COMP2521 24T1
Graphs (VII)
Minimum Spanning Trees

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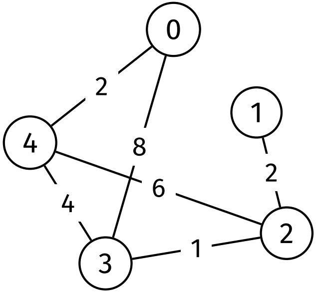
minimum spanning trees
kruskal's algorithm
prim's algorithm

Minimum Spanning Trees

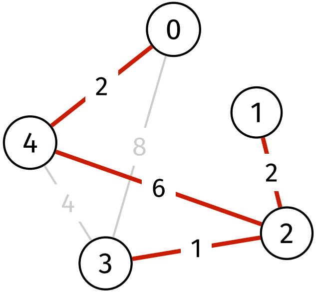
A spanning tree of an undirected graph G
is a subgraph of G that contains all vertices of G,
that is connected and contains no cycles

A minimum spanning tree of an undirected weighted graph G
is a spanning tree of G that has
minimum total edge weight among all spanning trees of G

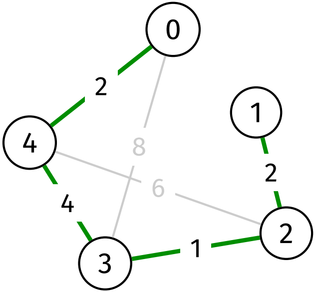
Applications:
Electrical grids, networks
Any situation where we want to connect nodes as cheaply as possible



Original graph



Spanning tree



Minimum spanning tree

Minimum Spanning Tree Algorithms

Basic minimum spanning tree algorithms:

- Kruskal's algorithm
- Prim's algorithm

Invented by
American mathematician, statistician, computer scientist
Joseph Kruskal in 1956

Kruskal's Algorithm

Algorithm:

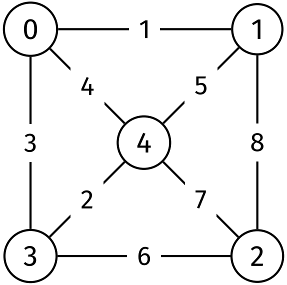
1. Start with an empty graph
   - With same vertices as original graph
2. Consider edges in increasing weight order
   - Add edge if it does not form a cycle in the MST
3. Repeat until $V - 1$ edges have been added

Critical operations:

- Iterating over edges in weight order
- Checking if adding an edge would form a cycle

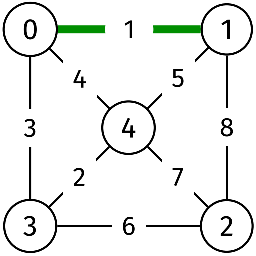
Kruskal's Algorithm
Example

Run Kruskal's algorithm on this graph:

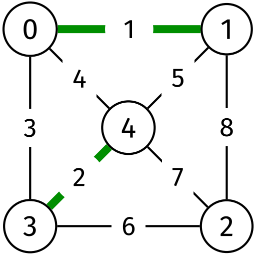


Kruskal's Algorithm

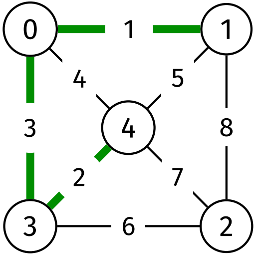
Example



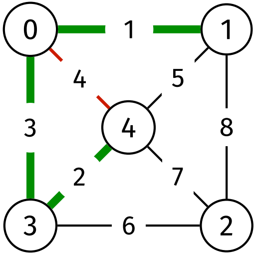
Add 0-1



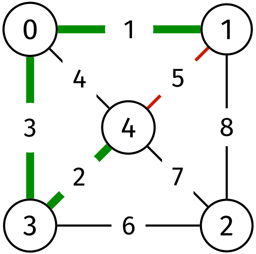
Add 3-4



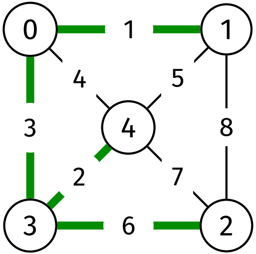
Add 0-3



Don't add 0-4



Don't add 1-4

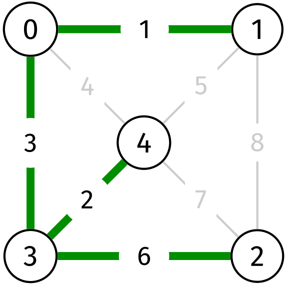


Add 2-3

Kruskal's Algorithm

Example

MST:



Kruskal's Algorithm

Pseudocode (Version 1)

```
kruskalMst(G):
    Input:  graph G with V vertices
    Output: minimum spanning tree of G

    mst = empty graph with V vertices

    sortedEdges = sort edges of G by weight
    for each edge e in sortedEdges:
        add e to mst
        if mst has a cycle:
            remove e from mst

        if mst has V - 1 edges:
            return mst
```

Kruskal's Algorithm

Pseudocode (Version 2)

```
kruskalMst(G):
    Input:  graph G with V vertices
    Output: minimum spanning tree of G

    mst = empty graph with V vertices

    sortedEdges = sort edges of G by weight
    for each edge (v, w, weight) in sortedEdges:
        if there is no path between v and w in mst:
            add edge (v, w, weight) to mst

        if mst has V − 1 edges:
            return mst
```

Proof by exchange argument.

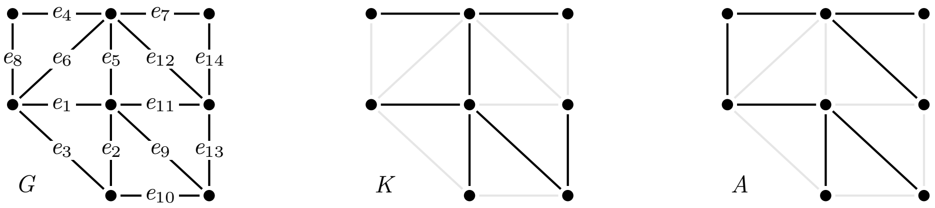
Idea:

- Suppose there exists another algorithm A which makes a different set of choices
  - In this case, chooses a different set of edges for the MST
- Identify *one* choice made by A which is not made by our algorithm
- Show that by exchanging that choice with one of the choices made by our algorithm, the solution does not become worse or less optimal
  - In this case, the "solution" is the spanning tree produced
  - In this case, an "optimal" solution is a spanning tree that costs as little as possible

Sort the edges of G in increasing order.

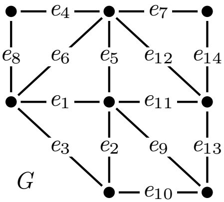
Let K be the set of edges selected by Kruskal's algorithm.
Let A be the set of edges selected by a different algorithm.

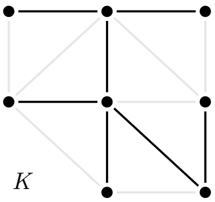
edges of G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A	e_1	e_2		e_4			e_7	e_8	e_9	$\cdots$

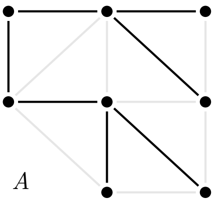


Consider the first edge that is chosen by K but *not* by A.

edges of G	e_1	e_2	e_3	e_4	$\boxed{e_5}$	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A	e_1	e_2		e_4			e_7	e_8	e_9	$\cdots$



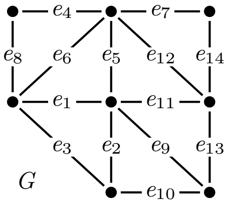


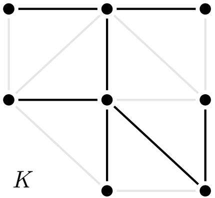


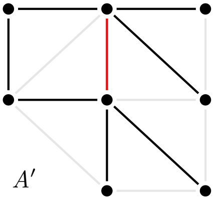
Consider the first edge that is chosen by K but *not* by A.

Add this edge to a copy of A (call it A'). This forms a cycle in A'.

edges of G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A'	e_1	e_2		e_4	e_5		e_7	e_8	e_9	$\cdots$







Kruskal's Algorithm

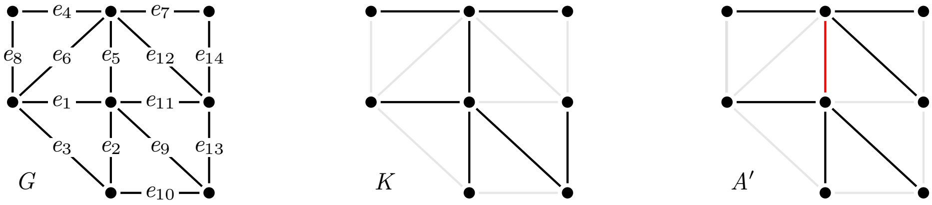
Analysis - Correctness

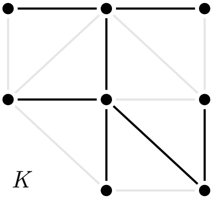
Consider the first edge that is chosen by K but *not* by A.

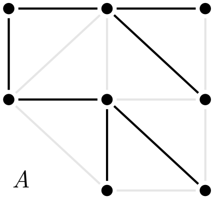
Add this edge to a copy of A (call it A'). This forms a cycle in A'.

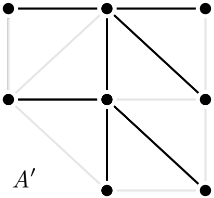
Now find the highest-weight edge in this cycle and *remove* it from A'.

edges of G	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	$\cdots$
edges of K	e_1	e_2		e_4	e_5		e_7		e_9	$\cdots$
edges of A'	e_1	e_2		e_4	e_5		e_7	e_8	e_9	$\cdots$

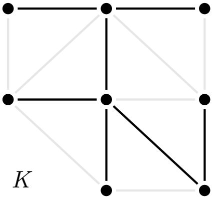


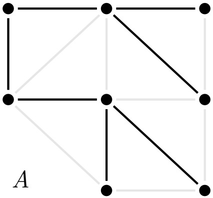


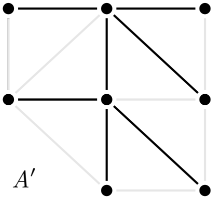




Now A' is once again a spanning tree,
but it is more similar to K than A and it costs no more than A.







Now A' is once again a spanning tree,
but it is more similar to K than A and it costs no more than A.

Repeat until A' is identical to K. Each time we perform an exchange, the spanning tree does not increase in cost.

Therefore, K is an optimal spanning tree (MST).

Analysis:

- Sorting edges is $O(E \cdot \log E)$
- Main loop has at most E iterations
- Different ways to check if adding an edge would form a cycle
  - Cycle/path checking is $O(V)$ in the worst case (adjacency list)
    $\Rightarrow$ overall cost $= O(E \cdot \log E + E \cdot V) = O(E \cdot V)$
  - Using union-find data structure is close to $O(1)$ in the worst case
    $\Rightarrow$ overall cost $= O(E \cdot \log E + E) = O(E \cdot \log E) = O(E \cdot \log V)$

Developed by Vojtěch Jarník in 1930
and rediscovered by Robert C. Prim in 1957

Vojtěch Jarník

Robert C. Prim

Algorithm:

1. Start with an empty graph
2. Start from any vertex, add it to the MST
3. Choose cheapest edge s–t such that:
   - s has been added to the MST, and
   - t has not been added to the MST

   and add this edge and the vertex t to the MST
4. Repeat previous step until $V - 1$ edges have been added
   - Or until all vertices have been added

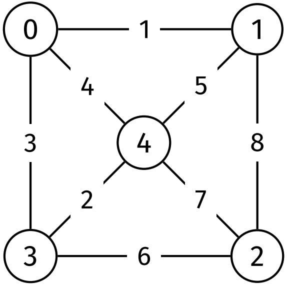
Critical operations:

- Finding the cheapest edge s–t such that
  s has been added to the MST and t has not been added to the MST

Prim's Algorithm

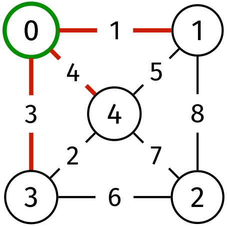
Example

Run Prim's algorithm on this graph (starting at 0):

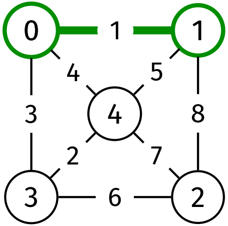


Prim's Algorithm

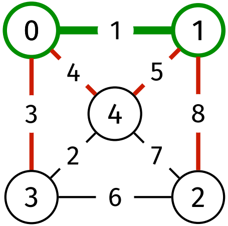
Example



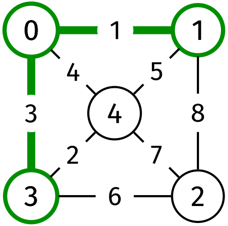
Start of step 1



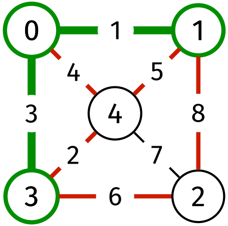
End of step 1



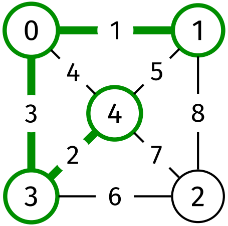
Start of step 2



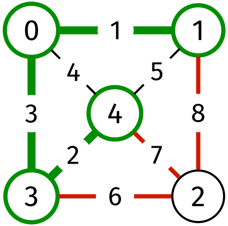
End of step 2



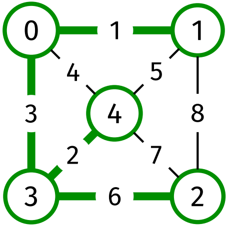
Start of step 3



End of step 3



Start of step 4

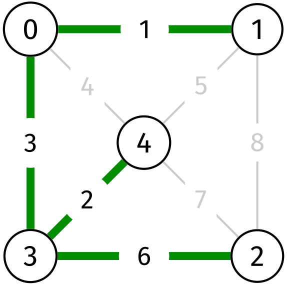


End of step 4

Prim's Algorithm

Example

MST:



Prim's Algorithm

Pseudocode

```
primMst(G):
    Input:  graph G with V vertices
    Output: minimum spanning tree of G

    mst = empty graph with V vertices
    usedV = {0}
    unusedE = edges of G
    while |usedV| < V:
        find cheapest edge e (s, t, weight) in unusedE such that
                s ∈ usedV and t ∉ usedV

        add e to mst
        add t to usedV
        remove e from unusedE

    return mst
```

Analysis:

- Algorithm considers at most E edges $\Rightarrow O(E)$
- Loop has V iterations
- In each iteration, finding the minimum-weighted edge:
  - With set of edges is $O(E)$
    $\Rightarrow$ overall cost $= O(E + V \cdot E) = O(V \cdot E)$
  - With Fibonacci heap is $O(\log E) = O(\log V)$
    $\Rightarrow$ overall cost $= O(E + V \cdot \log V)$

Comparison

Kruskal's algorithm vs Prim's algorithm

Kruskal's algorithm...

- is $O(E \cdot \log V)$
- uses array-based data structures
- performs better on sparse graphs

Prim's algorithm...

- is $O(E + V \cdot \log V)$
- uses complex linked data structures
  - in its most efficient implementation (Fibonacci heap)
- performs better on dense graphs

Other MST Algorithms

- Boruvka's algorithm
  - Oldest MST algorithm
  - Start with V separate components
  - Join components using min cost links
  - Continue until only a single component
  - Worst-case time complexity: $O(E \cdot \log V)$
- Karger, Klein and Tarjan
  - Based on Boruvka's algorithm, but non-deterministic
  - Randomly selects subset of edges to consider
  - Time complexity: $O(E)$ on average

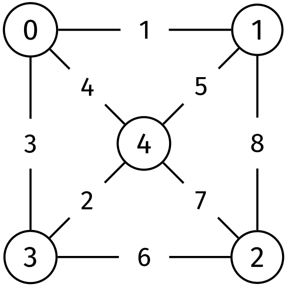
Feedback

https://forms.office.com/r/5c0fb4tvMb

Appendix

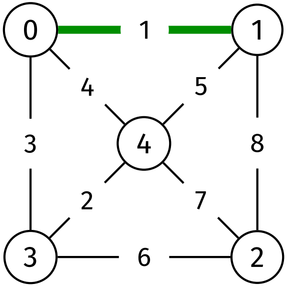
Kruskal's Algorithm Example

Original graph



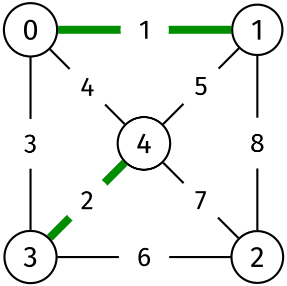
Kruskal's Algorithm Example

Adding 0-1 would not create a cycle



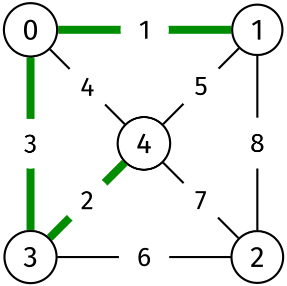
Kruskal's Algorithm Example

Adding 3-4 would not create a cycle



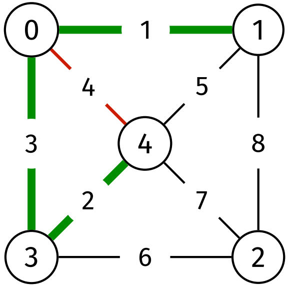
Kruskal's Algorithm Example

Adding 0-3 would not create a cycle



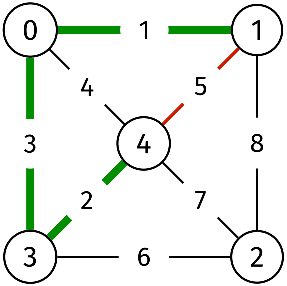
Kruskal's Algorithm Example

Adding 0-4 would create a cycle



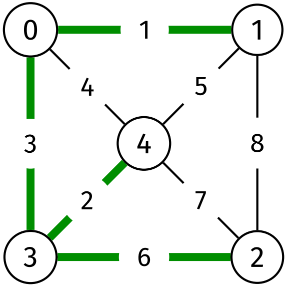
Kruskal's Algorithm Example

Adding 1-4 would create a cycle



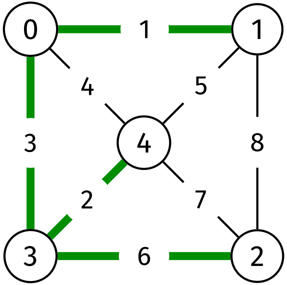
Kruskal's Algorithm Example

Adding 2-3 would not create a cycle



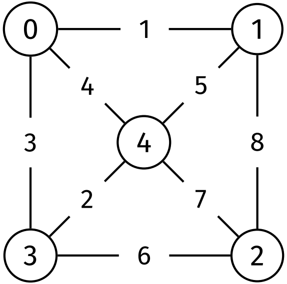
Kruskal's Algorithm Example

Done - MST has 4 edges



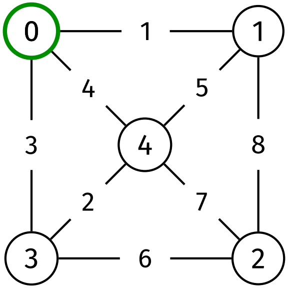
Prim's Algorithm Example

Original graph



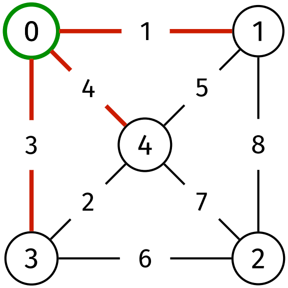
Prim's Algorithm Example

Start at vertex 0



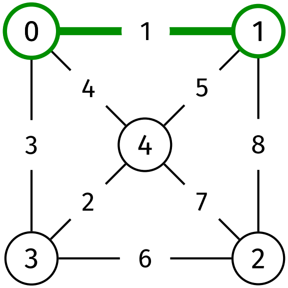
Prim's Algorithm Example

Choose cheapest edge out of these (in red)



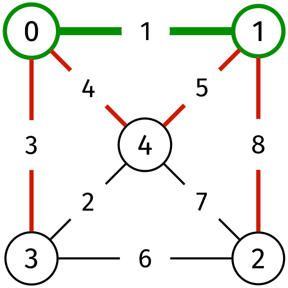
Prim's Algorithm Example

Add 0-1 to MST



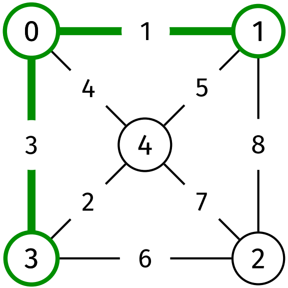
Prim's Algorithm Example

Choose cheapest edge out of these (in red)



Prim's Algorithm Example

Add 0-3 to MST



Prim's Algorithm Example

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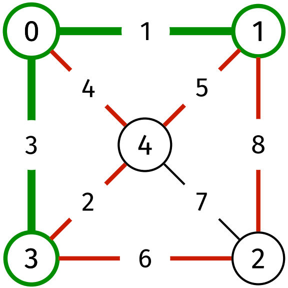
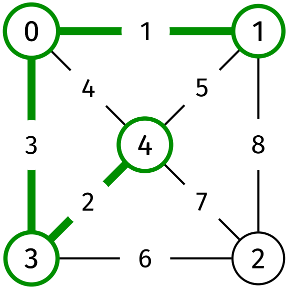
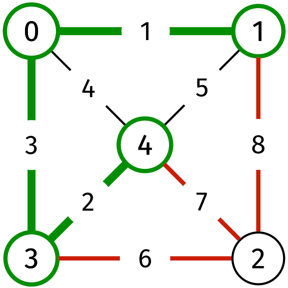


Prim's Algorithm Example

Add 3-4 to MST

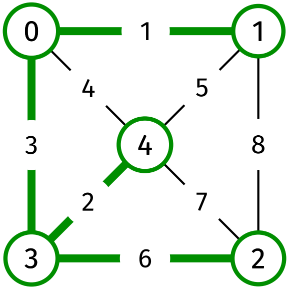


Choose cheapest edge out of these (in red)



Prim's Algorithm Example

Add 3-2 to MST



Prim's Algorithm Example

Done - MST has 4 edges