COMP2521 23T3
Digraph Algorithms

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digraph traversal
cycle checking
warshall's algorithm

Directed Graphs (Digraphs)

Reminder: directed graphs are graphs where...

- Each edge (v, w) has a source v and a destination w
- Unlike undirected graphs, $v \to w \neq w \to v$

Digraph Applications

domain	vertex is...	edge is...
WWW	web page	hyperlink
chess	board state	legal move
scheduling	task	precedence
program	function	function call
journals	article	citation
`make`	target	dependency

Same as for undirected graphs:

```
bfs(G, src):
    initialise visited array
    mark src as visited
    enqueue src into Q
    while Q is not empty:
        v = dequeue from Q
        for each edge (v, w) in G:
            if visited[w] = false:
                mark w as visited
                enqueue w into Q
```

```
dfs(G, src):
    initialise visited array
    dfsRec(G, src, visited)

dfsRec(G, v, visited):
    mark v as visited
    for each edge (v, w) in G:
        if w has not been visited:
            dfsRec(G, w, visited)
```

Web crawling

Visit a subset of the web...
...to index
...to cache locally

Which traversal method? BFS or DFS?

Note: we can't use a visited array, as we don't know how many webpages there are. Instead, use a visited set.

Web crawling algorithm:

```
webCrawl(startingUrl, maxPagesToVisit):
    create visited set
    add startingUrl to visited set
    enqueue startingUrl into Q

    numPagesVisited = 0
    while Q is not empty and numPagesVisited < maxPagesToVisit:
        currPage = dequeue from Q

        visit currPage
        numPagesVisited = numPagesVisited + 1

        for each hyperlink on currPage:
            if hyperlink not in visited set:
                add hyperlink to visited set
                enqueue hyperlink into Q
```

Digraph Traversal

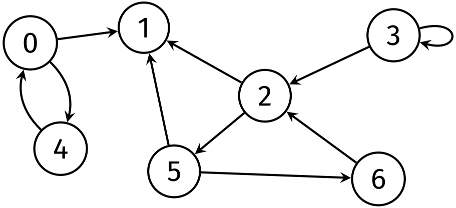
Application - Web Crawling

Wiki Game

Given two Wikipedia articles,
navigate from the first article to the second
in as few clicks as possible.

Cycle Checking

In directed graphs,
a cycle is a directed path
where the start vertex = end vertex



This graph has three distinct cycles:
0-4-0, 2-5-6-2, 3-3

Cycle Checking

Recall: Cycle checking for undirected graphs:

```
hasCycle(G):
    initialise visited array to false
    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, v, visited):
                return true

    return false

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each edge (v, w) in G:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

Does this work for directed graphs?

Cycle Checking

Recall: Cycle checking for undirected graphs:

```
hasCycle(G):
    initialise visited array to false
    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, v, visited):
                return true

    return false

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each edge (v, w) in G:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

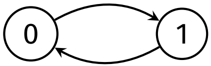
Does this work for directed graphs?

No

Cycle Checking

Problem #1

Algorithm ignores edge to previous vertex
and therefore does not detect the following cycle:



Simple fix: Don’t ignore edge to previous vertex

Cycle Checking


```
hasCycle(G):
    initialise visited array to false
    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, visited):
                return true

    return false

dfsHasCycle(G, v, visited):
    visited[v] = true

    for each edge (v, w) in G:
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, visited):
            return true

    return false
```

Does this work for directed graphs?

Cycle Checking


```
hasCycle(G):
    initialise visited array to false
    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, visited):
                return true

    return false

dfsHasCycle(G, v, visited):
    visited[v] = true

    for each edge (v, w) in G:
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, visited):
            return true

    return false
```

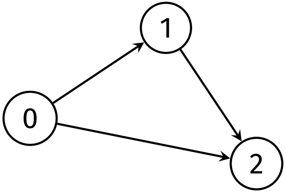
Does this work for directed graphs?

No!

Cycle Checking

Problem #2

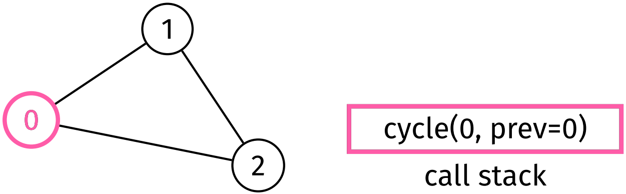
Algorithm can detect cycles when there is none, for example:



Algorithm starts at 0, recurses into 1 and 2, backtracks to 0, sees that 2 has been visited, and concludes there is a cycle

Cycle Checking

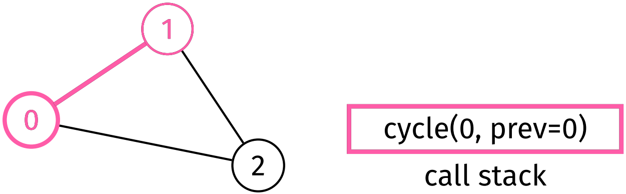
Consider a cycle check on this graph (starting at 0):



	[0]	[1]	[2]
visited	1	0	0

Cycle Checking

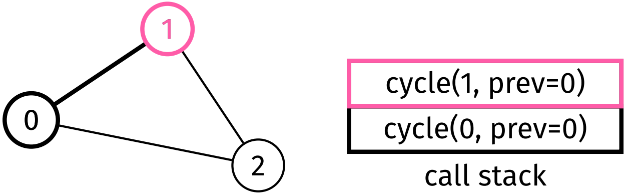
Consider a cycle check on this graph (starting at 0):



	[0]	[1]	[2]
visited	1	0	0

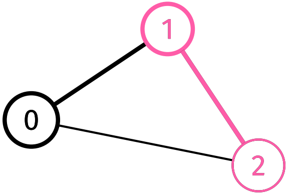
Cycle Checking

Consider a cycle check on this graph (starting at 0):



	[0]	[1]	[2]
visited	1	1	0

Consider a cycle check on this graph (starting at 0):

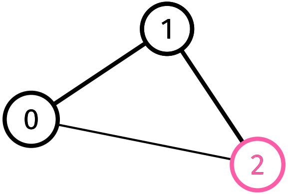


cycle(1, prev=0)
cycle(0, prev=0)

call stack

	[0]	[1]	[2]
visited	1	1	0

Consider a cycle check on this graph (starting at 0):



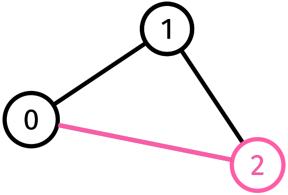
cycle(2, prev=1)
cycle(1, prev=0)
cycle(0, prev=0)

call stack

	[0]	[1]	[2]
visited	1	1	1

Cycle Checking

Consider a cycle check on this graph (starting at 0):



call stack
cycle(2, prev=1)
cycle(1, prev=0)
cycle(0, prev=0)

call stack

	[0]	[1]	[2]
visited	1	1	1

Cycle Checking

Idea:

To properly detect a cycle,
check if neighbour is already on the call stack

When the graph is undirected,
this can be done by checking the visited array,
but this doesn't work for directed graphs!

Need to use separate array to
keep track of when a vertex is on the call stack

Cycle Checking

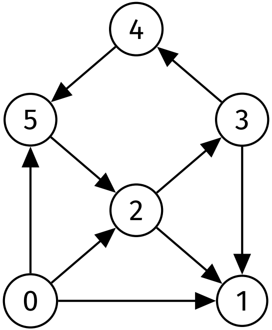
Pseudocode

```
hasCycle(G):
    create visited array, initialised to false
    create onStack array, initialised to false

    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, visited, onStack):
                return true

    return false

dfsHasCycle(G, v, visited, onStack):
    visited[v] = true
    onStack[v] = true

    for each edge (v, w) in G:
        if onStack[w] = true:
            return true
        else if visited[w] = false:
            if dfsHasCycle(G, w, visited, onStack):
                return true

    onStack[v] = false
    return false
```

Cycle Checking

Example

Check if a cycle exists in this graph:



Problem: computing reachability

Given a digraph G it is potentially useful to know:

- Is vertex t reachable from vertex s?

Transitive Closure

One way to implement a reachability check:

- Use BFS or DFS starting at s
  - This is $O(V + E)$ in the worst case
  - Only feasible if reachability is an infrequent operation

What about applications that frequently need to check reachability?

Transitive Closure

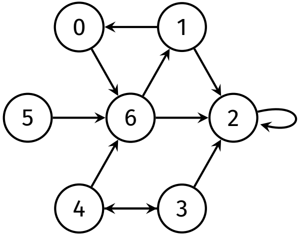
Idea

Construct a $V \times V$ matrix
that tells us whether there is a path (not edge)
from s to t, for $s, t \in V$

This matrix is called the transitive closure (tc) matrix
(or reachability matrix)

`tc`$[s]$$[t]$ is true if there is a path from s to t, false otherwise

Transitive Closure



	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	0	0	0	0	0	0	1
[1]	1	0	1	0	0	0	0
[2]	0	0	1	0	0	0	0
[3]	0	0	1	0	1	0	0
[4]	0	0	0	1	0	0	1
[5]	0	0	0	0	0	0	1
[6]	0	1	1	0	0	0	0

adjacency matrix

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	1	1	1	0	0	0	1
[1]	1	1	1	0	0	0	1
[2]	0	0	1	0	0	0	0
[3]	1	1	1	1	1	0	1
[4]	1	1	1	1	1	0	1
[5]	1	1	1	0	0	0	1
[6]	1	1	1	0	0	0	1

reachability matrix

Transitive Closure

One way to compute reachability matrix:

- Perform BFS/DFS from every vertex

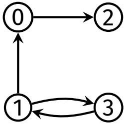
Another way $\Rightarrow$ Warshall's algorithm:

- Simple algorithm that does not require a graph traversal

Warshall's Algorithm

Main idea:

- There is a path from s to t if:
  - There is an edge from s to t, or

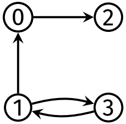


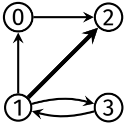
0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

Warshall's Algorithm

Main idea:

- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0, or





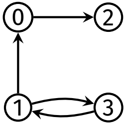
0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

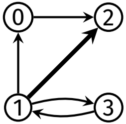
0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0

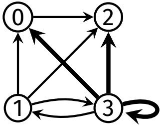
Warshall's Algorithm

Main idea:

- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0, or
  - There is a path from s to t via vertex 0 and/or 1, or







0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

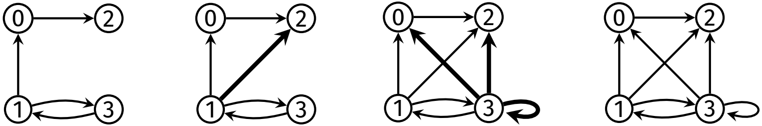
0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0

0	0	1	0
1	0	1	1
0	0	0	0
1	1	**1**	**1**

Warshall's Algorithm

Main idea:

- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0, or
  - There is a path from s to t via vertex 0 and/or 1, or
  - There is a path from s to t via vertex 0, 1 and/or 2, or



0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0

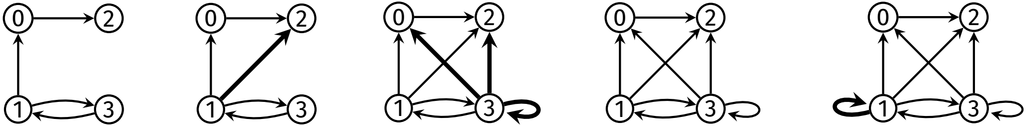
0	0	1	0
1	0	1	1
0	0	0	0
1	1	**1**	**1**

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1

Warshall's Algorithm

Main idea:

- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0, or
  - There is a path from s to t via vertex 0 and/or 1, or
  - There is a path from s to t via vertex 0, 1 and/or 2, or
  - ...
  - There is a path from s to t via any of the other vertices



0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0

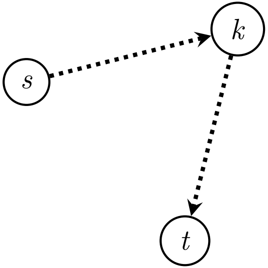
0	0	1	0
1	0	1	1
0	0	0	0
1	1	**1**	**1**

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1

0	0	1	0
1	**1**	1	1
0	0	0	0
1	1	1	1

Warshall's Algorithm

On the k-th iteration, the algorithm determines if a path exists between two vertices s and t using just 0, ... , k as intermediate vertices

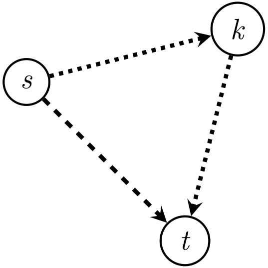


On the k-th iteration

If we have:
(1) a path from s to k
(2) a path from k to t
(using only vertices 0 to $k - 1$)

Warshall's Algorithm

On the k-th iteration, the algorithm determines if a path exists between two vertices s and t using just 0, ... , k as intermediate vertices



On the k-th iteration

If we have:
(1) a path from s to k
(2) a path from k to t
(using only vertices 0 to $k - 1$)

Then we have a path from s to t
using vertices from 0 to k

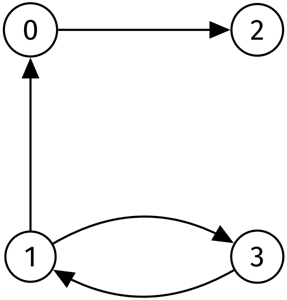
```
if tc[s][k] and tc[k][t]:
    tc[s][t] = true
```

Warshall's Algorithm

Pseudocode

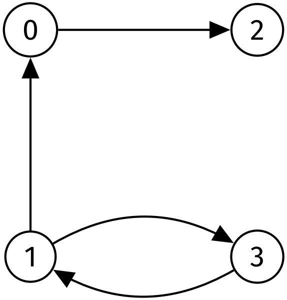

```
warshall(A):
    Inputs: n × n adjacency matrix A
    Output: n × n reachability matrix

    create tc matrix, initialised to false

    for each vertex s in G:
        for each vertex t in G:
            if A[s][t]:
                tc[s][t] = true

    for each vertex k in G: // from 0 to n - 1
        for each vertex s in G:
            for each vertex t in G:
                if tc[s][k] and tc[k][t]:
                    tc[s][t] = true

    return tc
```

Find transitive closure of this graph



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	0	1
[2]	0	0	0	0
[3]	0	1	0	0

Initialise tc with edges of original graph

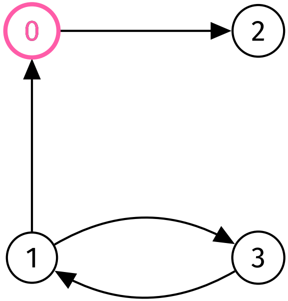


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	0	1
[2]	0	0	0	0
[3]	0	1	0	0

Warshall's Algorithm

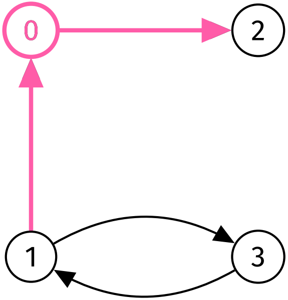
Example

First iteration: $k = 0$



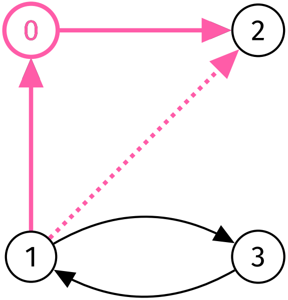
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	0	1
[2]	0	0	0	0
[3]	0	1	0	0

First iteration: $k = 0$
There is a path $1 \rightarrow 0$ and a path $0 \rightarrow 2$



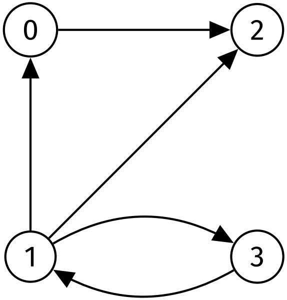
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	0	1
[2]	0	0	0	0
[3]	0	1	0	0

First iteration: $k = 0$
There is a path $1 \to 0$ and a path $0 \to 2$
So there is a path $1 \to 2$



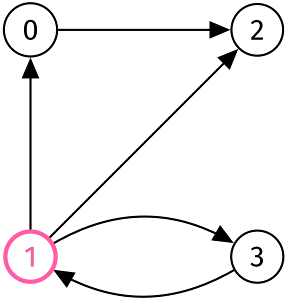
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	0	1	0	0

First iteration: $k = 0$
Done



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	0	1	0	0

Second iteration: $k = 1$

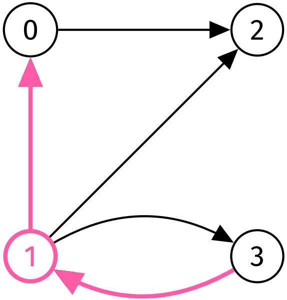


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	0	1	0	0

Warshall's Algorithm

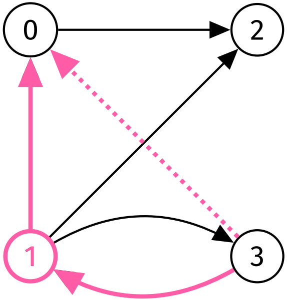
Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 0$



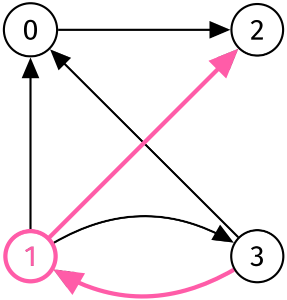
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	0	1	0	0

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 0$
So there is a path $3 \to 0$



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	0	0

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 2$

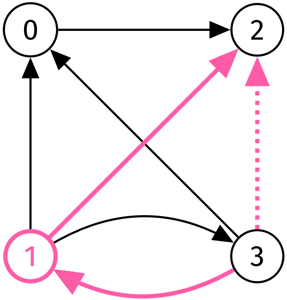


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	0	0

Warshall's Algorithm

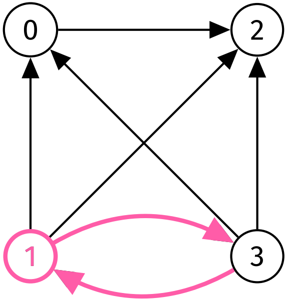
Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 2$
So there is a path $3 \to 2$



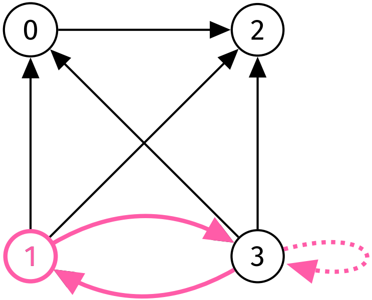
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	0

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 3$



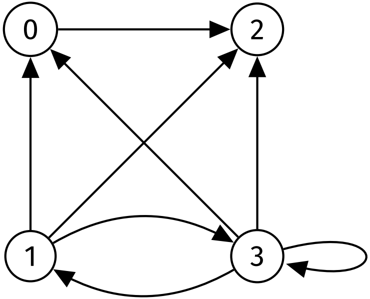
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	0

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 3$
So there is a path $3 \to 3$



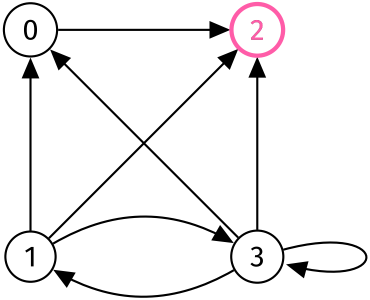
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Second iteration: $k = 1$
Done



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

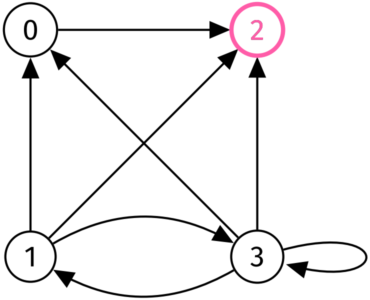
Third iteration: $k = 2$



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

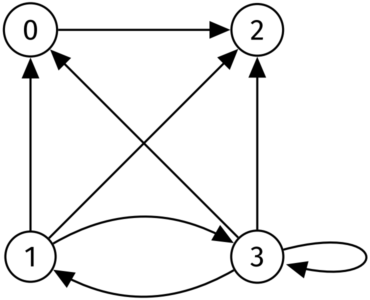
Third iteration: $k = 2$

No pairs (s, t) such that there are paths $s \to 2$ and $2 \to t$



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Third iteration: $k = 2$
Done

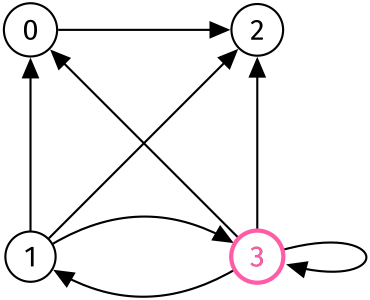


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Example

Fourth iteration: $k = 3$

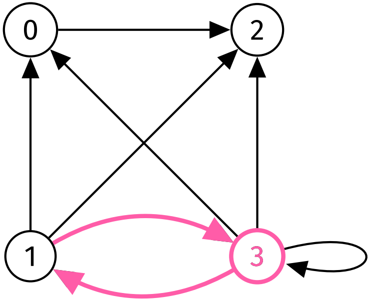


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

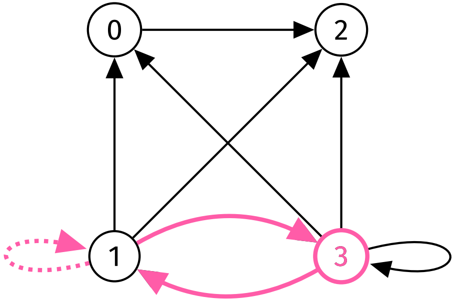
Example

Fourth iteration: $k = 3$
There is a path $1 \rightarrow 3$ and a path $3 \rightarrow 1$



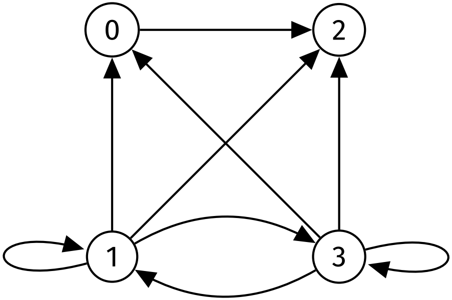
	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Fourth iteration: $k = 3$
There is a path $1 \to 3$ and a path $3 \to 1$
So there is a path $1 \to 1$



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	1	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Fourth iteration: $k = 3$
Done

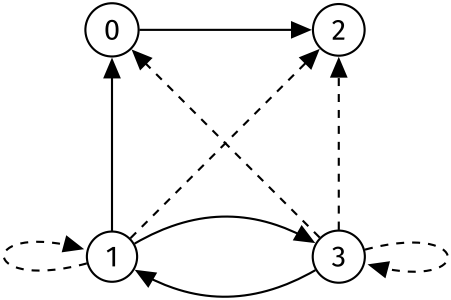


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	1	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Example

Finished



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	1	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Analysis

Analysis:

- Time complexity: $O(V^3)$
  - Three nested loops iterating over all vertices
- Space complexity: $O(V^2)$
  - Reachability matrix is $V \times V$
- Benefit: checking reachability between vertices is now $O(1)$
  - Makes up for slow setup ($O(V^3)$) if reachability is a very frequent operation

Feedback

https://forms.office.com/r/aPF09YHZ3X