COMP2521 23T3
Sorting Algorithms (IV)

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non-comparison-based sorts

The $n \log n$ Lower Bound

All of the sorting algorithms so far have been comparison-based sorts.

That is, they work by comparing whole keys.
Knowing how to compare whole keys is *all* they need to be able to sort.

It can be shown that these algorithms require $\Omega(n \log n)$ comparisons.
That is, they require at least $kn \log n$ comparisons for some constant k.

Why?

The $n \log n$ Lower Bound

Suppose we need to sort 3 items.

Obviously, one comparison is not sufficient to sort them.

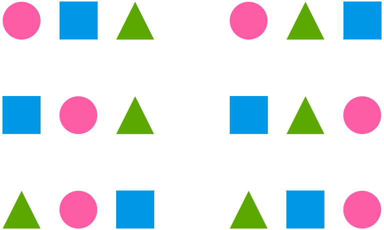
The $n \log n$ Lower Bound

Suppose we need to sort 3 items.

Even two comparisons are not sufficient to sort them. Why?

The $n \log n$ Lower Bound

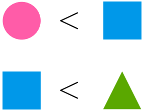
If we have 3 items, there are $3! = 6$ ways to order them:



Assuming items are unique, one of these permutations is in sorted order.

The $n \log n$ Lower Bound

Suppose we performed the following comparisons:

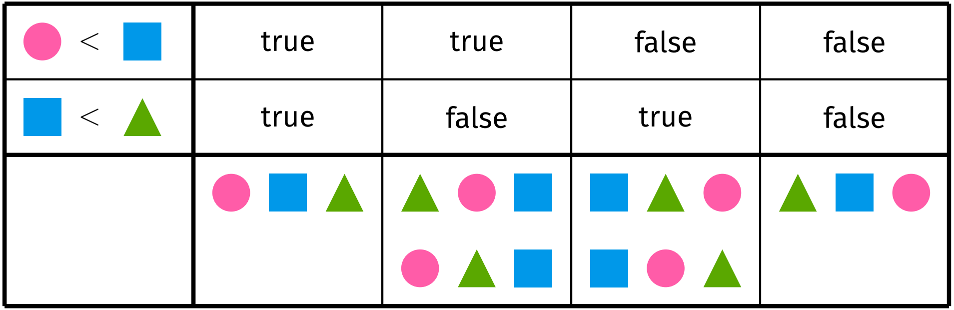


Four combinations of results are possible:

(`true`, `true`), (`true`, `false`), (`false`, `true`), (`false`, `false`)

The $n \log n$ Lower Bound

The two comparisons create four buckets, and
each permutation of items belongs to one of these buckets



The $n \log n$ Lower Bound

Mathematically,

If we have 3 items, then there are $3! = 6$ ways to order them.
In other words, 6 possible permutations.

But if we only perform 2 comparisons, then there are only $2^2 = 4$ buckets,
so at least one bucket will contain more than one permutation.

We need at least 3 comparisons, because this creates $2^3 = 8$ buckets,
so each permutation can sit in its own bucket.

If we have n items, then there are $n!$ permutations.

If we perform k comparisons, that creates up to 2^k buckets.

So given n items, we must perform enough comparisons k such that $2^k \geq n!$

The $n \log n$ Lower Bound

So given n items, we must perform enough comparisons k such that

$$2^k \geq n!$$

Taking the $\log_2$ of both sides gives

$$\log_2 2^k \geq \log_2 n!$$

Since $\log_2 2^k = k$, we get

$$k \geq \log_2 n!$$

Using Stirling's approximation, we get

$$k \geq n \log_2 n - n \log_2 e + O(\log_2 n)$$

Removing lower-order terms gives

$$k = \Omega(n \log_2 n)$$

The $n \log n$ Lower Bound

Therefore:

The theoretical lower bound on
worst-case execution time
for comparison-based sorts is $\Omega(n \log n)$.

Non-Comparison-Based Sorting

If we aren't limited to just comparing keys,
we can achieve better than $O(n \log n)$ worst-case time.

Non-comparison-based sorting algorithms exploit specific properties of the data to sort it.

Radix Sort

Radix sort is a non-comparison-based sorting algorithm.

It requires us to be able to decompose our keys into individual symbols (digits, characters, bits, etc.), for example:

- The key 372 is decomposed into (3, 7, 2)
- The key “sydney” is decomposed into (‘s’, ‘y’, ‘d’, ‘n’, ‘e’, ‘y’)

Formally, each key k is decomposed into a tuple (k_1, k_2, k_3, ..., k_m).

Radix Sort

Ideally, the range of possible symbols is reasonably small, for example:

- Numeric: 0-9
- Alphabetic: a-z

The number of possible symbols is known as the radix, and is denoted by R.

- Numeric: $R = 10$ (for base 10)
- Alphabetic: $R = 26$

If the keys have different lengths, pad them with a suitable character, for example:

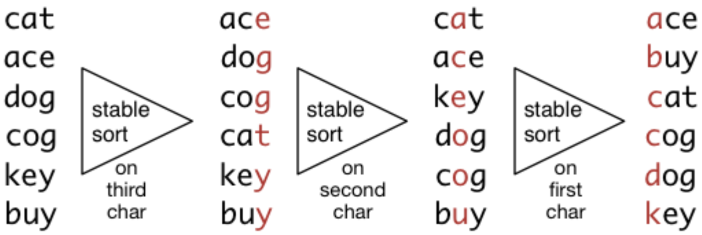
- Numeric: 123, 015, 007
- Alphabetic: "`abc`", "`zz␣`", "`t␣␣`"

Radix Sort

Method:

- Perform stable sort on k_m
- Perform stable sort on k_{m-1}
- ...
- Perform stable sort on k_1

Example:



Radix Sort

Pseudocode

```
radixSort(A):
    Input: array A of keys where
           each key consists of m symbols from an "alphabet"

    initialise m buckets // one for each symbol

    for i = m down to 1 do
        empty all buckets
        for key in A do
            append key to bucket key[i]
        end for

        clear A
        for each bucket (in order) do
            for each key in bucket do
                append key to A
            end for
        end for
    end for
```

Assume alphabet is {‘a’, ‘b’, ‘c’}, so $R = 3$.

We want to sort the array:

[“abc”, “cab”, “baa”, “a”, “ca”]

First, pad keys with blank characters:

[“abc”, “cab”, “baa”, “a␣␣”, “ca␣”]

Each key contains three characters, so $m = 3$.

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c

Radix Sort

Example

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c

Radix Sort
Example

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c

Radix Sort

Example

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c
			"abc"

Radix Sort
Example

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c
			"abc"

Radix Sort
Example

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c
		"cab"	"abc"

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c
		"cab"	"abc"

Radix Sort

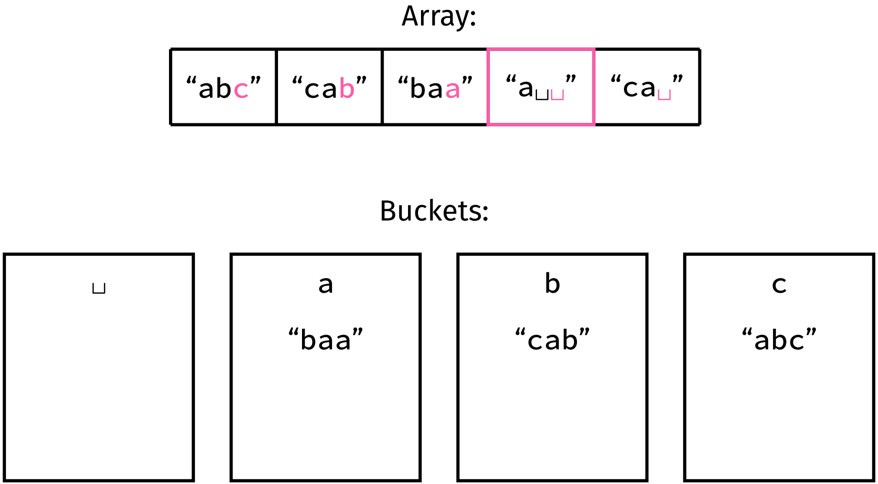
Example

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c
	"baa"	"cab"	"abc"


Array:
“abc”
“cab”
“baa”
“a␣␣”
“ca␣”
Buckets:
␣
a
“baa”
b
“cab”
c
“abc”

Radix Sort
Example

Array:

“abc”	“cab”	“baa”	“a␣␣”	“ca␣”

Buckets:

␣: “a␣␣”

a: “baa”

b: “cab”

c: “abc”

Array:

"abc"	"cab"	"baa"	"a␣␣"	"ca␣"

Buckets:

␣	a	b	c
"a␣␣"	"baa"	"cab"	"abc"

Radix Sort

Example

Array:

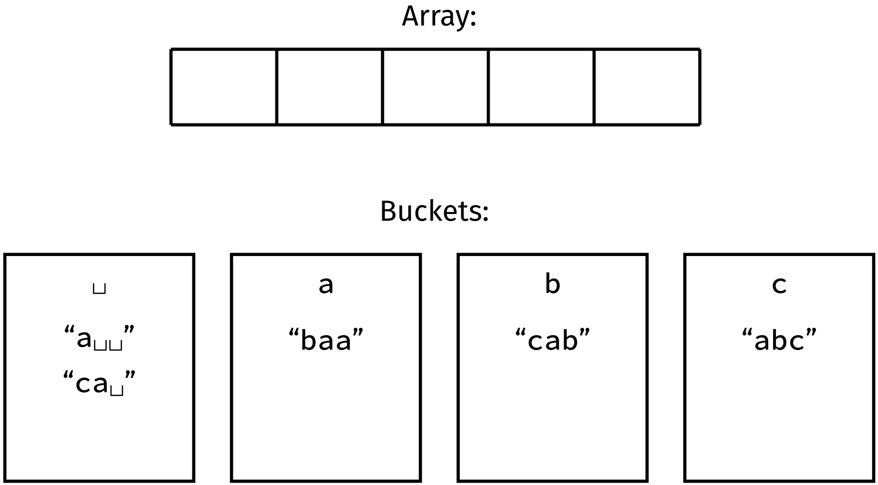
“abc”	“cab”	“baa”	“a␣␣”	“ca␣”

Buckets:

␣	a	b	c
“a␣␣”	“baa”	“cab”	“abc”
“ca␣”			

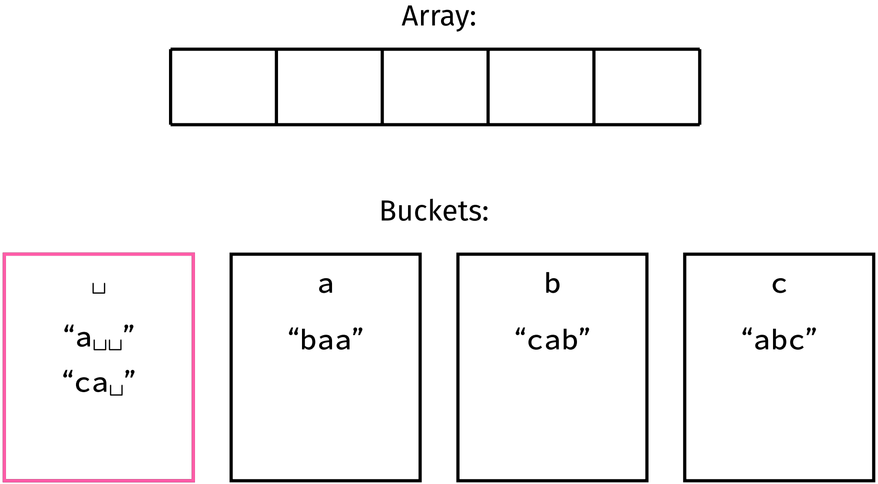
Radix Sort

Example



Radix Sort

Example



Radix Sort
Example

Array:

"a⊔⊔"				

Buckets:

⊔	a	b	c
"a⊔⊔"	"baa"	"cab"	"abc"
"ca⊔"			

Array:

"a␣␣"	"ca␣"			

Buckets:

␣	a	b	c
"a␣␣"	"baa"	"cab"	"abc"
"ca␣"			

Radix Sort

Example

Array:

“a␣␣”	“ca␣”			

Buckets:

␣	a	b	c
“a␣␣”	“baa”	“cab”	“abc”
“ca␣”			

Radix Sort

Example

Array:

"a␣␣"	"ca␣"	"baa"		

Buckets:

␣	a	b	c
"a␣␣"	"baa"	"cab"	"abc"
"ca␣"			

Radix Sort

Example

Array:

“a␣␣”	“ca␣”	“baa”		

Buckets:

␣	a	b	c
“a␣␣”	“baa”	“cab”	“abc”
“ca␣”			

Radix Sort
Example

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	

Buckets:

␣	a	b	c
“a␣␣”	“baa”	“cab”	“abc”
“ca␣”			

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	

Buckets:

␣	a	b	c
"a␣␣"	"baa"	"cab"	"abc"
"ca␣"			

Radix Sort

Example

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
"a␣␣"	"baa"	"cab"	"abc"
"ca␣"			

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
"a␣␣"			

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣: "a␣␣"

a:

b:

c:

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"		

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
“a␣␣”	“ca␣”		

Radix Sort

Example

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"		
	"baa"		

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"		
	"baa"		

Radix Sort

Example

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
“a␣␣”	“ca␣”		
	“baa”		
	“cab”		

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣
- “a␣␣”

a
- “ca␣”
- “baa”
- “cab”

b

c

Radix Sort
Example

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"	"abc"	
	"baa"		
	"cab"		

Radix Sort

Example

Array:

Buckets:

␣	a	b	c
“a␣␣”	“ca␣”	“abc”	
	“baa”		
	“cab”		

Array:

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"	"abc"	
	"baa"		
	"cab"		

Radix Sort

Example

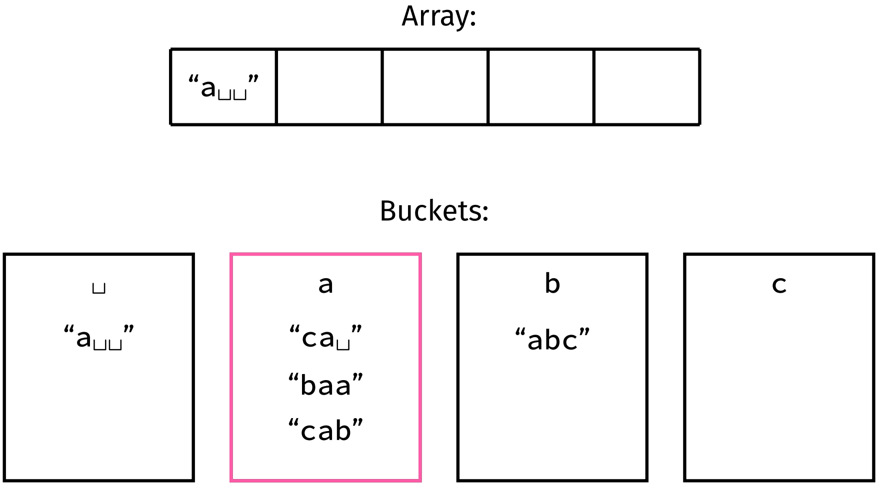
Array:

"a␣␣"				

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"	"abc"	
	"baa"		
	"cab"		

Radix Sort
Example



Array:

“a␣␣”	“ca␣”			

Buckets:

␣	a	b	c
“a␣␣”	“ca␣”	“abc”	
	“baa”		
	“cab”		

Array:

“a␣␣”	“ca␣”	“baa”		

Buckets:

␣	a	b	c
“a␣␣”	“ca␣”	“abc”	
	“baa”		
	“cab”		

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	

Buckets:

␣	a	b	c
“a␣␣”	“ca␣”	“abc”	
	“baa”		
	“cab”		

Radix Sort
Example

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"	"abc"	
	"baa"		
	"cab"		

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
"a␣␣"	"ca␣"	"abc"	
	"baa"		
	"cab"		

Radix Sort

Example

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
“a␣␣”	“ca␣”	“abc”	
	“baa”		
	“cab”		

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c

Radix Sort

Example

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c

Radix Sort

Example

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

- ␣
- a
- b
- c

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
	“a␣␣”		

Radix Sort

Example

Array:

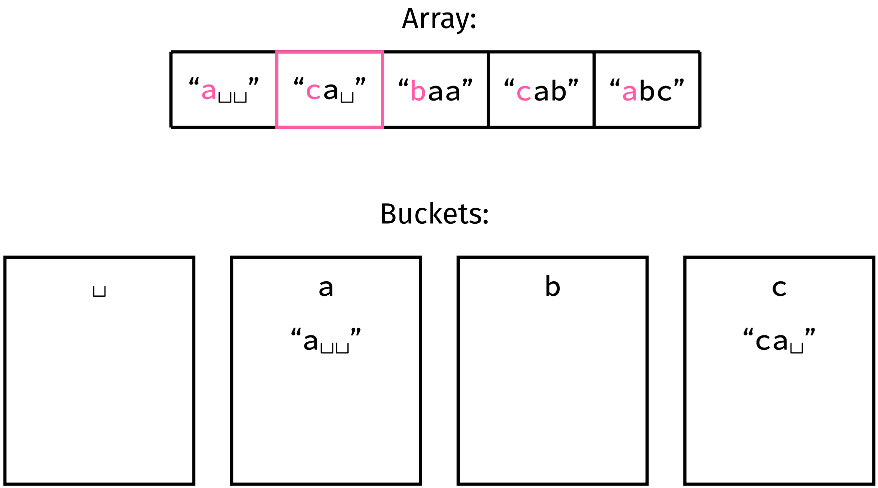
"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
	"a␣␣"		

Radix Sort

Example



Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
	“a␣␣”		“ca␣”

Radix Sort

Example

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
	“a␣␣”	“baa”	“ca␣”

Radix Sort

Example

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
	“a␣␣”	“baa”	“ca␣”

Radix Sort

Example

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
			"cab"

Radix Sort
Example

Array:

“a␣␣”	“ca␣”	“baa”	“cab”	“abc”

Buckets:

␣	a	b	c
	“a␣␣”	“baa”	“ca␣”
			“cab”

Array:

"a␣␣"	"ca␣"	"baa"	"cab"	"abc"

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
	"abc"		"cab"

Radix Sort
Example

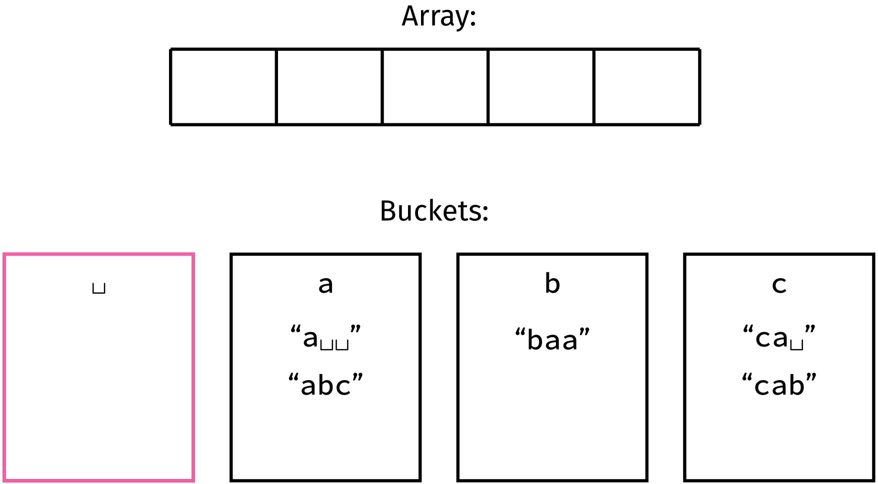
Array:

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
	"abc"		"cab"

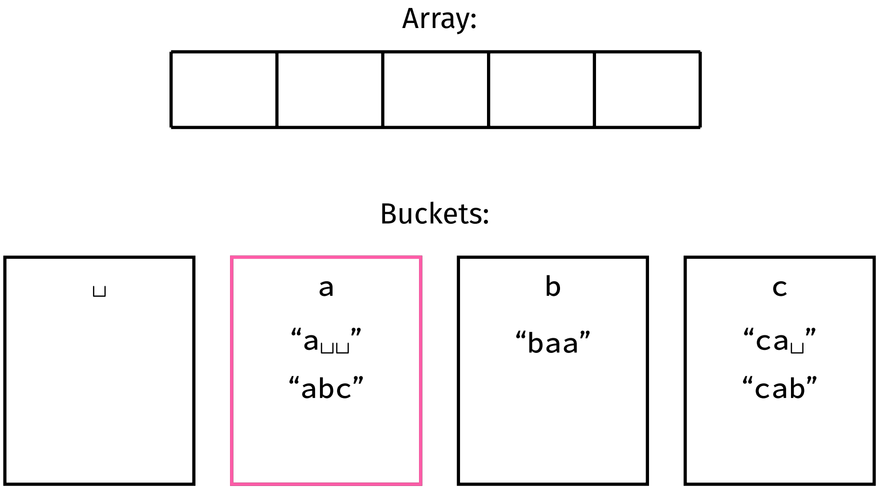
Radix Sort

Example



Radix Sort

Example



Radix Sort

Example

Array:

"a␣␣"				

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
	"abc"		"cab"

Array:

"a␣␣"	"abc"			

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
	"abc"		"cab"

Array:

"a␣␣"	"abc"			

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
	"abc"		"cab"

Array:

"a␣␣"	"abc"	"baa"		

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
	"abc"		"cab"

Array:

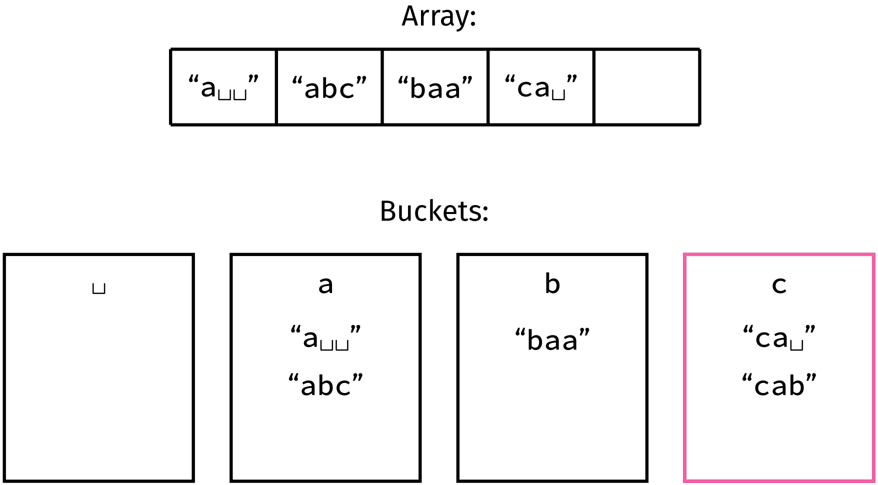
"a␣␣"	"abc"	"baa"		

Buckets:

␣	a	b	c
	"a␣␣"	"baa"	"ca␣"
	"abc"		"cab"

Radix Sort

Example



Array:

"a␣␣"	"abc"	"baa"	"ca␣"	"cab"

Buckets:

␣

a
"a␣␣"
"abc"

b
"baa"

c
"ca␣"
"cab"

Radix Sort

Example

Array:

"a␣␣"	"abc"	"baa"	"ca␣"	"cab"

Buckets:

␣	a	b	c

Radix Sort

Analysis

Analysis:

- Array contains n keys
- Each key contains m symbols
- Radix sort uses R buckets
- A single stable sort runs in time $O(n + R)$
- Radix sort uses stable sort m times

Hence, time complexity for radix sort is $O(m(n + R))$.

- $\approx O(mn)$, assuming R is small

Therefore, radix sort performs better than comparison-based sorting algorithms:

- When keys are short (i.e., m is small) and arrays are large (i.e., n is large)

Stable
All sub-sorts performed are stable

Non-adaptive
Same steps performed, regardless of sortedness

Not in-place
Uses $O(R + n)$ additional space for buckets
and storing keys in buckets

Other Non-Comparison-Based Sorts

- Bucket sort
- MSD Radix Sort
  - The version shown was LSD
- Key-indexed counting sort
- ...and others

Feedback

https://forms.office.com/r/aPF09YHZ3X