COMP2521 23T3
Priority Queues and Heaps

Kevin Luxa
cs2521@cse.unsw.edu.au

priority queues
binary heaps
We have learned about types of collections where items are inserted and then deleted based on insertion order.

- **stack**
  last in, first out

- **queue**
  first in, first out
There are applications where we want to process items based on priority

Examples:
Huffman coding
Dijkstra’s algorithm
Prim’s algorithm
A **priority queue** is an abstract data type where each item has an associated **priority**.

It supports the following operations:

- **insert**
  insert an item with an associated priority

- **delete**
  delete (and return) the item with the highest priority

- **peek**
  get the item with the highest priority, without deleting it

- **is empty**
  check if the priority queue is empty
Priority is often given by an integer value.

Depending on the application, either a large priority value or small priority value could be taken to mean “high priority”.

Here we’ll take a larger priority value to mean higher priority.
typedef struct pq *Pq;

/** Creates a new, empty pq */
Pq PqNew(void);

/** Frees memory allocated to a pq */
void PqFree(Pq pq);

/** Adds an item with priority to a pq */
void PqInsert(Pq pq, Item item, int priority);

/** Deletes and returns the item with the highest priority */
Item PqDelete(Pq pq);

/** Returns the item with the highest priority */
Item PqPeek(Pq pq);

/** Returns true if the pq is empty, false otherwise */
bool PqIsEmpty(Pq pq);
```c
Pq pq = PqNew();

PqInsert(pq, "alice", 4);
PqInsert(pq, "bob", 3);
PqInsert(pq, "andrew", 30);
PqInsert(pq, "jas", 35);

printf("%s\n", PqDelete(pq)); // jas
printf("%s\n", PqDelete(pq)); // andrew

PqInsert(pq, "jake", 23);
PqInsert(pq, "sasha", 25);

printf("%s\n", PqPeek(pq)); // sasha
printf("%s\n", PqDelete(pq)); // sasha
printf("%s\n", PqDelete(pq)); // jake
printf("%s\n", PqDelete(pq)); // alice
printf("%s\n", PqDelete(pq)); // bob

if (PqIsEmpty(pq)) {
    printf("the queue is empty\n");
}

PqFree(pq);
```
How to implement a priority queue?

unordered array

ordered array

linked list (unordered/ordered)
# Priority Queue

## Unordered array implementation

### unordered array

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>alice</td>
<td>bob</td>
<td>andrew</td>
<td>jas</td>
<td>jake</td>
<td>sasha</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>30</td>
<td>35</td>
<td>23</td>
<td>25</td>
</tr>
</tbody>
</table>

### Performance?

- **Insert**: $O(1)$
- **Delete**: $O(n)$
- **Peek**: $O(n)$
- **Is empty**: $O(1)$
Priority Queue
Ordered array implementation

- **Motivation**
- **Priority Queues**
- **Implementations**
- **Heaps**
- **PQ Summary**

### Ordered Array

<table>
<thead>
<tr>
<th>index (0)</th>
<th>index (1)</th>
<th>index (2)</th>
<th>index (3)</th>
<th>index (4)</th>
<th>index (5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>bob 3</td>
<td>alice 4</td>
<td>jake 23</td>
<td>sasha 25</td>
<td>andrew 30</td>
<td>jas 35</td>
</tr>
</tbody>
</table>

**Performance?**
- Insert: $O(n)$
- Delete: $O(1)$
- Peek: $O(1)$
- Is empty: $O(1)$
Unordered linked list implementation

alice 4 → bob 3 → andrew 30 → jas 35 → jake 23 → sasha 25 → NULL

Performance?
- Insert: $O(1)$
- Delete: $O(n)$
- Peek: $O(n)$
- Is empty: $O(1)$
Priority Queue

Ordered linked list implementation

Performance?
- Insert: $O(n)$
- Delete: $O(1)$
- Peek: $O(1)$
- Is empty: $O(1)$
### Summary (so far)

<table>
<thead>
<tr>
<th>Data Structure</th>
<th>Insert</th>
<th>Delete</th>
<th>Peek</th>
<th>Is Empty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unordered array</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Ordered array</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Unordered linked list</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Ordered linked list</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
A heap is a tree-based data structure which satisfies the heap property.

The heap property specifies how values in the heap should be ordered, and depends on the kind of heap:

In a max heap, the value in each node must be greater than or equal to the values in its children.

In a min heap, the value in each node must be less than or equal to the values in its children.
In this lecture we will focus on *max heaps*
(min heaps can be implemented very similarly)
There are many variants of heaps, for example:

binary heap, binomial heap, Fibonacci heap, leftist heap, pairing heap, soft heap, ...

We will consider just the binary heap.
A binary heap is a heap that takes the form of a binary tree, and satisfies the following properties:

- **Heap property**
  as defined above

- **Completeness property**
  all levels of the tree (except possibly the last) must be fully filled and the last level must be filled from left to right
Binary Heaps

Motivation
Priority
Queues
Heaps
Insertion
Deletion
PQ implementation
PQ Summary

- Binary Heaps

13

\[ \begin{array}{c}
7 \\
5 \\
4 \\
9 \\
2 \\
\end{array} \]

satisfies heap property
satisfies completeness
\Rightarrow \text{is a binary heap}

13

\[ \begin{array}{c}
7 \\
5 \\
2 \\
9 \\
8 \\
\end{array} \]

satisfies heap property
\text{does not satisfy completeness}
\Rightarrow \text{is not a binary heap}
A result of the completeness property is that binary heaps always contain $\lceil \log_2 n \rceil + 1$ levels where $n$ is the number of nodes. This will be relevant for analysis.

<table>
<thead>
<tr>
<th>$n$</th>
<th>number of levels</th>
<th>heap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>[ ]</td>
</tr>
<tr>
<td>2-3</td>
<td>2</td>
<td>[ ]</td>
</tr>
<tr>
<td>4-7</td>
<td>3</td>
<td>[ ]</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Heaps are usually implemented with an array.

For a binary heap, index 1 of the array contains the root item, the next two indices contain the root’s children, the next four indices contain the children of the root’s children, and so on.
This arrangement gives rise to a useful property:

- For an item at index \( i \):
  - Its left child is located at index \( 2i \)
  - Its right child is located at index \( 2i + 1 \)
  - Its parent is located at index \( \lfloor i/2 \rfloor \)

This makes it efficient to move “up” and “down” the tree.
Consider this max heap:
The heap as an array:
Assuming integer items:

```c
struct heap {
    int *items;
    int numItems;
    int capacity;
};
```
struct heap *heapNew(void) {
    struct heap *heap = malloc(sizeof(struct heap));

    heap->numItems = 0;
    heap->capacity = INITIAL_CAPACITY;
    heap->items = malloc((heap->capacity + 1) * sizeof(int));

    return heap;
}

Insertion is a two-step process:

1. **Add new item at next available position on bottom level** i.e., after the last item
   - New item may violate the heap property

2. **Fix up**: While new item is greater than its parent (and not at the root), swap with its parent
   - This re-organises items along the path to the root and restores the heap property
Example: Insert 26

![Binary Heap Insertion Example](image)

Insert 26 after the last item (8)

26 is greater than its parent (11) → swap

26 is greater than its parent (20) → swap

Done
Example: Insert 26

Insert 26 after the last item (8)
Example: Insert 26

Fix up

Binary Heap Insertion
Example: Insert 26

Fix up

26 is greater than its parent (11) ⇒ swap
Example: Insert 26

Fix up
26 is greater than its parent (11) ⇒ swap
Example: Insert 26

Fix up

26 is greater than its parent (20) ⇒ swap
Example: Insert 26

Fix up
26 is greater than its parent (20) ⇒ swap
Example: Insert 26

Done

![Binary Heap Insertion Diagram]

- Insert 26 after the last item (8)
- Fix up
  - 26 is greater than its parent (11) ⇒ swap
  - 26 is greater than its parent (20) ⇒ swap
- Done
Insert the following items into an initially empty max heap:

17 25 8 6 30 13
Insert the following items into an initially empty max heap:

17 25 8 6 30 13
Insert the following items into an initially empty max heap:

17 25 8 6 30 13
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

Add 17 to the heap

17

[0] [1] [2] [3] [4] [5] [6]

17
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

17 is at the root - done
Insert the following items into an initially empty max heap:

17 25 8 6 30 13
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

Add 25 after the last item
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

25 is greater than its parent (17) - swap
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

25 is greater than its parent (17) - swap
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

25 is at the root - done
Insert the following items into an initially empty max heap:

17 25 8 6 30 13
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

Add 8 after the last item
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

8 is not greater than its parent (25) - done
Insert the following items into an initially empty max heap:

17  25  8  6  30  13
Binary Heap Insertion

Example

Insert the following items into an initially empty max heap:

17 25 8 6 30 13

Add 6 after the last item
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

6 is not greater than its parent (17) - done
Insert the following items into an initially empty max heap:

17 25 8 6 30 13
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

Add 30 after the last item
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

30 is greater than its parent (17) - swap
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

30 is greater than its parent (17) - swap
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

30 is greater than its parent (25) - swap
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

30 is greater than its parent (25) - swap
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

30 is at the root - done
Insert the following items into an initially empty max heap:

17  25  8  6  30  13
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

Add 13 after the last item
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

13 is greater than its parent (8) - swap
Insert the following items into an initially empty max heap:

17 25 8 6 30 13

13 is greater than its parent (8) - swap
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

13 is not greater than its parent (30) - done
Insert the following items into an initially empty max heap:

17  25  8  6  30  13

```
    30
   /  \
  25   13
 /    /  \
6  17  8   
```

```
[0] [1] [2] [3] [4] [5] [6]
30 25 13  6 17  8 ...
```
void heapInsert(struct heap *heap, Item it) {
    if (heap->numItems == heap->capacity) {
        // resize
    }
    heap->numItems++;
    heap->items[heap->numItems] = it;
    fixUp(heap->items, heap->numItems);
}

void fixUp(Item items[], int i) {
    // while index i is not the root and
    // item at index i is greater than its parent
    while (i > 1 && items[i] > items[i / 2]) {
        swap(items, i, i / 2);
        i = i / 2;
    }
}
Binary Heap Insertion

Cost of insertion:

- Add new item after last item ⇒ $O(1)$
- Fix up considers one item on each level in the worst case
- Heap is a complete tree ⇒ $O(\log n)$ levels
- Therefore, worst-case time complexity is $O(\log n)$
Deletion is a three-step process:

1. Replace root item with last item
   - Last item = bottom-most, rightmost item
   - Let this item be \( i \)

2. Remove last item

3. Fix down: While \( i \) is less than its greater child, swap it with its greater child
   - This restores the heap property
Example: Delete from this max heap

```
20
/    \
17    11
/ \   /  \
13 1 8
```

1. Delete 20, replace with 8
2. Fix down:
   - 8 is less than its greater child (17) ⇒ swap
   - 8 is less than its greater child (13) ⇒ swap
3. Done
Example: Delete from this max heap

Delete 20, replace with 8

```
Example: Delete from this max heap

Delete 20, replace with 8

```

```
20

17

13 11

13 1 8

Example
```
Example: Delete from this max heap

Delete 20, replace with 8
Example: Delete from this max heap

Fix down

```
Example
```

```
8

17

13

11

1

20
```

```
8 is less than its greater child (17)
⇒ swap

8 is less than its greater child (13)
⇒ swap

Done
```
Example: Delete from this max heap

Fix down
8 is less than its greater child (17) ⇒ swap
Example: Delete from this max heap

Fix down

8 is less than its greater child (17) ⇒ swap
Example: Delete from this max heap

Fix down
8 is less than its greater child (13) ⇒ swap
Example: Delete from this max heap

Fix down
8 is less than its greater child (13) ⇒ swap
Example: Delete from this max heap

Done
Delete from the following max heap until it is empty:

```
30  25  17  13  8
25  13  6
6  17  8
```

```
[0]  [1]  [2]  [3]  [4]  [5]  [6]
30  25  13  6  17  8
```
Delete from the following max heap until it is empty:

30

Deleting 30
Delete from the following max heap until it is empty:

30

Replace 30 with last item (8)
Delete from the following max heap until it is empty:

30

8 is less than its greater child (25) - swap
Delete from the following max heap until it is empty:

30

8 is less than its greater child (25) - swap
Delete from the following max heap until it is empty:

30

8 is less than its greater child (17) - swap
Delete from the following max heap until it is empty:

30

8 is less than its greater child (17) - swap
Delete from the following max heap until it is empty:

30

8 is at a leaf - done

[0] 25 17 13 6 8
Delete from the following max heap until it is empty:

30  25

Deleting 25

[0]  25  17  13  6  8  [6]
Delete from the following max heap until it is empty:

30 25

Replace 25 with last item (8)
Delete from the following max heap until it is empty:

30  25

8 is less than its greater child (17) - swap
Delete from the following max heap until it is empty:

30  25

8 is less than its greater child (17) - swap
Delete from the following max heap until it is empty:

30  25

8 is not less than its greater child (6) - done
Delete from the following max heap until it is empty:

30  25  17

Deleting 17

[8  13  6]
[17]
Delete from the following max heap until it is empty:

30  25  17

Replace 17 with last item (6)
Delete from the following max heap until it is empty:

30  25  17

6 is less than its greater child (13) - swap
Delete from the following max heap until it is empty:

30  25  17

6 is less than its greater child (13) - swap
Delete from the following max heap until it is empty:

30 25 17

6 is at a leaf - done
Delete from the following max heap until it is empty:

30 25 17 13

Deleting 13
Delete from the following max heap until it is empty:

30 25 17 13

Replace 13 with last item (6)
Delete from the following max heap until it is empty:

30  25  17  13

6 is less than its greater child (8) - swap
Delete from the following max heap until it is empty:

30  25  17  13

6 is less than its greater child (8) - swap
Delete from the following max heap until it is empty:

30 25 17 13

6 is at a leaf - done
Delete from the following max heap until it is empty:

30 25 17 13 8

Deleting 8
Delete from the following max heap until it is empty:

30 25 17 13 8

Replace 8 with last item (6)
Delete from the following max heap until it is empty:

30  25  17  13  8

6 is at a leaf - done
Delete from the following max heap until it is empty:

30 25 17 13 8 6

Deleting 6
Delete from the following max heap until it is empty:

30  25  17  13  8  6

Delete 6
Delete from the following max heap until it is empty:

30 25 17 13 8 6

Heap is now empty
Item heapDelete(struct heap *heap) {
    Item item = heap->items[1];
    heap->items[1] = heap->items[heap->numItems];
    heap->numItems--;
    fixDown(heap->items, 1, heap->numItems);
    return item;
}
void fixDown(Item items[], int i, int N) {
    // while index i has at least one child
    while (2 * i <= N) {
        // let j be the index of index i's left child
        int j = 2 * i;

        // if index i's right child is greater than its left child
        if (j < N && items[j] < items[j + 1]) j++;

        // if the item at index i is greater than or equal to both children
        if (items[i] >= items[j]) break;

        swap(items, i, j);

        // move one level down the heap
        i = j;
    }
}
Cost of deletion:

- Replace root by item at end of array $\Rightarrow O(1)$
- Fix down considers two items on each level in the worst case
- Heap is a complete tree $\Rightarrow O(\log n)$ levels
- Therefore, worst-case time complexity is $O(\log n)$
struct pq {
    struct pqItem *items; // array of items
    int numItems; // number of items stored
    int capacity; // max number of items
};

struct pqItem {
    Item item;
    int priority;
};
Pq PQNew(void) {
    Pq pq = malloc(sizeof(struct pq));
    pq->numItems = 0;
    pq->capacity = INITIAL_CAPACITY;
    pq->items = malloc((pq->capacity + 1) * sizeof(struct pqItem));
    return pq;
}
void PqInsert(Pq pq, Item it, int priority) {
    if (pq->numItems == pq->capacity) {
        // resize array
    }
    pq->numItems++;
pq->items[pq->numItems] = (struct pqItem){it, priority};
    fixUp(pq->items, pq->numItems);
}

void fixUp(struct pqItem items[], int i) {
    while (i > 1 && items[i].priority > items[i / 2].priority) {
        swap(items, i, i / 2);
        i = i / 2;
    }
}
**PQ Implementation**

**Deletion**

```c
Item PqDelete(Pq pq) {
    Item item = pq->items[1].item;
    pq->items[1] = pq->items[pq->numItems];
    pq->numItems--;
    fixDown(pq->items, 1, pq->numItems);
    return item;
}

void fixDown(struct pqItem items[], int i, int N) {
    while (2 * i <= N) {
        int j = 2 * i;
        if (j < N && items[j].priority < items[j + 1].priority) j++;
        if (items[i].priority >= items[j].priority) break;
        swap(items, i, j);
        i = j;
    }
}
```
## Priority Queue ADT Summary

<table>
<thead>
<tr>
<th>Data Structure</th>
<th>Insert</th>
<th>Delete</th>
<th>Peek</th>
<th>Is Empty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unordered array</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Ordered array</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Unordered linked list</td>
<td>$O(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Ordered linked list</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Binary heap</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
https://forms.office.com/r/aPF09YHZ3X