COMP2521 24T1

Balancing Binary Search Trees

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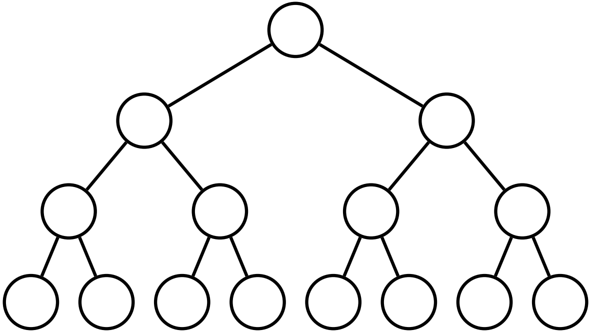
balancing operations

balancing methods

The structure, height, and hence
performance
of a binary search tree
depends on the order of insertion.

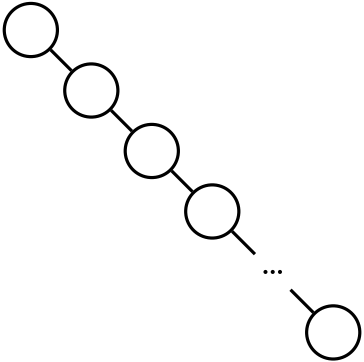
Best case

Items are inserted evenly on the left and right throughout the tree
Height of tree will be $O(\log n)$



Worst case

Items are inserted in ascending or descending order
such that tree consists of a single branch
Height of tree will be $O(n)$



Binary Search Trees

A binary tree of n nodes is said to be **balanced** if it has (close to) minimal height ($O(\log n)$), and **degenerate** if it has (close to) maximal height ($O(n)$).

Types of Balance

SIZE-BALANCED

a *size-balanced* tree has,
for every node,

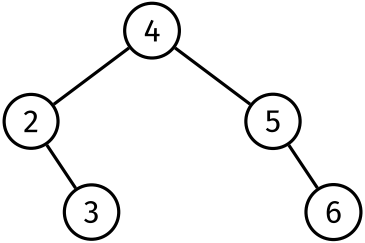
$$|\text{SIZE}(l) - \text{SIZE}(r)| \leq 1$$

HEIGHT-BALANCED

a *height-balanced* tree has,
for every node,

$$|\text{HEIGHT}(l) - \text{HEIGHT}(r)| \leq 1$$

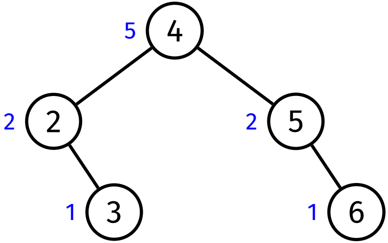
Types of Balance
Example



Size-balanced?

Height-balanced?

Types of Balance
Example

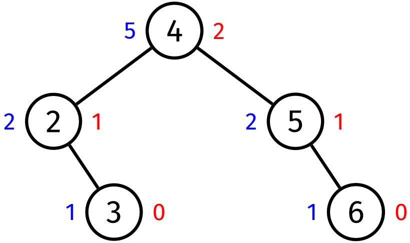


Size-balanced?
Yes

For every node,
$|\text{SIZE}(l) - \text{SIZE}(r)| \leq 1$

Height-balanced?

Types of Balance
Example



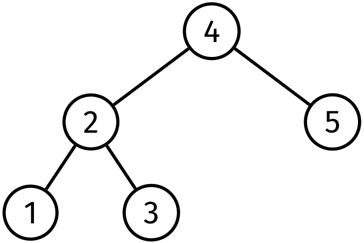
Size-balanced?
Yes

For every node,
$|\text{SIZE}(l) - \text{SIZE}(r)| \leq 1$

Height-balanced?
Yes

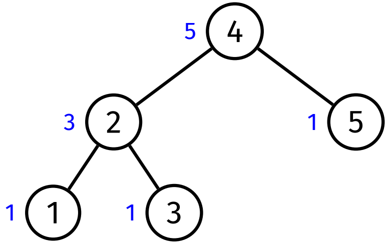
For every node,
$|\text{HEIGHT}(l) - \text{HEIGHT}(r)| \leq 1$

Types of Balance
Example



Size-balanced?

Height-balanced?



Size-balanced?

No

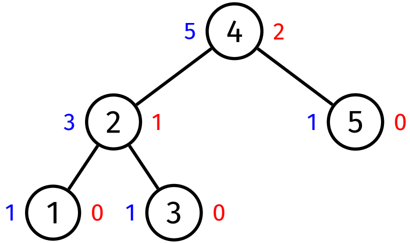
At node 4,

$$|\text{SIZE}(l) - \text{SIZE}(r)|$$
$$= |3 - 1| = 2 > 1$$

Height-balanced?

Types of Balance

Example



Size-balanced?
No

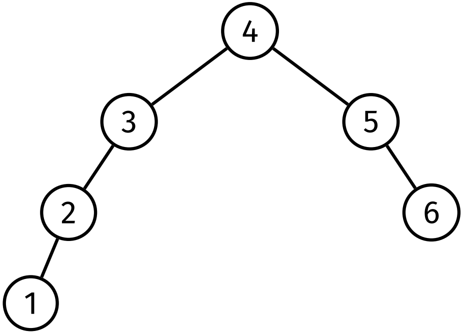
At node 4,
$|\text{SIZE}(l) - \text{SIZE}(r)|$
$= |3 - 1| = 2 > 1$

Height-balanced?
Yes

For every node,
$|\text{HEIGHT}(l) - \text{HEIGHT}(r)| \leq 1$

Types of Balance

Example

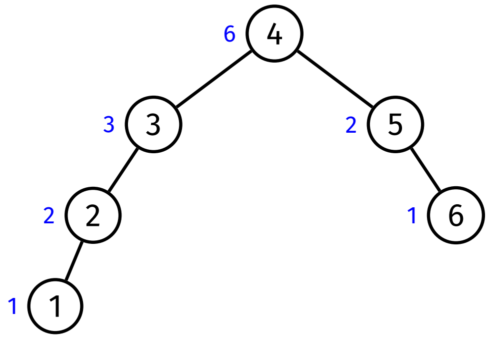


Size-balanced?

Height-balanced?

Types of Balance

Example



Size-balanced?

No

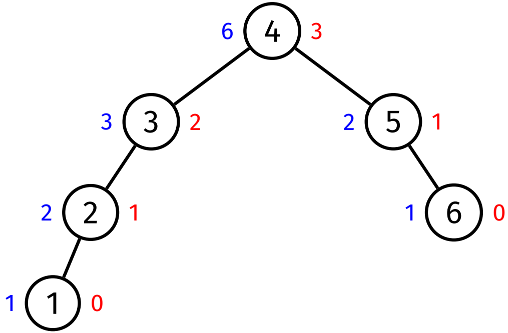
At node 3,

$$|\text{SIZE}(l) - \text{SIZE}(r)| = |2 - 0| = 2 > 1$$

Height-balanced?

Types of Balance

Example



Size-balanced?

No

At node 3,

$$|\text{SIZE}(l) - \text{SIZE}(r)| = |2 - 0| = 2 > 1$$

Height-balanced?

No

At node 3,

$$|\text{HEIGHT}(l) - \text{HEIGHT}(r)| = |1 - (-1)| = 2 > 1$$

Balancing Operations

Rotation

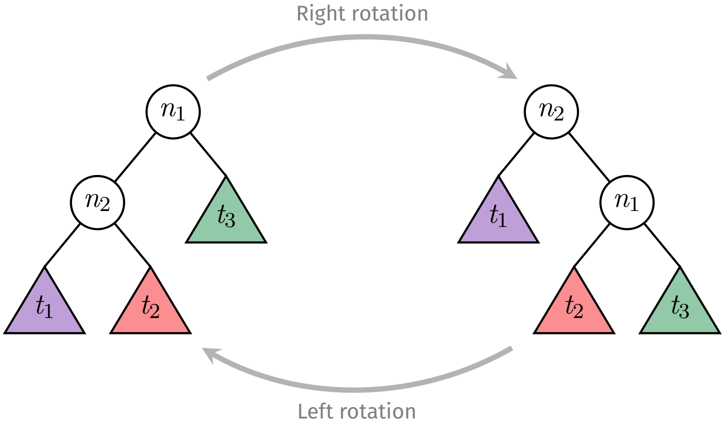
- Left rotation
- Right rotation

Partition

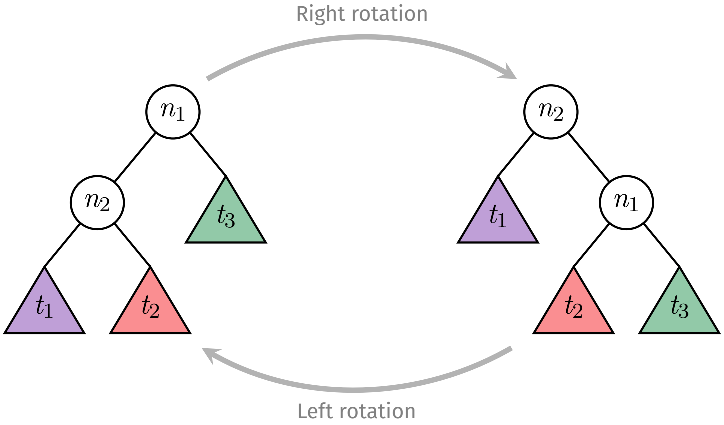
- Rearrange tree around a specified node by rotating it up to the root

Rotations

LEFT ROTATION and **RIGHT ROTATION**:
a pair of operations
that change the balance of a tree



Rotations maintain the order of a search tree:

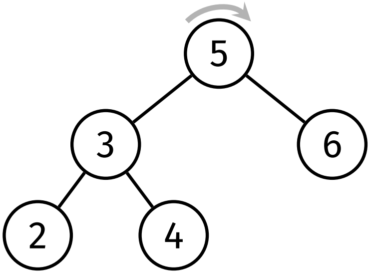


(all values in t_1) $< n_2 <$ (all values in t_2) $< n_1 <$ (all values in t_3)

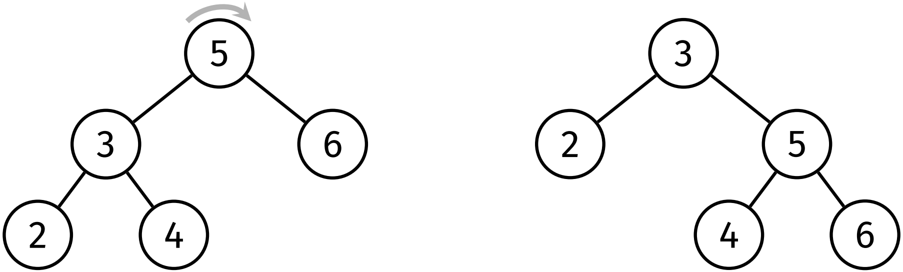
Rotations

Example

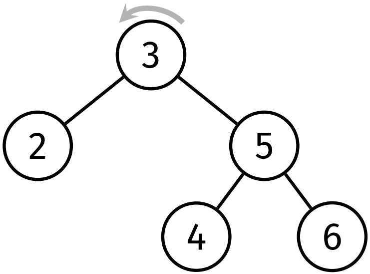
Rotate right at 5



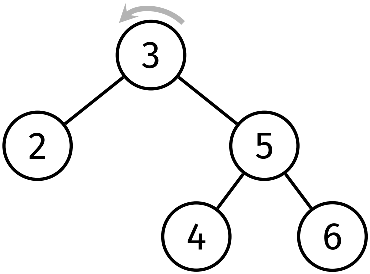
Rotate right at 5

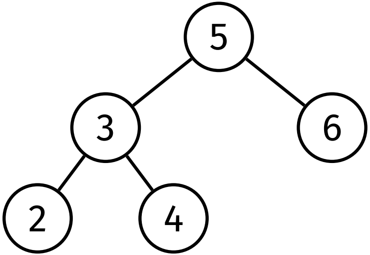


Rotate left at 3

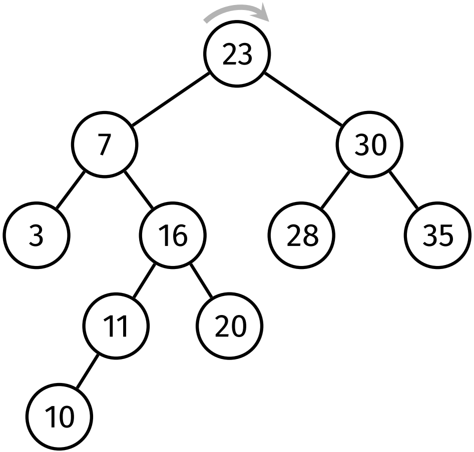


Rotate left at 3





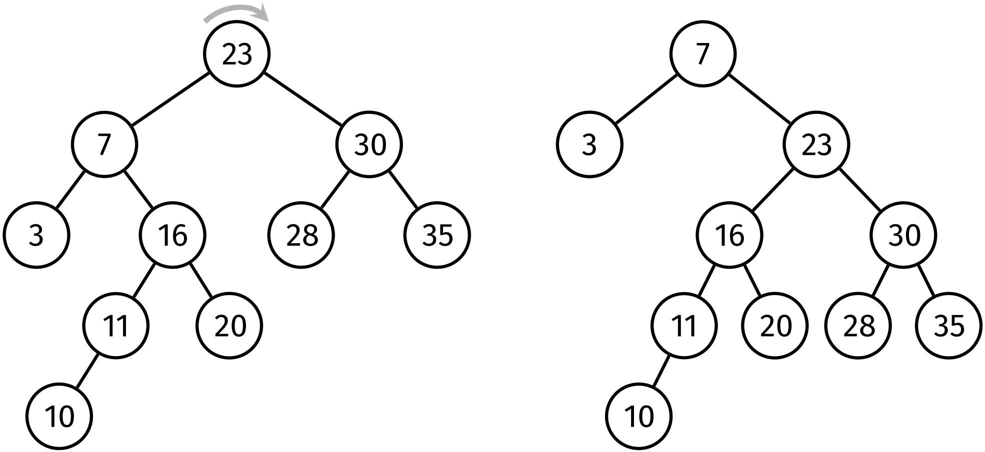
Rotate right at 23



Rotations

Example

Rotate right at 23



Rotations

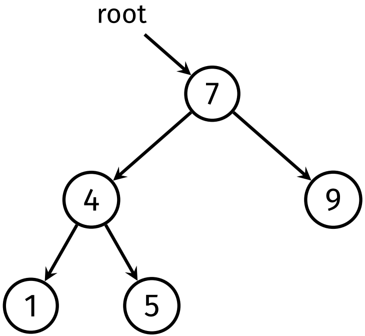
Implementation

```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;
    newRoot->right = root;
    return newRoot;
}
```

```
struct node *rotateLeft(struct node *root) {
    if (root == NULL || root->right == NULL) return root;
    struct node *newRoot = root->right;
    root->right = newRoot->left;
    newRoot->left = root;
    return newRoot;
}
```

Rotations
Implementation

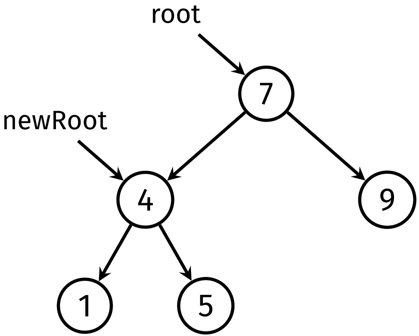
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;




}
```



Rotations

Implementation

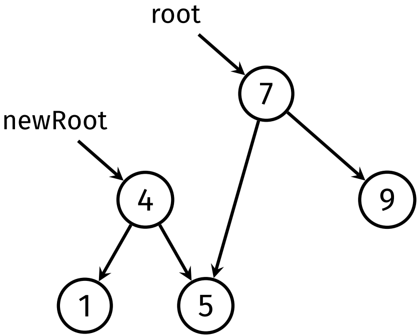
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;



}
```



Rotations

Implementation

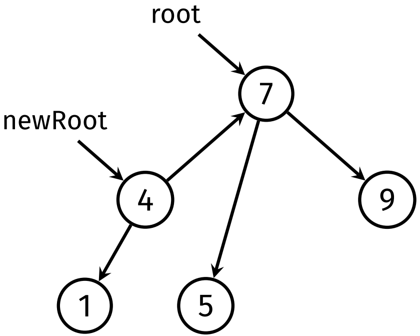
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;


}
```



Rotations

Implementation

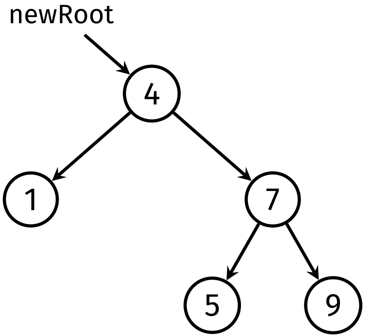
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;
    newRoot->right = root;

}
```



Rotations

Implementation

```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;
    newRoot->right = root;
    return newRoot;
}
```



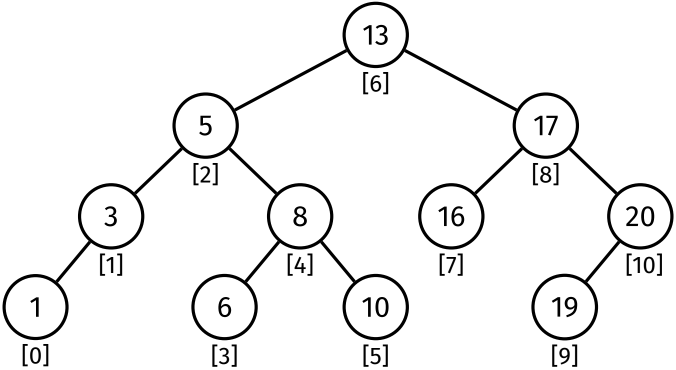
Time complexity: $O(1)$

- Rotation requires only a few localised pointer re-arrangements

Partition

`partition(tree, i)`

Rearrange the tree so that the element with index i becomes the root

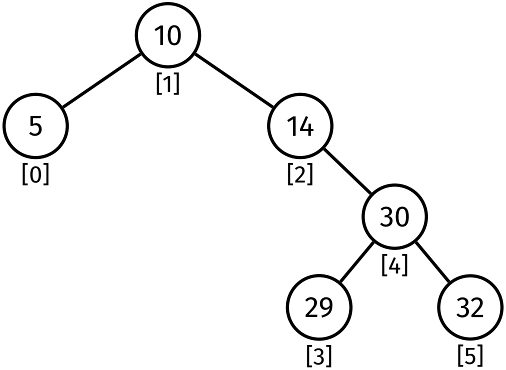


Partition

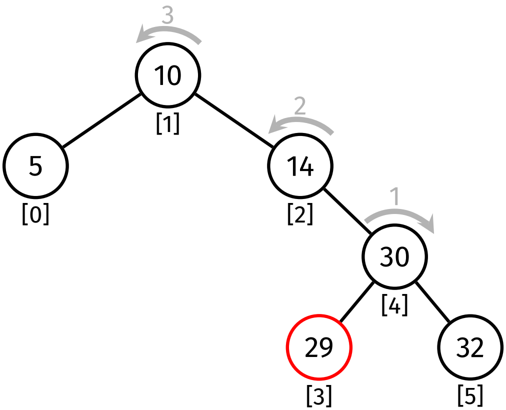
Method:

- Find element with index i
- Perform rotations to lift it to the root
  - If it is the left child of its parent, perform right rotation at its parent
  - If it is the right child of its parent, perform left rotation at its parent
  - Repeat until it is at the root of the tree

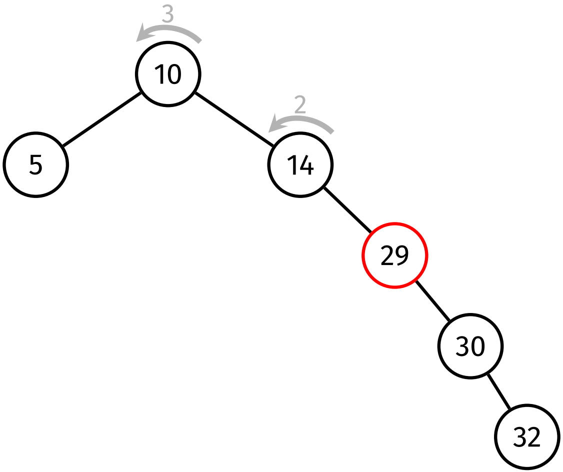
Partition this tree around index 3:



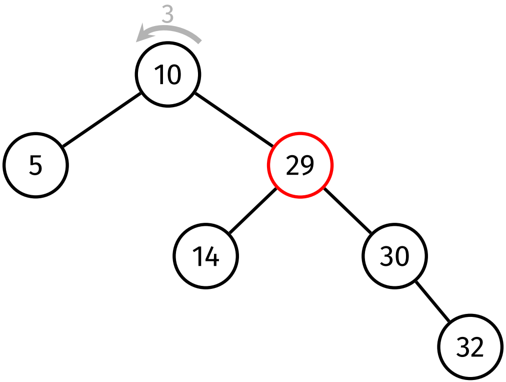
Partition this tree around index 3:



After right rotation at 30:



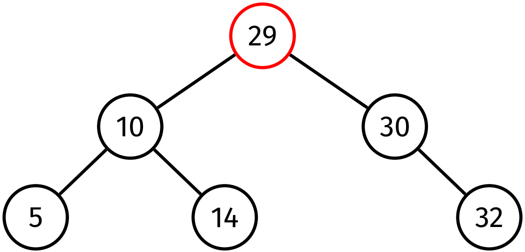
After left rotation at 14:



Partition

Example

After left rotation at 10:



Partition

Pseudocode

```
partition(t, i):
    Input:  tree t, index i
    Output: tree with i-th item moved to root

    m = size(t->left)
    if i < m:
        t->left = partition(t->left, i)
        t = rotateRight(t)
    else if i > m:
        t->right = partition(t->right, i - m - 1)
        t = rotateLeft(t)

    return t
```

Analysis:

- size() operation is expensive
  - needs to traverse whole subtree
- can cause partition to be $O(n^2)$ in the worst case
- to improve efficiency, can change node structure so that each node stores the size of its subtree in the node itself
  - however, this will require extra work in other functions to maintain

```
struct node {
    int item;
    struct node *left;
    struct node *right;
    int size;
};
```

Balancing Methods

- Global Rebalancing
- Root Insertion
- Randomised Insertion

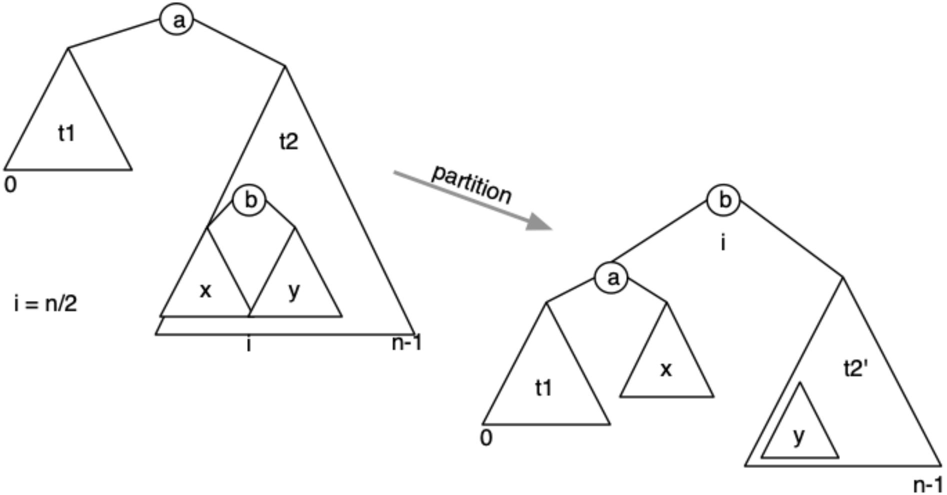
Global Rebalancing

Idea:
Completely rebalance whole tree so it is size-balanced

Method:
Lift the median node to the root
by partitioning on $\textsc{size}(t)/2$,
then rebalance both subtrees (recursively)

Global Rebalancing

First, partition on index $n/2$...



...then rebalance both subtrees

Global Rebalancing

Pseudocode

```
rebalance(t):
    Input:  tree t
    Output: rebalanced t

    if size(t) < 3:
        return t

    t = partition(t, size(t) / 2)
    t->left = rebalance(t->left)
    t->right = rebalance(t->right)
    return t
```

Global Rebalancing

Analysis

Worst-case time complexity: $O(n \log n)$

- Assume nodes store the size of their subtrees
- First step: partition entire tree on index $n/2$
  - This takes at most n recursive calls, n rotations $\Rightarrow$ n steps
  - Result is two subtrees of size $\approx n/2$
- Then partition both subtrees
  - Partitioning these subtrees takes $n/2$ steps each $\Rightarrow$ n steps in total
  - Result is four subtrees of size $\approx n/4$
- ...and so on...
- About $\log_2 n$ levels of partitioning in total, each requiring n steps $\Rightarrow O(n \log n)$

What if we insert more items?

- Options:
  - Rebalance on every insertion
    - Not feasible
  - Rebalance every k insertions; what k is good?
  - Rebalance when imbalance exceeds threshold.
- It's a tradeoff...
  - We either have more costly insertions
  - Or we have degraded performance for periods of time

Periodic Rebalancing


```
bstInsert(t, v):
    Input:  tree t, value v
    Output: t with v inserted

    t = insertAtLeaf(t, v)

    if size(t) mod k = 0:
        t = rebalance(t)

    return t
```

Periodic Rebalancing

Remarks

- Good if tree is not modified very often
- Otherwise...
  - Insertion will be slow occasionally due to rebalancing
  - Performance will gradually degrade until next rebalance

Global vs Local Rebalancing

GLOBAL REBALANCING

walks every node, balances its subtree;
$\Rightarrow$ perfectly balanced tree — at cost.

LOCAL REBALANCING

do small, incremental operations
to improve the overall balance of the tree
... at the cost of imperfect balance

Root Insertion

Idea:

Rotations change the structure of a tree

If we perform some rotations every time we insert,
that may restructure the tree randomly enough
such that it is more balanced

One systematic way to perform these rotations:
Insert new values at the root

Root Insertion

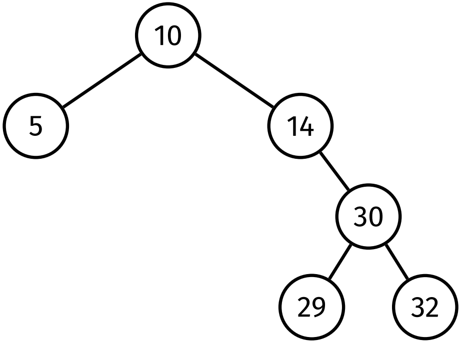
Method:

Insert new value normally (at the leaf) ...

... and then rotate the new node up to the root.

Root Insertion
Example

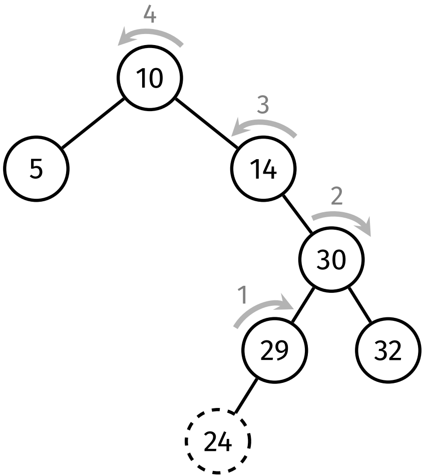
Insert 24 at the root of this tree:



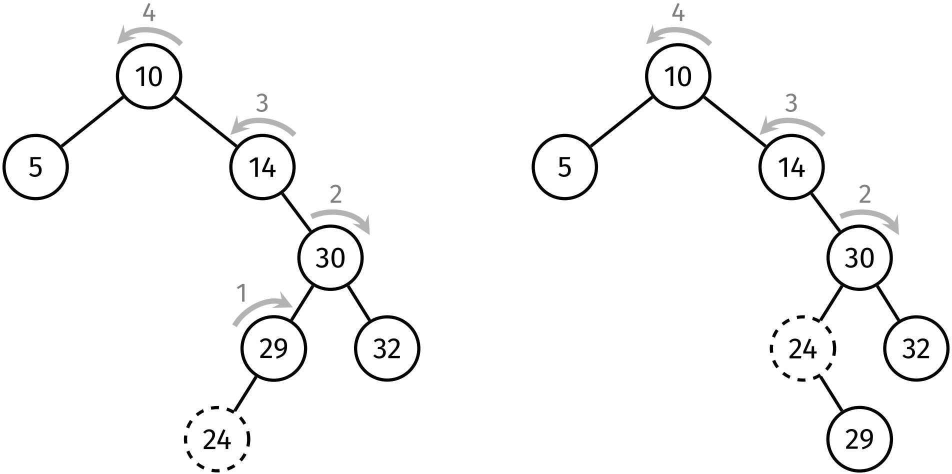
Root Insertion

Example

Insert 24 at the root of this tree:



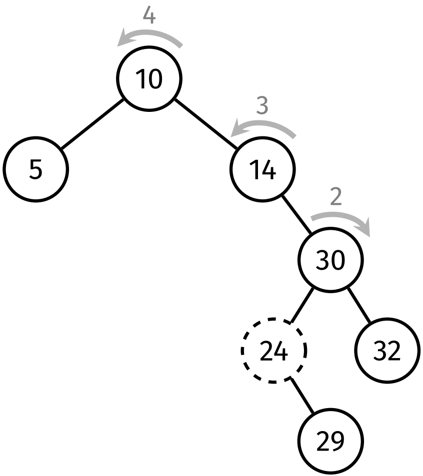
Rotate right at 29

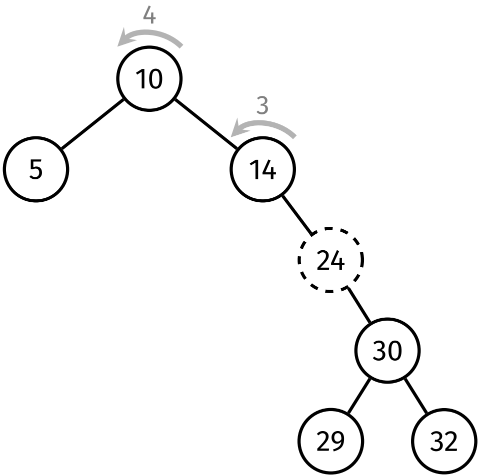


Root Insertion

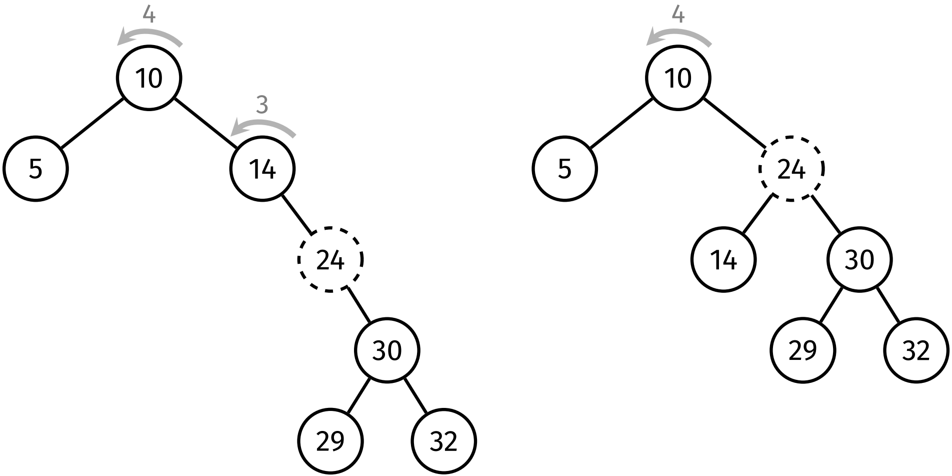
Example

Rotate right at 30





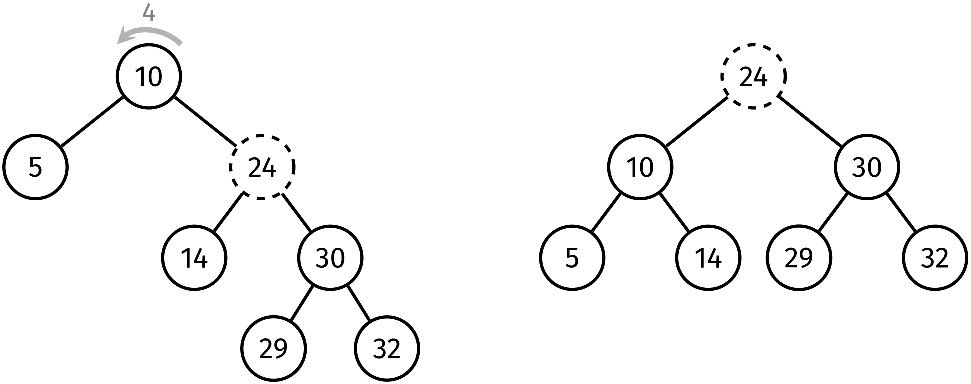
Rotate left at 14



Root Insertion

Example

Rotate left at 10



Root Insertion

Pseudocode

```
insertAtRoot(t, v):
    Input:  tree t, value v
    Output: t with v inserted at the root

    if t is empty:
        return new node containing v
    else if v < t->item:
        t->left = insertAtRoot(t->left, v)
        t = rotateRight(t)
    else if v > t->item:
        t->right = insertAtRoot(t->right, v)
        t = rotateLeft(t)

    return t
```

Analysis:

- Same time complexity as normal insertion: $O(h)$
- Tree is more likely to be balanced, but no guarantee
- Root insertion ensures recently inserted items are close to the root
  - Useful for applications where recently added items are more likely to be searched
- Major problem: ascending-ordered and descending-ordered data is still a worst case for root insertion

Randomised Insertion

BSTs don't have control over insertion order.
Worst cases — (partially) ordered data — are common.

Idea:
Introduce some randomness into insertion algorithm:
Randomly choose whether to insert normally or insert at root

Randomised Insertion

Pseudocode

```
insertRandom(t, v):
    Input:  tree t, value v
    Output: t with v inserted

    if t is empty:
        return new node containing v

    // p/q chance of inserting at root
    if random() mod q < p:
        return insertAtRoot(t, v)
    else:
        return insertAtLeaf(t, v)
```

Note: `random()` is a pseudo-random number generator
30% chance of root insertion ⇒ choose p = 3, q = 10

Randomised Insertion

Remarks

Randomised insertion creates similar results to inserting items in random order.

Tree is more likely to be balanced (but no guarantee)

Final Remarks

The balancing methods we have covered
are either inefficient (global rebalancing),
or don't guarantee a balanced tree (root/randomised insertion)

Feedback

https://forms.office.com/r/5c0fb4tvMb