COMP2521 24T3

Sorting Algorithms (II)
Elementary Sorting Algorithms

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selection sort
bubble sort
insertion sort
shell sort

Selection Sort

Method:

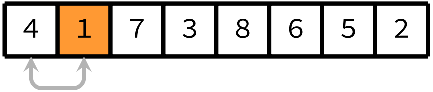
- Find the smallest element, swap it with the first element
- Find the second-smallest element, swap it with the second element
- ...
- Find the second-largest element, swap it with the second-last element

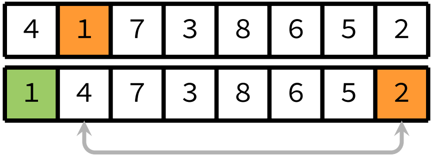
Each iteration improves the “sortedness” of the array by one element.

Example

4	1	7	3	8	6	5	2

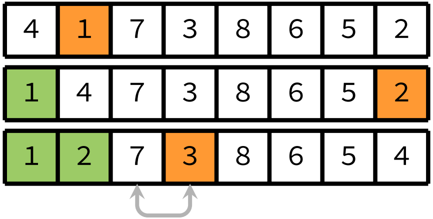
Selection Sort
Example




4
1
7
3
8
6
5
2
1
4
7
3
8
6
5
2

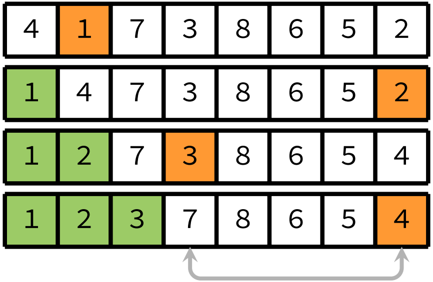
Selection Sort

Example



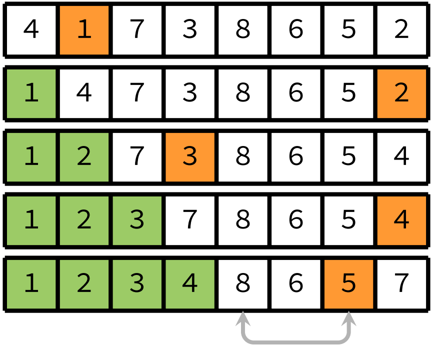
Selection Sort

Example



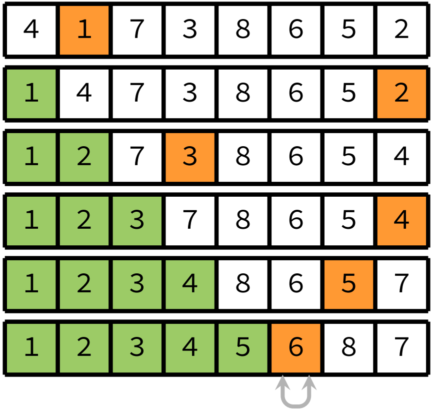
Selection Sort

Example



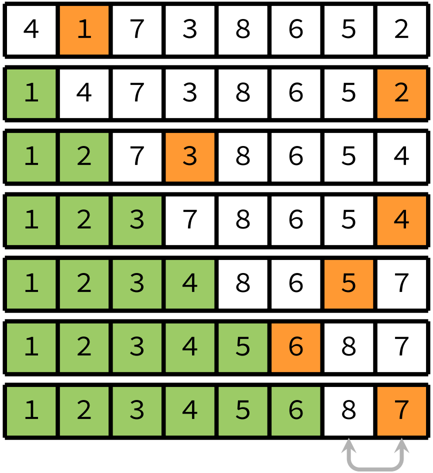
Selection Sort

Example



Selection Sort

Example



Selection Sort

Example

4	1	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	2	7	3	8	6	5	4
1	2	3	7	8	6	5	4
1	2	3	4	8	6	5	7
1	2	3	4	5	6	8	7
1	2	3	4	5	6	8	7
1	2	3	4	5	6	7	8

```c
void selectionSort(Item items[], int lo, int hi) {
    for (int i = lo; i < hi; i++) {
        int min = i;
        for (int j = i + 1; j <= hi; j++) {
            if (lt(items[j], items[min])) {
                min = j;
            }
        }
        swap(items, i, min);
    }
}
```

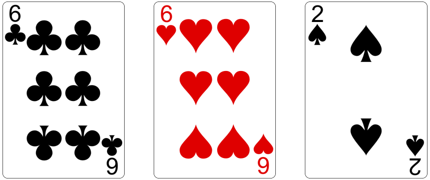
Cost analysis:

- In the first iteration, $n-1$ comparisons, 1 swap
- In the second iteration, $n-2$ comparisons, 1 swap
- ...
- In the final iteration, 1 comparison, 1 swap
- $C = (n-1) + (n-2) + \ldots + 1 = \frac{1}{2}n(n-1) \Rightarrow O(n^2)$
- $S = n - 1$

Cost is the same, regardless of the sortedness of the original array.

Selection sort is unstable

- Due to long-range swaps
- For example, sort these cards by value:



Unstable
Due to long-range swaps

Non-adaptive
Performs same steps, regardless of sortedness of original array

In-place
Sorting is done within original array; does not use temporary arrays

Bubble Sort

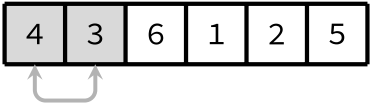
Method:

- Make multiple passes from left (`lo`) to right
- On each pass, swap any out-of-order adjacent pairs
- Elements “bubble up” until they meet a larger element
- Stop if there are no swaps during a pass
  - This means the array is sorted

Example

4	3	6	1	2	5

First pass



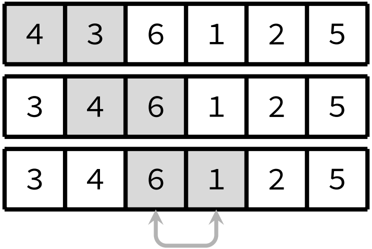
Bubble Sort
Example

First pass

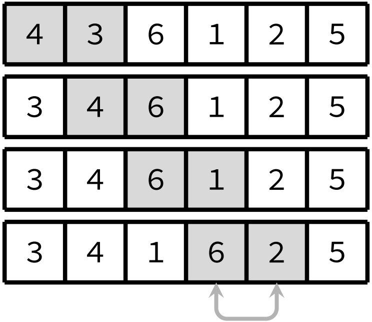
4	3	6	1	2	5

3	4	6	1	2	5

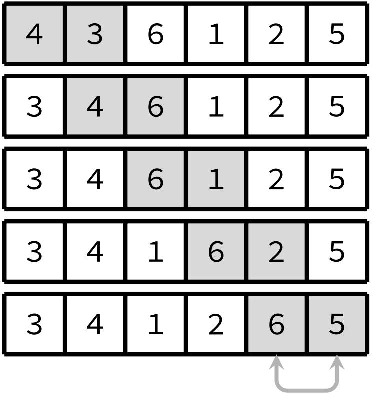
First pass



First pass



First pass



First pass

4	3	6	1	2	5
3	4	6	1	2	5
3	4	6	1	2	5
3	4	1	6	2	5
3	4	1	2	6	5
3	4	1	2	5	6

Second pass

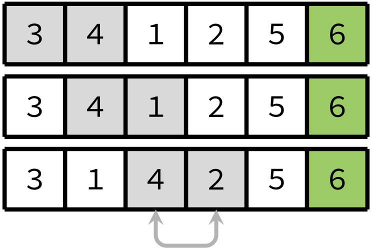
3	4	1	2	5	6

Bubble Sort
Example

Second pass

3	4	1	2	5	6
3	4	1	2	5	6

Second pass



Second pass

3	4	1	2	5	6

3	4	1	2	5	6

3	1	4	2	5	6

3	1	2	4	5	6

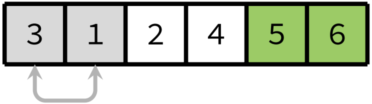
Bubble Sort

Example

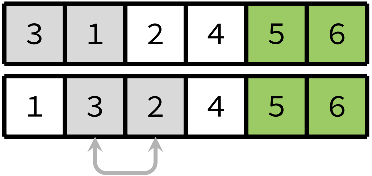
Second pass

3	4	1	2	5	6
3	4	1	2	5	6
3	1	4	2	5	6
3	1	2	4	5	6
3	1	2	4	5	6

Third pass



Third pass



Third pass

3	1	2	4	5	6

1	3	2	4	5	6

1	2	3	4	5	6

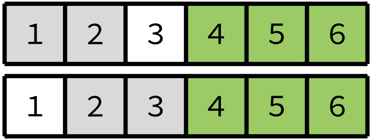
Third pass

3	1	2	4	5	6
1	3	2	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6

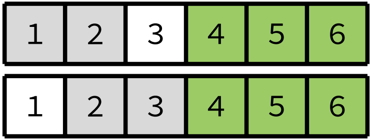
Fourth pass

1	2	3	4	5	6

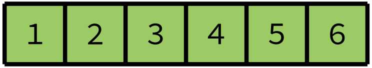
Fourth pass



Fourth pass



No swaps made; stop



Bubble Sort

C Implementation

```
void bubbleSort(Item items[], int lo, int hi) {
    for (int i = hi; i > lo; i--) {
        bool swapped = false;
        for (int j = lo; j < i; j++) {
            if (gt(items[j], items[j + 1])) {
                swap(items, j, j + 1);
                swapped = true;
            }
        }
        if (!swapped) break;
    }
}
```

Best case: Array is sorted

- Only a single pass required
- $n - 1$ comparisons, no swaps
- Best-case time complexity: $O(n)$

1	2	3	4	5	6

Worst case: Array is reverse-sorted

- $n - 1$ passes required
  - First pass: $n - 1$ comparisons
  - Second pass: $n - 2$ comparisons
  - ...
  - Final pass: 1 comparison
- Total comparisons: $(n-1) + (n-2) + \ldots + 1 = \frac{1}{2}n(n-1)$
- Every comparison leads to a swap $\Rightarrow \frac{1}{2}n(n-1)$ swaps
- Worst-case time complexity: $O(n^2)$

6	5	4	3	2	1

Average-case time complexity: $O(n^2)$

- It can be proven that for a randomly ordered array, bubble sort needs to perform $\frac{1}{4}n(n-1)$ swaps on average $\Rightarrow O(n^2)$
  - See appendix for details
- Can show empirically by generating random sequences and sorting them

Stable

Comparisons are between adjacent elements only
Elements are only swapped if out of order

Adaptive

Bubble sort is $O(n^2)$ on average, $O(n)$ if input array is sorted

In-place

Sorting is done within original array; does not use temporary arrays

Method:

- Take first element and treat as sorted array (of length 1)
- Take next element and insert into sorted part of array so that order is preserved
  - This increases the length of the sorted part by one
- Repeat for remaining elements

Example

4	1	7	3	8	6	5	2

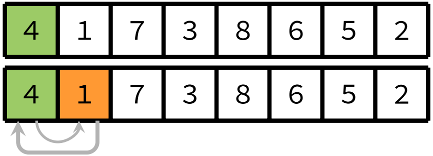
Insertion Sort

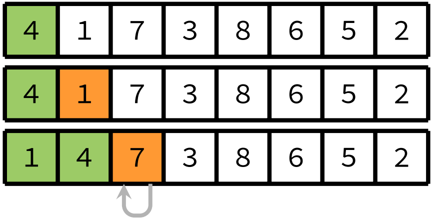
Example

4	1	7	3	8	6	5	2

Insertion Sort

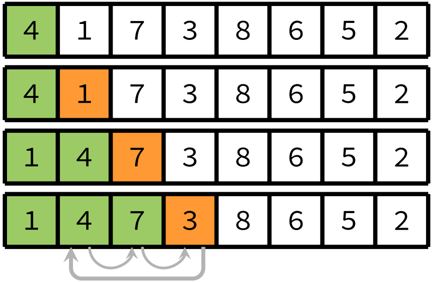
Example




4 1 7 3 8 6 5 2
4 1 7 3 8 6 5 2
1 4 7 3 8 6 5 2

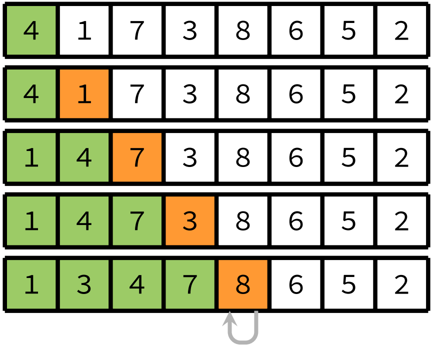
Insertion Sort

Example



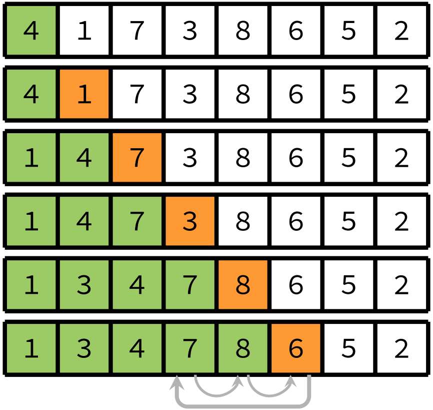
Insertion Sort

Example



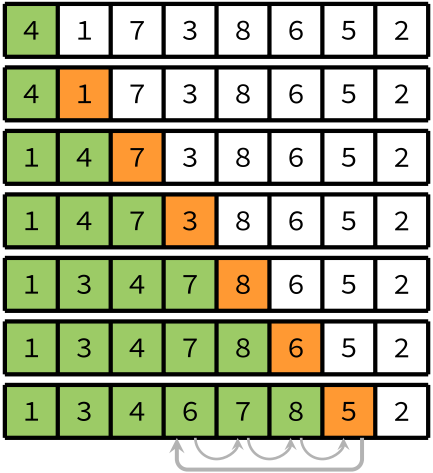
Insertion Sort

Example



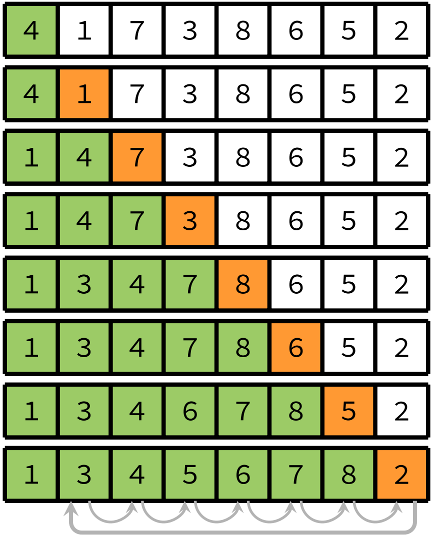
Insertion Sort

Example



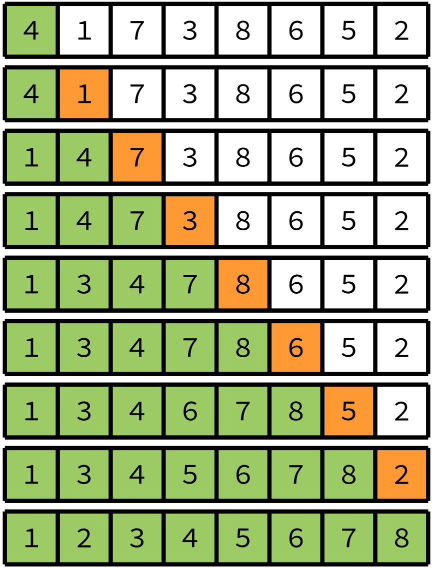
Insertion Sort

Example



Insertion Sort

Example



Insertion Sort

C Implementation

```
void insertionSort(Item items[], int lo, int hi) {
    for (int i = lo + 1; i <= hi; i++) {
        Item item = items[i];
        int j = i;
        for (; j > lo && lt(item, items[j - 1]); j--) {
            items[j] = items[j - 1];
        }
        items[j] = item;
    }
}
```

Best case: Array is sorted

- Inserting each element requires one comparison
- $n - 1$ comparisons
- Best-case time complexity: $O(n)$

1	2	3	4	5	6	7	8

Worst case: Array is reverse-sorted

- Inserting i-th element requires i comparisons
  - Inserting index 1 element requires 1 comparison
  - Inserting index 2 element requires 2 comparisons
  - ...
- Total comparisons: $1 + 2 + \ldots + (n-1) = \frac{1}{2}n(n-1)$
- Worst-case time complexity: $O(n^2)$

8	7	6	5	4	3	2	1

Insertion Sort

Analysis

Average-case time complexity: $O(n^2)$

- Same reason as for bubble sort
- Can show empirically by generating random sequences and sorting them

Stable

Elements are always inserted to the right of any equal elements

Adaptive

Insertion sort is $O(n^2)$ on average, $O(n)$ if input array is sorted

In-place

Sorting is done within original array; does not use temporary arrays

Bubble sort and insertion sort
move elements by shifting them up/down
one space at a time.

If we make longer-distance exchanges,
can we be more efficient?

What if we consider elements that are some distance apart?

Shell sort, invented by Donald Shell

Idea:

- An array is h-sorted if taking every h-th element yields a sorted array
- An h-sorted array is made up of $\frac{n}{h}$ interleaved sorted arrays
- Shell sort: h-sort the array for progressively smaller h, ending with $h = 1$

Example of h-sorted arrays:

3-sorted

0	1	2	3	4	5	6	7	8	9
4	1	0	5	3	2	7	6	9	8

2-sorted

0	1	2	3	4	5	6	7	8	9
1	0	3	2	4	5	7	6	9	8

1-sorted

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

Shell Sort
Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2

Shell Sort

Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2
$h = 3$ passes	3			4			5	
		1			2			8
			6			7		
3-sorted	3	1	6	4	2	7	5	8

Shell Sort

Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2
$h = 3$ passes	3			4			5	
		1			2			8
			6			7		
3-sorted	3	1	6	4	2	7	5	8
$h = 2$ passes	2		3		5		6	
		1		4		7		8
2-sorted	2	1	3	4	5	7	6	8

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2
$h = 3$ passes	3			4			5	
		1			2			8
			6			7		
3-sorted	3	1	6	4	2	7	5	8
$h = 2$ passes	2		3		5		6	
		1		4		7		8
2-sorted	2	1	3	4	5	7	6	8
$h = 1$ pass	1	2	3	4	5	6	7	8

Shell Sort

C Implementation

```
void shellSort(int a[], int lo, int hi)
{
    int hvals[8] = {701, 301, 132, 57, 23, 10, 4, 1};
    int g, h, start, i, j, val;
    for (g = 0; g < 8; g++) {
        h = hvals[g];
        start = lo + h;
        for (i = start+1; i <= hi; i++) {
            val = a[i];
            for (j = i; j >= start; j -= h) {
                if (!less(val,a[j-h]) break;
                a[j] = a[j-h];
            }
            a[j] = val;
        }
    }
```

- Efficiency of shell sort depends on the h-sequence
- Effective h-sequences have been determined empirically
- Many h-sequences have been found to be $O(n^{\frac{3}{2}})$
  - For example: 1, 4, 13, 40, 121, 364, 1093, ...
    - $h_{i+1} = 3h_i + 1$
- Some h-sequences have been found to be $O(n^{\frac{4}{3}})$
  - For example: 1, 8, 23, 77, 281, 1073, 4193, ...

Unstable
Due to long-range swaps

Adaptive
Shell sort applies a generalisation of insertion sort
(which is adaptive)

In-place
Sorting is done within original array; does not use temporary arrays

Summary of Elementary Sorts

	Time complexity			Properties	
	Best	Average	Worst	Stable	Adaptive
Selection sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	No	No
Bubble sort	$O(n)$	$O(n^2)$	$O(n^2)$	Yes	Yes
Insertion sort	$O(n)$	$O(n^2)$	$O(n^2)$	Yes	Yes
Shell sort	depends	depends	depends	No	Yes

Aside: Sorting Linked Lists

Selection sort:

- Let $L =$ original list, $S =$ sorted list (initially empty)
- Repeat the following until L is empty:
  - Find the node V containing the largest value in L, and unlink it
  - Insert V at the front of S

Bubble sort:

- Traverse the list, comparing adjacent values
  - If value in current node is greater than value in next node, swap values
- Repeat the above until no swaps required in one traversal

Insertion sort:

- Let $L =$ original list, $S =$ sorted list (initially empty)
- For each node in L:
  - Insert the node into S in order

Aside: Sorting Linked Lists

Shell sort:

- Difficult to implement efficiently
- Can’t access specific index in constant time
  - Have to traverse from the beginning

Feedback

https://forms.office.com/r/zEqxUXvmLR

Appendix

Bubble Sort - Proof of $O(n^2)$ Average Case

New concept: inversion

An inversion is a pair of elements from a sequence where the left element is greater than the right element.

For example, consider the following array:

4	2	1	5	3

The array contains 5 inversions:
(4, 2), (4, 1), (4, 3), (2, 1), (5, 3)

Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

Observation:

- In bubble sort, every swap reduces the number of inversions by 1

The goal of the proof: Show that the average number of inversions in a randomly sorted array is $O(n^2)$.

- This implies the number of swaps required by bubble sort is $O(n^2)$ ...
- Which implies that the average-case time complexity of bubble sort is $O(n^2)$ or slower
  - (but we know that it can't be slower than $O(n^2)$ since the worst-case time complexity of bubble sort is $O(n^2)$)

Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

In a randomly sorted array:

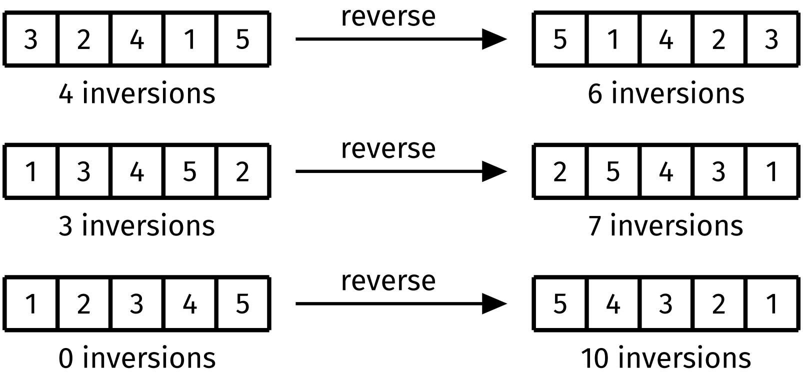
- The minimum possible number of inversions is 0 (sorted array)
- The maximum possible number of inversions is $\frac{1}{2}n(n-1)$ (reverse-sorted array)

Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

Let k be the number of inversions in a random permutation.
By reversing this permutation, one can obtain a permutation with $\frac{1}{2}n(n-1) - k$ inversions.

For example, suppose $n = 5$:



Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

Thus, if we take all the possible permutations of an array and pair each permutation with its reverse, the total number of inversions in each pair is $\frac{1}{2}n(n-1)$.

This implies that the average number of inversions across all permutations is $\frac{1}{4}n(n-1)$, which is $O(n^2)$.