# COMP2521 23T3
## Graph Problems

Kevin Luxa
cs2521@cse.unsw.edu.au

cycle checking
connected components
hamiltonian paths/circuits
euler paths/circuits

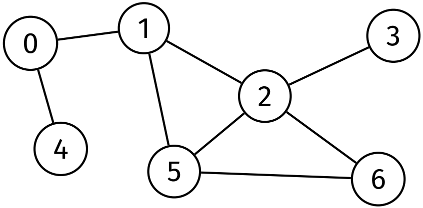
# Graph Problems

Basic graph problems:

- Is there a cycle in the graph?
- How many connected components are there in the graph?
- Is there a path that passes through all vertices?
- Is there a path that passes through all edges?

# Cycle Checking

A cycle is a path of length $> 2$
where the start vertex = end vertex
and no edge is used more than once




This graph has three distinct cycles:
1-2-5-1, 2-5-6-2, 1-2-6-5-1

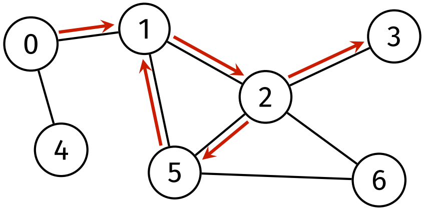
(two cycles are distinct if they have different sets of edges)

# Cycle Checking

## Attempt 1

Idea:

- Perform a DFS, starting from any vertex
- If at any point, the current vertex has an edge to an already-visited vertex, then there is a cycle



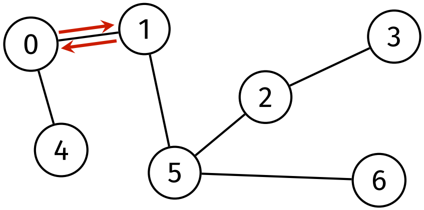
```
hasCycle(G):
    Inputs: graph G
    Output: true if G has a cycle, false otherwise

    pick any vertex v in G
    create visited array, initialised to false
    return dfsHasCycle(G, v, visited)

dfsHasCycle(G, v, visited):
    visited[v] = true

    for each neighbour w of v:
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, visited):
            return true

    return false
```

# Cycle Checking

## Attempt 1

Bug:

- The algorithm does not check whether the neighbour $w$ is the vertex that it just came from
- Therefore, it considers moving back and forth along a single edge to be a cycle (e.g., 0-1-0)



Fix:

- Keep track of previous vertex during DFS
- Ignore this vertex when considering neighbours

# Cycle Checking

## Attempt 2

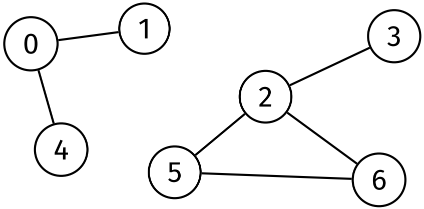
```
hasCycle(G):
    Inputs: graph G
    Output: true if G has a cycle, false otherwise

    pick any vertex v in G
    create visited array, initialised to false
    return dfsHasCycle(G, v, v, visited)

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each neighbour w of v:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

# Cycle Checking

## Attempt 2

Bug:

- The algorithm only checks one connected component
    - The connected component that the initially chosen vertex belongs to



Fix:

- After initial DFS, if any vertex has not been visited, perform DFS on that vertex
- Repeat until all vertices have been visited

```
hasCycle(G):
    Inputs: graph G
    Output: true if G has a cycle, false otherwise

    create visited array, initialised to false

    for each vertex v in G:
        if visited[v] = false:
            if dfsHasCycle(G, v, v, visited):
                return true

    return false

dfsHasCycle(G, v, prev, visited):
    visited[v] = true

    for each neighbour w of v:
        if w = prev:
            continue
        if visited[w] = true:
            return true
        else if dfsHasCycle(G, w, v, visited):
            return true

    return false
```

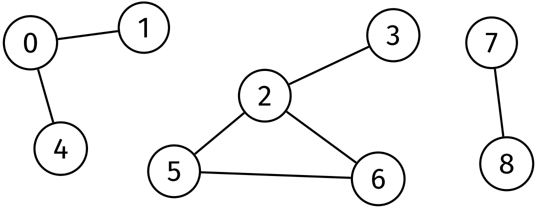
Analysis for adjacency list representation:

- Algorithm is a slight modification of DFS
- A full DFS traversal is $O(V + E)$
- Thus, worst-case time complexity of cycle checking is $O(V + E)$

# Connected Components

A connected component
is a maximally connected subgraph

For example, this graph has three connected components:



# Connected Components

## Definitions:

subgraph
a subset of vertices and edges of original graph

connected subgraph
there is a path between every pair of vertices in the subgraph

maximally connected subgraph
no way to include more edges/vertices from original graph into the subgraph such that subgraph is still connected

# Connected Components

Problems:

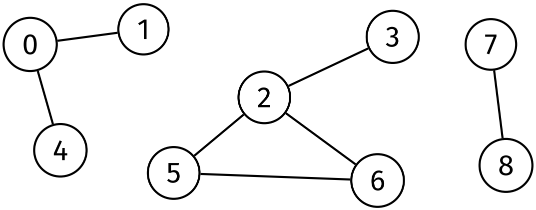
How many connected components are there?

Are two vertices in the same connected component?

# Connected Components

Idea:

- Compute an array which indicates which connected component each vertex is in
  - Let this array be called `componentOf`
  - `componentOf[`$v$`]` contains the component number of vertex $v$
- For example:




| [0] | [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 2 | 2 |

# Connected Components

Idea:

- Choose a vertex and perform DFS starting at that vertex
  - During the DFS, assign all vertices visited to component 0
- After intial DFS, if any vertex has not been assigned a component, perform DFS on that vertex
  - During this DFS, assign all vertices visited to component 1
- Repeat until all vertices are assigned a component, increasing the component number each time

# Connected Components



```
components(G):
    Inputs: graph G
    Output: componentOf array

    create componentOf array, initialised to -1

    compNo = 0
    for each vertex v in G:
        if componentOf[v] = -1:
            dfsComponents(G, v, componentOf, compNo)
            compNo = compNo + 1

    return componentOf

dfsComponents(G, v, componentOf, compNo):
    componentOf[v] = compNo
    for each neighbour w of v in G:
        if componentOf[w] = -1:
            dfsComponents(G, w, componentOf, compNo)
```

# Connected Components

## Analysis

Analysis for adjacency list representation:

- Algorithm performs a full DFS, which is $O(V + E)$

# Connected Components

Suppose we frequently need to answer the following questions:

- How many connected components are there?
- Are $v$ and $w$ in the same connected component?
- Is there a path between $v$ and $w$?

Note: The last two questions are actually equivalent in an undirected graph.

Solution:

- Cache the components array in the graph struct

```
struct graph {
    ...
    int nC;  // number of connected components
    int *cc; // componentOf array
};
```

This allows us to answer the questions very easily:

```
// How many connected components are there?
int numComponents(Graph g) {
    return g->nC;
}

// Are v and w in the same connected component?
bool inSameComponent(Graph g, Vertex v, Vertex w) {
    return g->cc[v] == g->cc[w];
}

// Is there a path between v and w?
bool hasPath(Graph g, Vertex v, Vertex w) {
    return g->cc[v] == g->cc[w];
}
```

# Connected Components

However, this information needs to be maintained as the graph changes:

- Inserting an edge may reduce nC
    - If the endpoint vertices were in different components
- Removing an edge may increase nC
    - If the endpoint vertices were in the same component *and* there is no other path between them

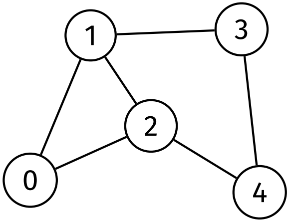
# Hamiltonian Path and Circuit

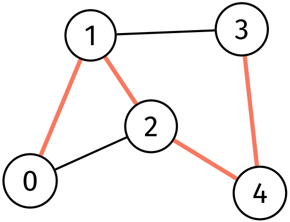
A Hamiltonian path is
a path that includes each vertex exactly once

A Hamiltonian circuit is
a cycle that includes each vertex exactly once

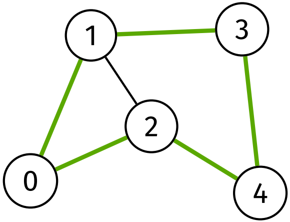
# Hamiltonian Path and Circuit

Consider the following graph:




Hamiltonian path




Hamiltonian circuit

# Hamiltonian Path

How to check if a graph has a Hamiltonian path?

Idea:

- Brute force
- Use DFS to check all possible paths
  - Recursive DFS is perfect, as it naturally allows backtracking
- Keep track of the number of vertices left to visit
- Stop when this number reaches 0

# Hamiltonian Path

```
hasHamiltonianPath(G):
    Inputs: graph G
    Output: true if G has a Hamiltonian path
            false otherwise

    create visited array, initialised to false
    for each vertex v in G:
        if dfsHamiltonianPath(G, v, visited, #vertices(G)):
            return true

    return false
```

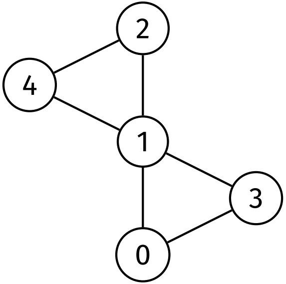
# Hamiltonian Path

```
dfsHamiltonianPath(G, v, visited, numVerticesLeft):
    visited[v] = true
    numVerticesLeft = numVerticesLeft - 1

    if numVerticesLeft = 0:
        return true

    for each neighbour w of v in G:
        if visited[w] = false:
            if dfsHamiltonianPath(G, w, visited, numVerticesLeft):
                return true

    visited[v] = false
    return false
```

# Hamiltonian Path

Why set `visited`$[v]$ to false at the end of `dfsHamiltonianPath`?

Function needs to check *all* possible paths –
if `visited`$[v]$ is not set back to false,
then function will not be able to return to $v$ when checking another path



# Hamiltonian Circuit

How to check if a graph has a Hamiltonian *circuit*?

- Similar approach as Hamiltonian path
- Don’t need to try all starting vertices
    - If the graph has a Hamiltonian circuit, then it can be found starting at any vertex
- After a Hamiltonian path is found, check if the final vertex is adjacent to the starting vertex

# Hamiltonian Circuit

```
hasHamiltonianCircuit(G):
    Inputs: graph G
    Output: true if G has a Hamiltonian circuit
            false otherwise

    create visited array, initialised to false
    return dfsHamiltonianCircuit(G, 0, visited, #vertices(G))

dfsHamiltonianCircuit(G, v, visited, numVerticesLeft):
    visited[v] = true
    numVerticesLeft = numVerticesLeft - 1

    if numVerticesLeft = 0 and adjacent(G, v, 0):
        return true

    for each neighbour w of v in G:
        if visited[w] = false:
            if dfsHamiltonianCircuit(G, w, visited, numVerticesLeft):
                return true

    visited[v] = false
    return false
```

# Hamiltonian Path and Circuit

## Analysis

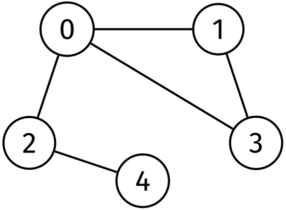
Analysis:

- Worst-case time complexity: $O(V!)$
- There are at most $V!$ paths to check ($\approx \sqrt{2\pi V}(V/e)^V$ by Stirling's approximation)
- There is no known polynomial time algorithm, so the Hamiltonian path problem is NP-hard

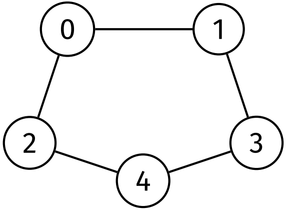
# Euler Path and Circuit

An Euler path is
a path that includes each edge exactly once

A Euler circuit is
a cycle that includes each edge exactly once




Euler path:
4-2-0-1-3-0



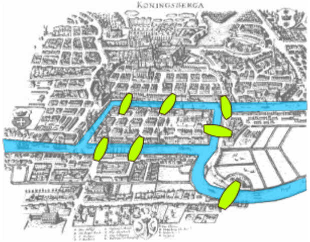

Euler circuit:
4-2-0-1-3-4

# Euler Path and Circuit

## Background

Problem is named after
Swiss mathematician, physicist, astronomer, logician and engineer
Leonhard Euler (1707-1783)

Problem was introduced by Euler while trying to solve the Seven Bridges of Konigsberg problem in 1736.

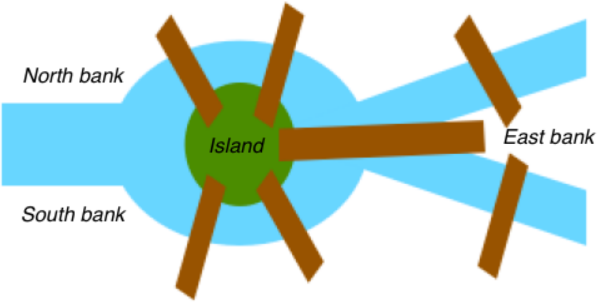



Is there a way to cross all the bridges exactly once on a walk through the town?

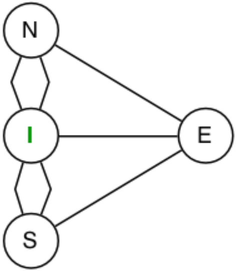
# Euler Path and Circuit

## Background

This is a graph problem:
vertices represent pieces of land
edges represent bridges




Bridges as schematic




Bridges as graph

# Euler Path and Circuit

How to check if a graph has an Euler path or circuit?

No need to try all possible paths!
Can use the following theorems:

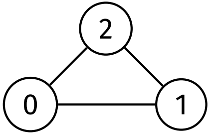
A graph has an Euler path if and only if
exactly zero or two vertices have odd degree,
and all vertices with non-zero degree belong to the same connected component

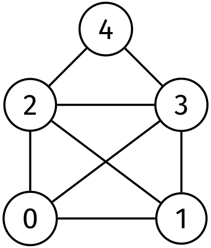
A graph has an Euler circuit if and only if
every vertex has even degree,
and all vertices with non-zero degree belong to the same connected component

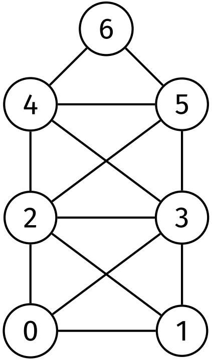
# Euler Path and Circuit



Which of these graphs have an Euler path? How about an Euler circuit?



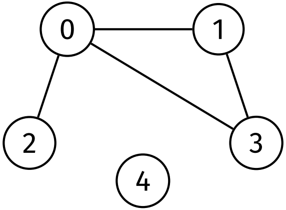



# Euler Path and Circuit

Why
"all vertices with non-zero degree belong to the same connected component"?

The following graph, even though not connected, has an Euler path:



```
hasEulerPath(G):
    Inputs: graph G
    Output: true if G has an Euler path
            false otherwise

    numOddDegree = 0
    for each vertex v in G:
        if degree(G, v) is odd:
            numOddDegree = numOddDegree + 1

    return (numOddDegree = 0 or numOddDegree = 2) and
           eulerConnected(G)
```

```
eulerConnected(G):
    Inputs: graph G
    Output: true if all vertices in G with non-zero degree
            belong to the same connected component
            false otherwise

    create visited array, initialised to false

    for each vertex v in G:
        if degree(G, v) > 0:
            dfsRec(G, v, visited)
            break

    for each vertex v in G:
        if degree(G, v) > 0 and visited[v] = false:
            return false

    return true
```

# Euler Circuit

```
hasEulerCircuit(G):
    Inputs: graph G
    Output: true if G has an Euler circuit
            false otherwise

    for each vertex v in G:
        if degree(G, v) is odd:
            return false

    return eulerConnected(G)
```

# Euler Path and Circuit

## Analysis

Analysis for adjacency list representation:

- Finding degree of every vertex is $O(V + E)$
- Checking connectivity requires a DFS which is $O(V + E)$
- Therefore, worst-case time complexity is $O(V + E)$

So unlike the Hamiltonian path problem, the Euler path problem can be solved in polynomial time (i.e., it is tractable).

# Other Graph Problems

## Tractable and Intractable



- tractable: can we find a simple path connecting two vertices in a graph?
  tractable: what's the shortest such path?
  intractable: what's the longest such path?

# Other Graph Problems

## Tractable and Intractable

- tractable: can we find a simple path connecting two vertices in a graph?
  tractable: what's the shortest such path?
  intractable: what's the longest such path?
- tractable: is there a clique in a given graph?
  intractable: what's the largest clique?

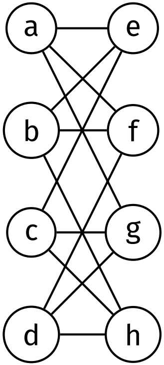
# Other Graph Problems

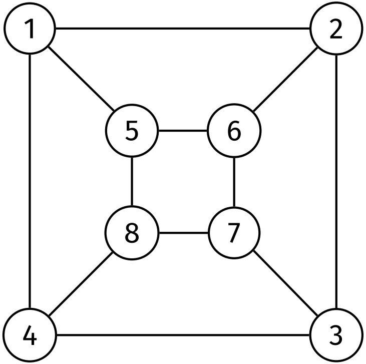
## Tractable and Intractable

- tractable: can we find a simple path connecting two vertices in a graph?
  tractable: what's the shortest such path?
  intractable: what's the longest such path?
- tractable: is there a clique in a given graph?
  intractable: what's the largest clique?
- tractable: given two colours, can we colour every vertex in a graph such that no two adjacent vertices are the same colour?
  intractable: what about three colours?

# Other Graph Problems

## Bonus Round!




Graph isomorphism:
Can we make two given graphs identical by renaming vertices?

# Feedback

https://forms.office.com/r/aPF09YHZ3X