COMP2521 24T1

Sorting Algorithms (II)
Elementary Sorting Algorithms

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selection sort
bubble sort
insertion sort
shell sort

Selection Sort

Method:

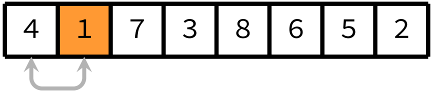
- Find the smallest element, swap it with the first element
- Find the second-smallest element, swap it with the second element
- ...
- Find the second-largest element, swap it with the second-last element

Each iteration improves the “sortedness” of the array by one element.

Example

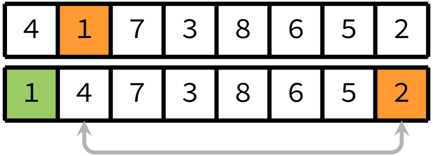
4	1	7	3	8	6	5	2

Selection Sort
Example



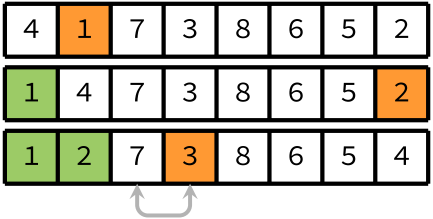
Selection Sort

Example



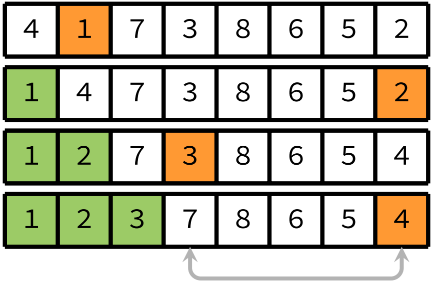
Selection Sort

Example



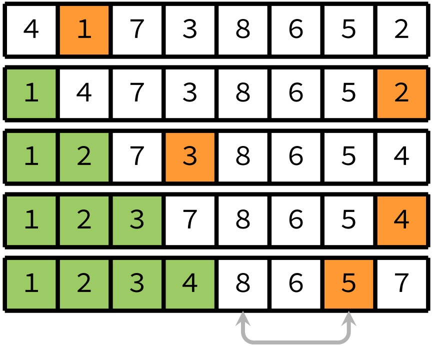
Selection Sort

Example



Selection Sort

Example



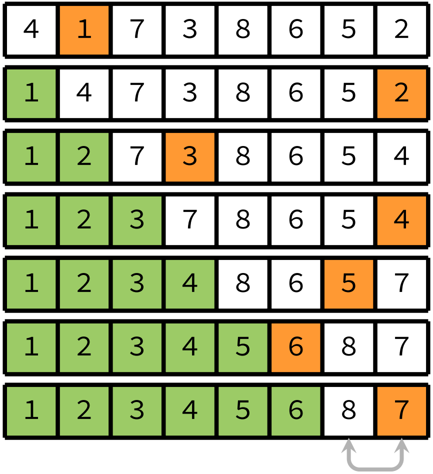
Selection Sort

Example

4	1	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	2	7	3	8	6	5	4
1	2	3	7	8	6	5	4
1	2	3	4	8	6	5	7
1	2	3	4	5	6	8	7

Selection Sort

Example



Selection Sort

Example

4	1	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	2	7	3	8	6	5	4
1	2	3	7	8	6	5	4
1	2	3	4	8	6	5	7
1	2	3	4	5	6	8	7
1	2	3	4	5	6	8	7
1	2	3	4	5	6	7	8

```c
void selectionSort(Item items[], int lo, int hi) {
    for (int i = lo; i < hi; i++) {
        int min = i;
        for (int j = i + 1; j <= hi; j++) {
            if (lt(items[j], items[min])) {
                min = j;
            }
        }
        swap(items, i, min);
    }
}
```

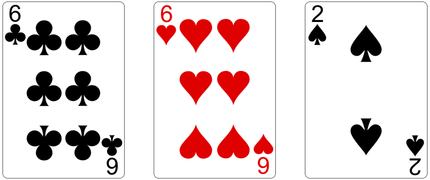
Cost analysis:

- In the first iteration, $n-1$ comparisons, 1 swap
- In the second iteration, $n-2$ comparisons, 1 swap
- ...
- In the final iteration, 1 comparison, 1 swap
- $C = (n-1) + (n-2) + \ldots + 1 = \frac{1}{2}n(n-1) \Rightarrow O(n^2)$
- $S = n - 1$

Cost is the same, regardless of the sortedness of the original array.

Selection sort is unstable

- Due to long-range swaps
- For example, sort these cards by value:



Unstable
Due to long-range swaps

Non-adaptive
Performs same steps, regardless of sortedness of original array

In-place
Sorting is done within original array; does not use temporary arrays

Bubble Sort

Method:

- Make multiple passes from left (`lo`) to right
- On each pass, swap any out-of-order adjacent pairs
- Elements “bubble up” until they meet a larger element
- Stop if there are no swaps during a pass
  - This means the array is sorted

Example

4	3	6	1	2	5

Bubble Sort
Example

First pass

4	3	6	1	2	5

Selection Sort
Bubble Sort
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Analysis
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Insertion Sort
Shell Sort
Summary
Sorting Lists
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First pass

4	3	6	1	2	5

3	4	6	1	2	5

Bubble Sort

Example

First pass

4	3	6	1	2	5

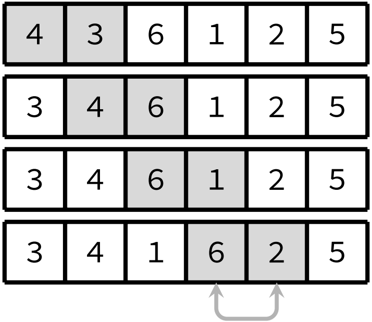
3	4	6	1	2	5

3	4	6	1	2	5

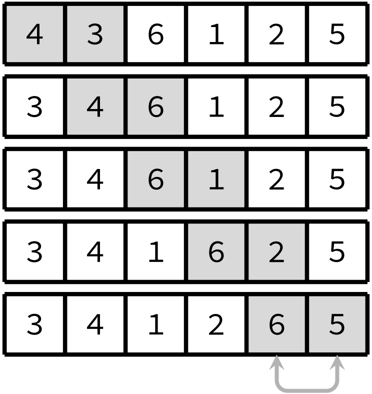
Bubble Sort

Example

First pass



First pass



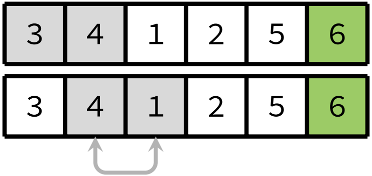
First pass

4	3	6	1	2	5
3	4	6	1	2	5
3	4	6	1	2	5
3	4	1	6	2	5
3	4	1	2	6	5
3	4	1	2	5	6

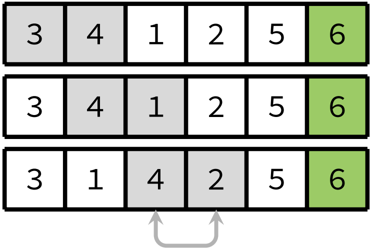
Second pass

3	4	1	2	5	6

Second pass



Second pass



Second pass

3	4	1	2	5	6
3	4	1	2	5	6
3	1	4	2	5	6
3	1	2	4	5	6

Bubble Sort

Example

Second pass

3	4	1	2	5	6

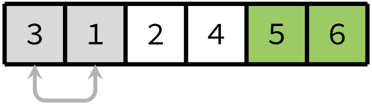
3	4	1	2	5	6

3	1	4	2	5	6

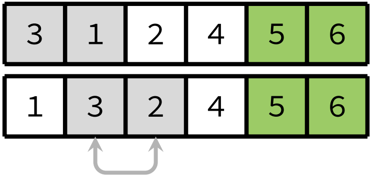
3	1	2	4	5	6

3	1	2	4	5	6

Third pass



Third pass



Third pass

3	1	2	4	5	6

1	3	2	4	5	6

1	2	3	4	5	6

Bubble Sort

Example

Third pass

3	1	2	4	5	6
1	3	2	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6

Fourth pass

1	2	3	4	5	6

Fourth pass

1	2	3	4	5	6
1	2	3	4	5	6

Bubble Sort

Example

Fourth pass

1	2	3	4	5	6

1	2	3	4	5	6

No swaps made; stop

1	2	3	4	5	6

Bubble Sort

C Implementation

```c
void bubbleSort(Item items[], int lo, int hi) {
    for (int i = hi; i > lo; i--) {
        bool swapped = false;
        for (int j = lo; j < i; j++) {
            if (gt(items[j], items[j + 1])) {
                swap(items, j, j + 1);
                swapped = true;
            }
        }
        if (!swapped) break;
    }
}
```

Best case: Array is sorted

- Only a single pass required
- $n - 1$ comparisons, no swaps
- Best-case time complexity: $O(n)$

1	2	3	4	5	6

Worst case: Array is reverse-sorted

- $n - 1$ passes required
  - First pass: $n - 1$ comparisons
  - Second pass: $n - 2$ comparisons
  - ...
  - Final pass: 1 comparison
- Total comparisons: $(n-1) + (n-2) + \ldots + 1 = \frac{1}{2}n(n-1)$
- Every comparison leads to a swap $\Rightarrow \frac{1}{2}n(n-1)$ swaps
- Worst-case time complexity: $O(n^2)$

6	5	4	3	2	1

Average-case time complexity: $O(n^2)$

- It can be proven that for a randomly ordered array, bubble sort needs to perform $\frac{1}{4}n(n-1)$ swaps on average $\Rightarrow O(n^2)$
  - See appendix for details
- Can show empirically by generating random sequences and sorting them

Bubble Sort

Properties

Stable

Comparisons are between adjacent elements only
Elements are only swapped if out of order

Adaptive

Bubble sort is $O(n^2)$ on average, $O(n)$ if input array is sorted

In-place

Sorting is done within original array; does not use temporary arrays

Insertion Sort

Method:

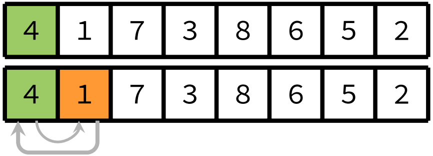
- Take first element and treat as sorted array (of length 1)
- Take next element and insert into sorted part of array so that order is preserved
  - This increases the length of the sorted part by one
- Repeat for remaining elements

Example

4	1	7	3	8	6	5	2

4	1	7	3	8	6	5	2

Insertion Sort
Example



Insertion Sort

Example

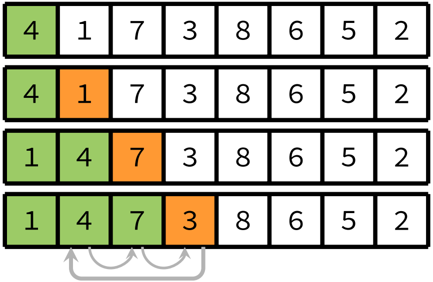
4	1	7	3	8	6	5	2

4	1	7	3	8	6	5	2

1	4	7	3	8	6	5	2

Insertion Sort

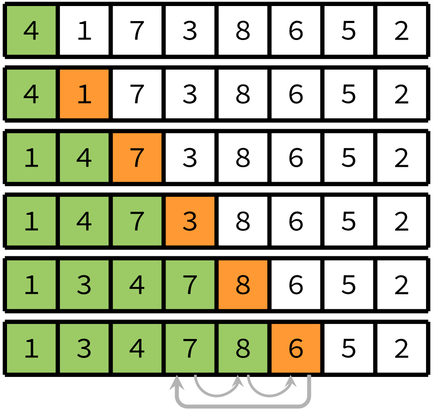
Example



4	1	7	3	8	6	5	2
4	1	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	3	4	7	8	6	5	2

Insertion Sort

Example



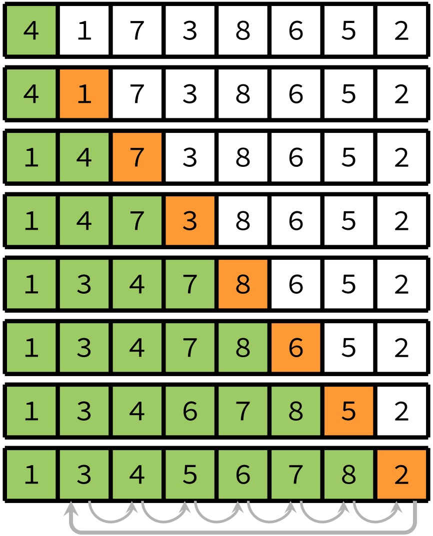
Insertion Sort

Example

4	1	7	3	8	6	5	2
4	1	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	3	4	7	8	6	5	2
1	3	4	7	8	6	5	2
1	3	4	6	7	8	5	2

Insertion Sort

Example



Insertion Sort
Example

4	1	7	3	8	6	5	2
4	1	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	4	7	3	8	6	5	2
1	3	4	7	8	6	5	2
1	3	4	7	8	6	5	2
1	3	4	6	7	8	5	2
1	3	4	5	6	7	8	2
1	2	3	4	5	6	7	8

```c
void insertionSort(Item items[], int lo, int hi) {
    for (int i = lo + 1; i <= hi; i++) {
        Item item = items[i];
        int j = i;
        for (; j > lo && lt(item, items[j - 1]); j--) {
            items[j] = items[j - 1];
        }
        items[j] = item;
    }
}
```

Best case: Array is sorted

- Inserting each element requires one comparison
- $n - 1$ comparisons
- Best-case time complexity: $O(n)$

1	2	3	4	5	6	7	8

Worst case: Array is reverse-sorted

- Inserting i-th element requires i comparisons
  - Inserting index 1 element requires 1 comparison
  - Inserting index 2 element requires 2 comparisons
  - ...
- Total comparisons: $1 + 2 + \ldots + (n-1) = \frac{1}{2}n(n-1)$
- Worst-case time complexity: $O(n^2)$

8	7	6	5	4	3	2	1

Average-case time complexity: $O(n^2)$

- Same reason as for bubble sort
- Can show empirically by generating random sequences and sorting them

Insertion Sort
Properties

Stable
Elements are always inserted to the right of any equal elements

Adaptive
Insertion sort is $O(n^2)$ on average, $O(n)$ if input array is sorted

In-place
Sorting is done within original array; does not use temporary arrays

Shell Sort

Bubble sort and insertion sort
move elements by shifting them up/down
one space at a time.

If we make longer-distance exchanges,
can we be more efficient?

What if we consider elements that are some distance apart?

Shell sort, invented by Donald Shell

Idea:

- An array is h-sorted if taking every h-th element yields a sorted array
- An h-sorted array is made up of $\frac{n}{h}$ interleaved sorted arrays
- Shell sort: h-sort the array for progressively smaller h, ending with $h = 1$

Example of h-sorted arrays:

	0	1	2	3	4	5	6	7	8	9
3-sorted	4	1	0	5	3	2	7	6	9	8

	0	1	2	3	4	5	6	7	8	9
2-sorted	1	0	3	2	4	5	7	6	9	8

	0	1	2	3	4	5	6	7	8	9
1-sorted	0	1	2	3	4	5	6	7	8	9

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2
$h = 3$ passes	3			4			5	
		1			2			8
			6			7		
3-sorted	3	1	6	4	2	7	5	8

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2
$h = 3$ passes	3			4			5	
		1			2			8
			6			7		
3-sorted	3	1	6	4	2	7	5	8
$h = 2$ passes	2		3		5		6	
		1		4		7		8
2-sorted	2	1	3	4	5	7	6	8

Shell Sort
Example

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
unsorted	4	1	7	3	8	6	5	2
$h = 3$ passes	3			4			5	
		1			2			8
			6			7		
3-sorted	3	1	6	4	2	7	5	8
$h = 2$ passes	2		3		5		6	
		1		4		7		8
2-sorted	2	1	3	4	5	7	6	8
$h = 1$ pass	1	2	3	4	5	6	7	8

Shell Sort

C Implementation

```
void shellSort(Item items[], int lo, int hi) {
    int size = hi - lo + 1;
    // find appropriate h-value to start with
    int h;
    for (h = 1; h <= (size - 1) / 9; h = (3 * h) + 1);

    for (; h > 0; h /= 3) {
        for (int i = lo + h; i <= hi; i++) {
            Item item = items[i];
            int j = i;
            for (; j >= lo + h && lt(item, items[j - h]); j -= h) {
                items[j] = items[j - h];
            }
            items[j] = item;
        }
    }
}
```

- Efficiency of shell sort depends on the h-sequence
- Effective h-sequences have been determined empirically
- Many h-sequences have been found to be $O(n^{\frac{3}{2}})$
  - For example: 1, 4, 13, 40, 121, 364, 1093, ...
    - $h_{i+1} = 3h_i + 1$
- Some h-sequences have been found to be $O(n^{\frac{4}{3}})$
  - For example: 1, 8, 23, 77, 281, 1073, 4193, ...

Unstable

Due to long-range swaps

Adaptive

Shell sort applies a generalisation of insertion sort (which is adaptive)

In-place

Sorting is done within original array; does not use temporary arrays

Summary of Elementary Sorts

	Time complexity			Properties	
	Best	Average	Worst	Stable	Adaptive
Selection sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	No	No
Bubble sort	$O(n)$	$O(n^2)$	$O(n^2)$	Yes	Yes
Insertion sort	$O(n)$	$O(n^2)$	$O(n^2)$	Yes	Yes
Shell sort	depends	depends	depends	No	Yes

Aside: Sorting Linked Lists

Selection sort:

- Let $L =$ original list, $S =$ sorted list (initially empty)
- Repeat the following until L is empty:
  - Find the node V containing the largest value in L, and unlink it
  - Insert V at the front of S

Bubble sort:

- Traverse the list, comparing adjacent values
  - If value in current node is greater than value in next node, swap values
- Repeat the above until no swaps required in one traversal

Insertion sort:

- Let $L =$ original list, $S =$ sorted list (initially empty)
- For each node in L:
  - Insert the node into S in order

Aside: Sorting Linked Lists

Shell sort:

- Difficult to implement efficiently
- Can’t access specific index in constant time
  - Have to traverse from the beginning

Feedback

https://forms.office.com/r/5c0fb4tvMb

Appendix

Bubble Sort - Proof of $O(n^2)$ Average Case

New concept: inversion

An inversion is a pair of elements from a sequence where the left element is greater than the right element.

For example, consider the following array:

4	2	1	5	3

The array contains 5 inversions:
(4, 2), (4, 1), (4, 3), (2, 1), (5, 3)

Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

Observation:

- In bubble sort, every swap reduces the number of inversions by 1

The goal of the proof: Show that the average number of inversions in a randomly sorted array is $O(n^2)$.

- This implies the number of swaps required by bubble sort is $O(n^2)$ ...
- Which implies that the average-case time complexity of bubble sort is $O(n^2)$ or slower
  - (but we know that it can't be slower than $O(n^2)$ since the worst-case time complexity of bubble sort is $O(n^2)$)

Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

In a randomly sorted array:

- The minimum possible number of inversions is 0 (sorted array)
- The maximum possible number of inversions is $\frac{1}{2}n(n-1)$ (reverse-sorted array)

Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

Let k be the number of inversions in a random permutation.
By reversing this permutation, one can obtain a permutation with $\frac{1}{2}n(n-1) - k$ inversions.

For example, suppose $n = 5$:

3	2	4	1	5	reverse →	5	1	4	2	3

4 inversions → 6 inversions

1	3	4	5	2	reverse →	2	5	4	3	1

3 inversions → 7 inversions

1	2	3	4	5	reverse →	5	4	3	2	1

0 inversions → 10 inversions

Bubble Sort - Proof of $O(n^2)$ Average Case

Continued

Thus, if we take all the possible permutations of an array and pair each permutation with its reverse, the total number of inversions in each pair is $\frac{1}{2}n(n-1)$.

This implies that the average number of inversions across all permutations is $\frac{1}{4}n(n-1)$, which is $O(n^2)$.