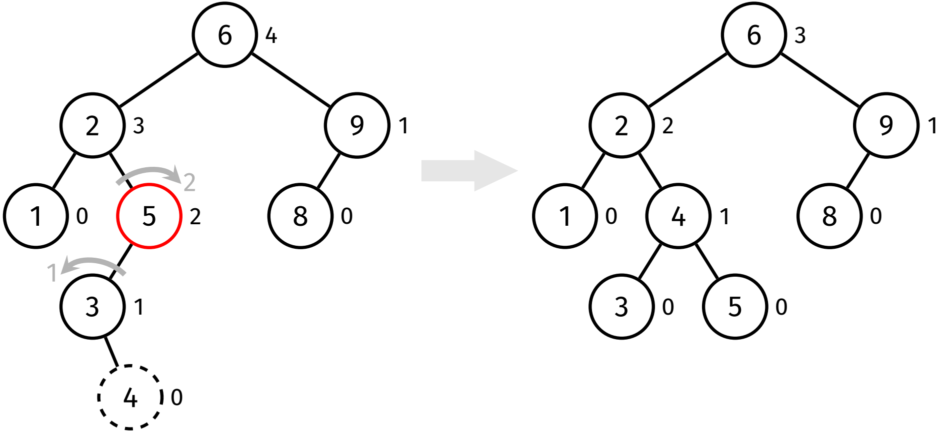


Check for balance at 3, then at 5, then at 2, then at 6.

5 is unbalanced.

AVL Tree Insertion

Rebalancing Example 2 - Left Right



AVL tree insertion requires balance checking
at each node on the insertion path...

...which requires the height of many subtrees to be computed

In an ordinary binary search tree, computing the height is $O(n)$!
(need to traverse whole (sub)tree)

AVL Tree Insertion

Storing Height Data

Solution:

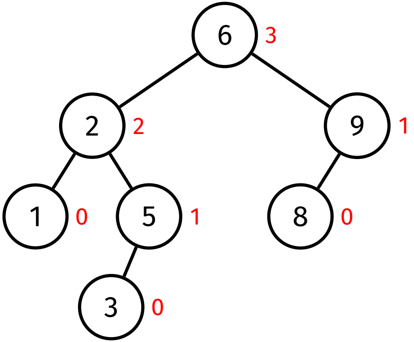
For each node, store the height of its subtree in the node itself:

```
struct node {
    int item;
    struct node *left;
    struct node *right;
    int height;
};
```

AVL Tree Insertion

Storing Height Data

Height of each node's subtree is stored in the node itself

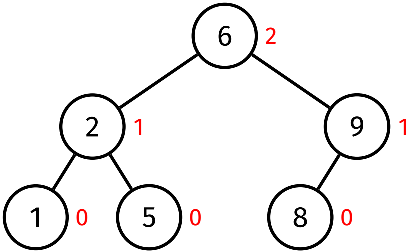


When does height data need to be maintained?

- Whenever a node is inserted
    - Heights of all ancestors may be affected
- Whenever a rotation is performed
    - Heights of original root and new root may be affected

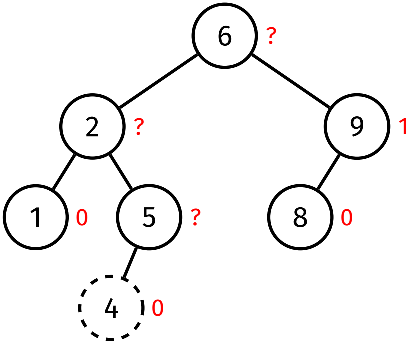
Whenever a node is inserted...
...heights of all ancestors may be affected

Example: Insert 4 into this tree

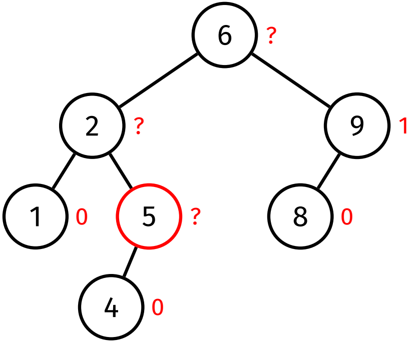


AVL Tree Insertion

Maintaining Height Data - Insertions



Recompute height of each ancestor (from bottom to top) using the heights stored in its children.

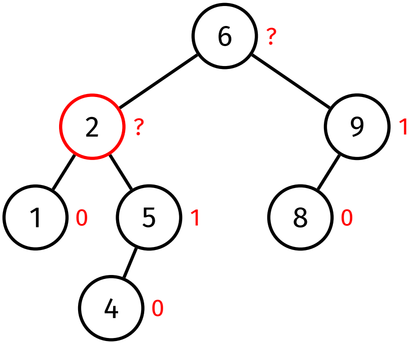


The heights of 5's children are 0 and -1 (empty tree).

Thus, the height of 5 is $\max(0, -1) + 1 = 1$.

AVL Tree Insertion

Maintaining Height Data - Insertions

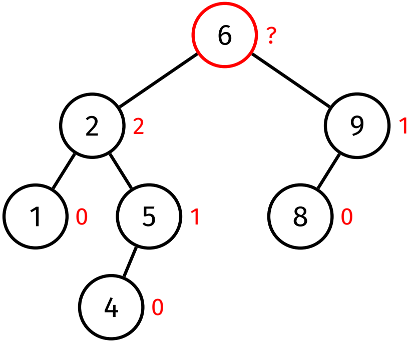


The heights of 2's children are 0 and 1.

Thus, the height of 2 is $\max(0, 1) + 1 = 2$.

AVL Tree Insertion

Maintaining Height Data - Insertions

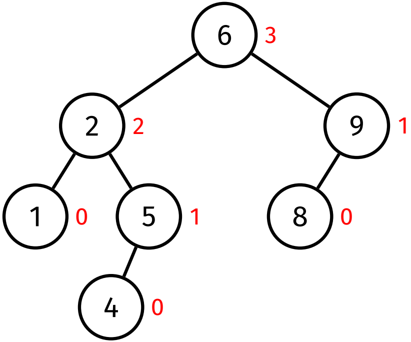


The heights of 6's children are 2 and 1.

Thus, the height of 6 is $\max(2, 1) + 1 = 3$.

AVL Tree Insertion

Maintaining Height Data - Insertions



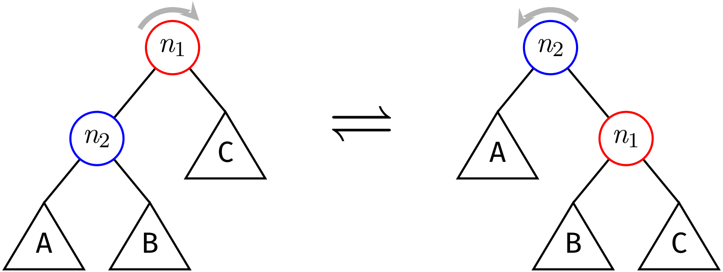
Done.

Note that recomputing the height of each node was done in $O(1)$ time.

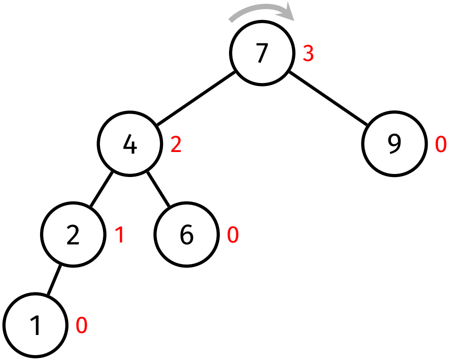
AVL Tree Insertion

Maintaining Height Data - Rotations

Whenever a rotation is performed...
...heights of original root and new root may be affected

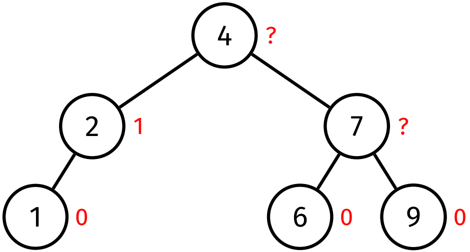


Example: Perform a right rotation at 7



AVL Tree Insertion

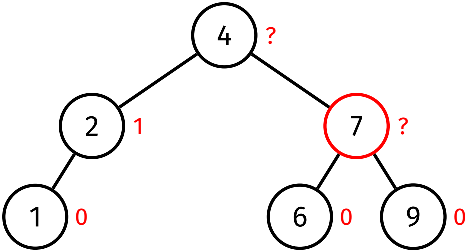
Maintaining Height Data - Rotations



Recompute height of original root
then recompute height of new root
using the heights stored in their children.

AVL Tree Insertion

Maintaining Height Data - Rotations

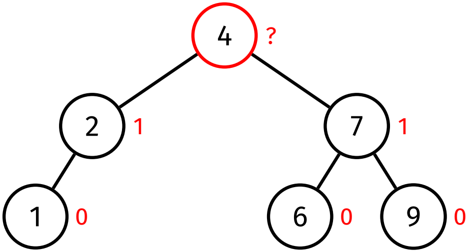


The height of 7's children are 0 and 0.

Thus, the height of 7 is $\max(0, 0) + 1 = 1$.

AVL Tree Insertion

Maintaining Height Data - Rotations

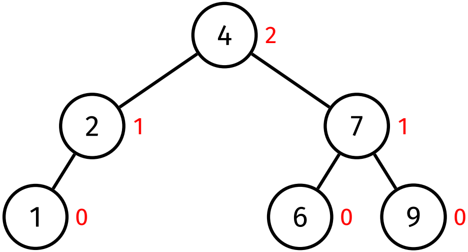


The height of 4's children are 1 and 1.

Thus, the height of 4 is $\max(1, 1) + 1 = 2$.

AVL Tree Insertion

Maintaining Height Data - Rotations



Done.

Every rotation, two height updates are performed, each in $O(1)$ time.

AVL Tree Insertion

Analysis

Analysis:

- Height of an AVL tree is $O(\log n)$
- In the worst case, length of insertion path is $O(\log n)$
- Have to maintain height data and check/fix balance at each node on insertion path
  - This is $O(1)$ per node
- Therefore, worst-case time complexity of AVL tree insertion is $O(\log n)$

AVL Tree Search

Exactly the same as for regular BSTs.

Worst-case time complexity is $O(\log n)$,
since AVL trees are height-balanced.

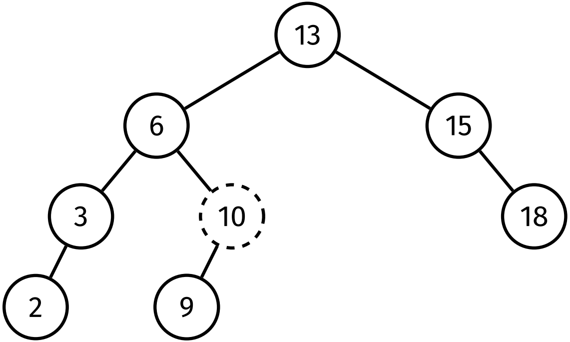
AVL Tree Deletion

Method:

- Delete item recursively
- Check balance at each node along the deletion path* in reverse
- Fix imbalances as they are found

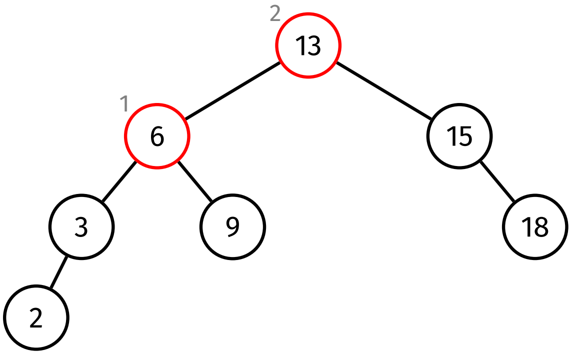
AVL Tree Deletion

Example: Delete 10 from this tree



AVL Tree Deletion

Example: Delete 10 from this tree



Balance must be checked at 6, then at 13

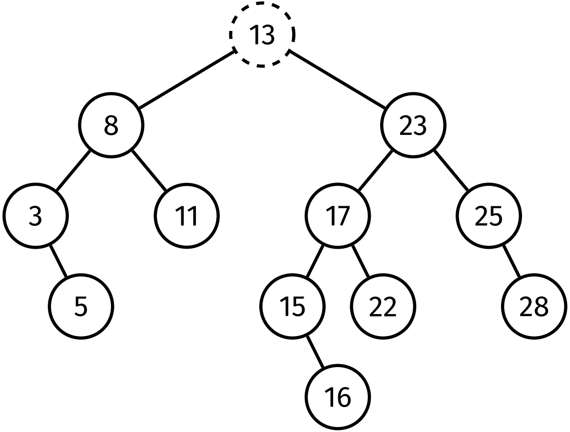
AVL Tree Deletion

Important:

If the item being deleted has two child nodes,
the deletion path includes the path to its successor
(the smallest value in its right subtree)

AVL Tree Deletion

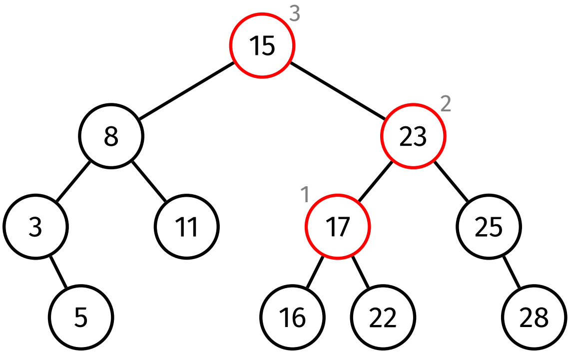
Example: Delete 13 from this tree



13 will be replaced by 15 (its in-order successor)

AVL Tree Deletion

Example: Delete 13 from this tree



Balance must be checked at 17, then at 23, then at 15

AVL Tree Deletion

Pseudocode


```
avlDelete(t, v):
    Input:  AVL tree t, item v
    Output: t with v deleted

    if t is empty:
        return empty tree
    else if v < t->item:
        t->left = avlDelete(t->left, v)
    else if v > t->item:
        t->right = avlDelete(t->right, v)
    else:
        if t->left is empty:
            temp = t->right
            free(t)
            return temp
        else if t->right is empty:
            temp = t->left
            free(t)
            return temp
        else:
            successor = minimum value in t->right
            t->item = successor
            t->right = avlDelete(t->right, successor)

    return avlRebalance(t)
```

Note: This is the same as in AVL tree insertion

```
avlRebalance(t):
    Input:  possibly unbalanced tree t
    Output: balanced t

    bal = balance(t)
    if bal > 1:
        if balance(t->left) < 0:
            t->left = rotateLeft(t->left)
        t = rotateRight(t)
    else if bal < -1:
        if balance(t->right) > 0:
            t->right = rotateRight(t->right)
        t = rotateLeft(t)

    return t

balance(t):
    Input:  tree t
    Output: balance factor of t

    return height(t->left) - height(t->right)
```

AVL Tree Deletion

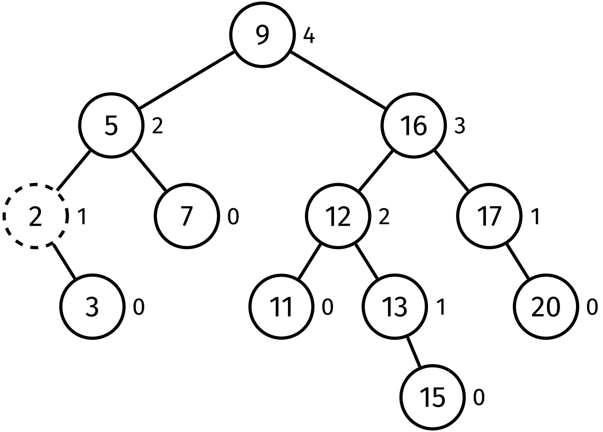
Rebalancing

AVL tree deletion
has the same rebalancing cases
as AVL tree insertion.

AVL Tree Deletion

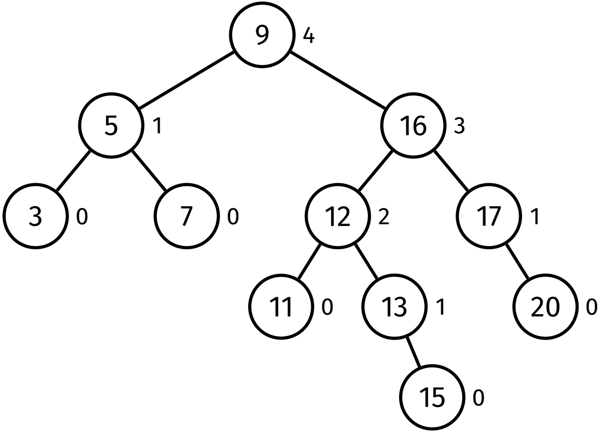
Rebalancing Example 1 - Right Left

Delete 2 from this tree:



AVL Tree Deletion

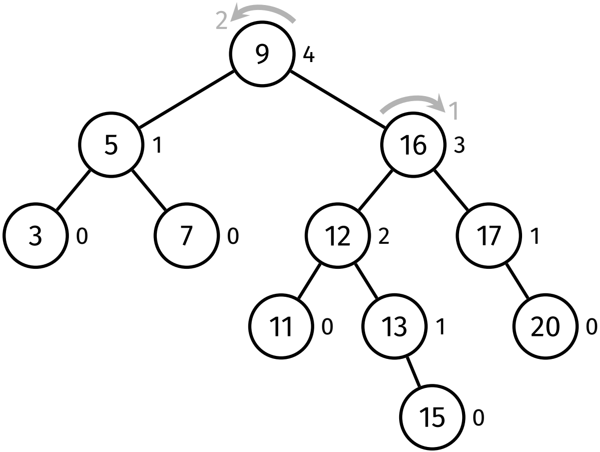
Rebalancing Example 1 - Right Left



Check for balance at 5 and 9

AVL Tree Deletion

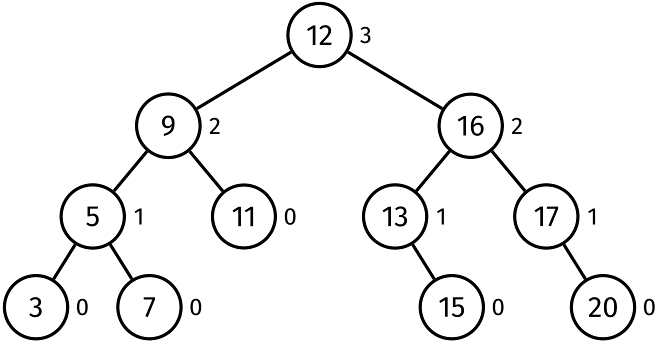
Rebalancing Example 1 - Right Left



9 is unbalanced

AVL Tree Deletion

Rebalancing Example 1 - Right Left

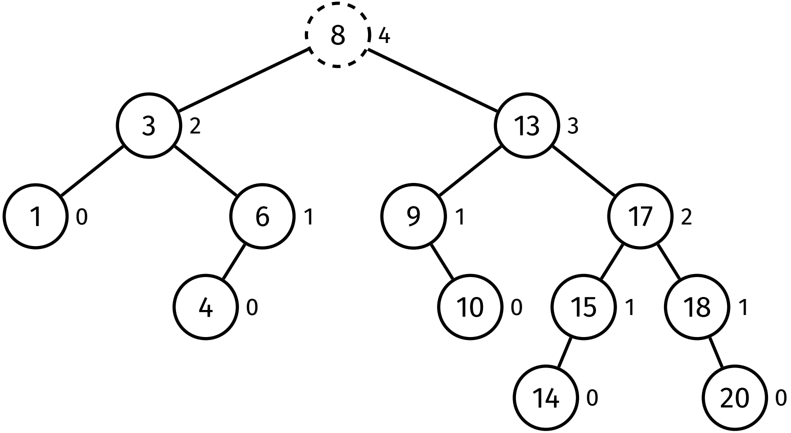


Balanced

AVL Tree Deletion

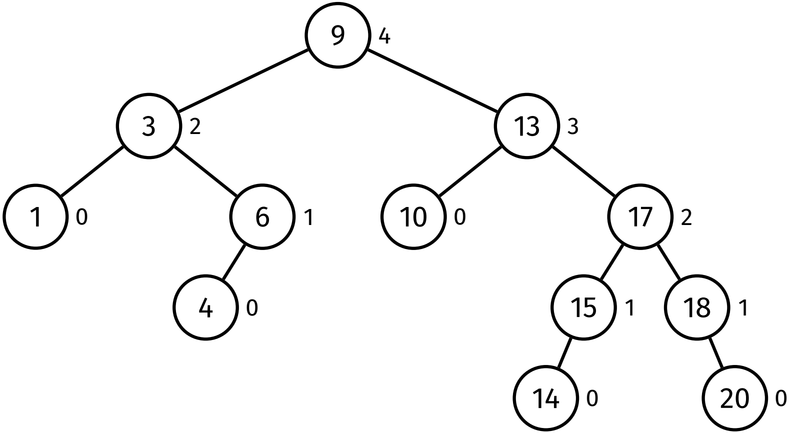
Rebalancing Example 2 - Right Right

Delete 8 from this tree:



AVL Tree Deletion

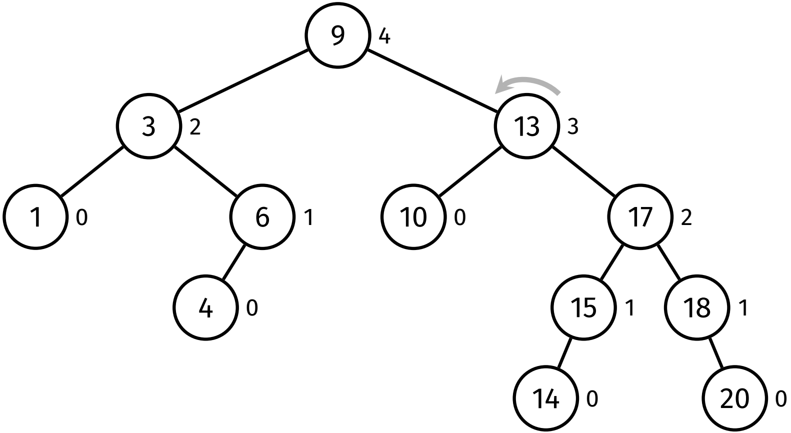
Rebalancing Example 2 - Right Right



Check for balance at 13 and 9

AVL Tree Deletion

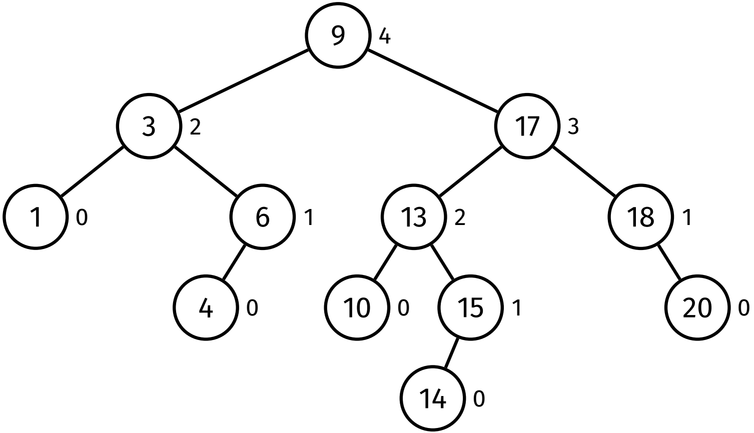
Rebalancing Example 2 - Right Right



13 is unbalanced

AVL Tree Deletion

Rebalancing Example 2 - Right Right

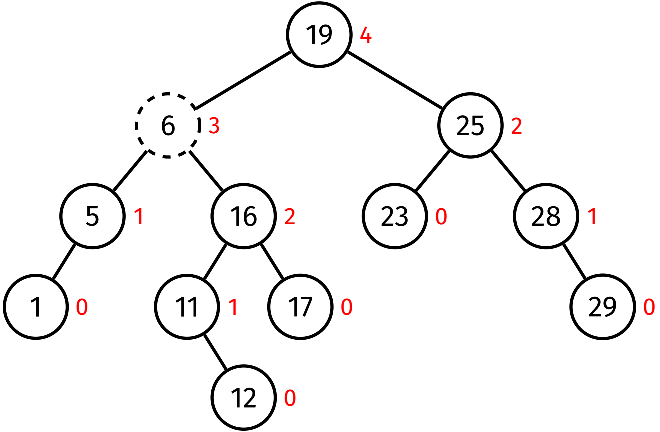


Balanced

Height data also needs to be maintained...

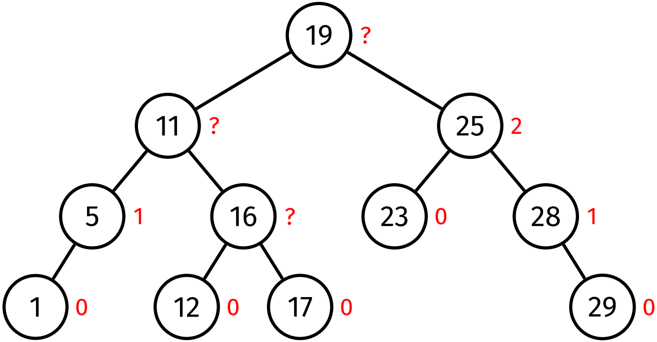
- Whenever a node is deleted
  - Heights of all nodes on deletion path may be affected

Example: Delete 6 from this tree



AVL Tree Deletion

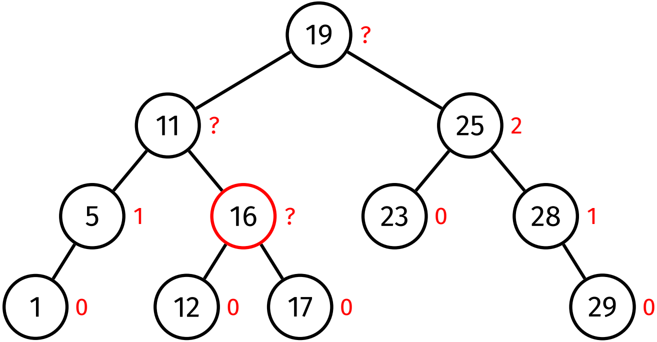
Maintaining Height Data - Deletions



Recompute height of each node on the deletion path using the heights stored in its children.

AVL Tree Deletion

Maintaining Height Data - Deletions

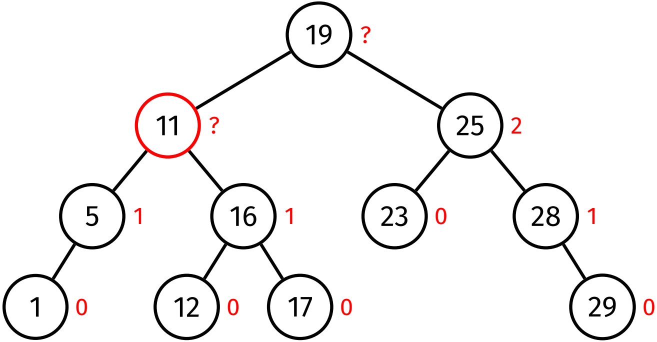


The heights of 16's children are 0 and 0.

Thus, the height of 16 is $\max(0, 0) + 1 = 1$.

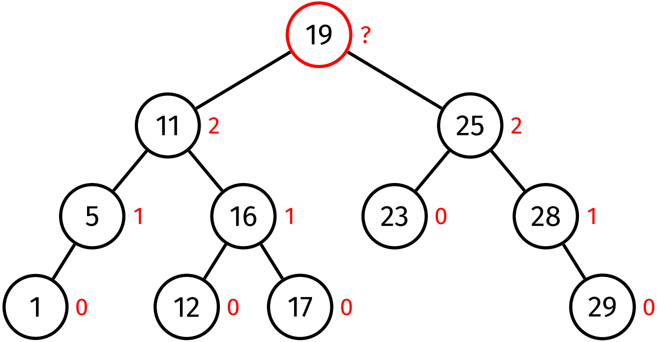
AVL Tree Deletion

Maintaining Height Data - Deletions



The heights of 11's children are 1 and 1.

Thus, the height of 11 is $\max(1, 1) + 1 = 2$.

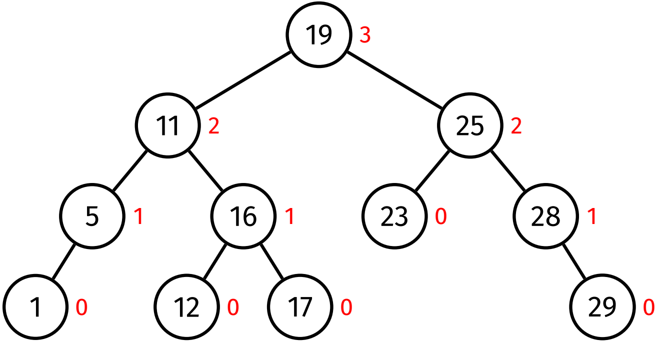


The heights of 19's children are 2 and 2.

Thus, the height of 19 is $\max(2, 2) + 1 = 3$.

AVL Tree Deletion

Maintaining Height Data - Deletions



Done.

Analysis:

- Height of an AVL tree is $O(\log n)$
- In the worst case, length of deletion path is $O(\log n)$
- Have to maintain height data and check/fix balance at each node on deletion path
  - This is $O(1)$ per node
- Therefore, worst-case time complexity of AVL tree deletion is $O(\log n)$

Summary

- AVL trees are always height-balanced
  - This means the height of an AVL tree is $O(\log n)$
- Rotations are used to fix imbalances during insertion and deletion
- Balance is checked efficiently by storing height data in each node, which needs to be maintained
- Worst-case time complexity of $O(\log n)$ for insertion, search and deletion

Feedback

https://forms.office.com/r/zEqxUXvmLR