# COMP2521 23T3
## Analysis of Algorithms

Kevin Luxa

cs2521@cse.unsw.edu.au

# Motivation

- Program runtime is critical for many applications:
    - Finance, robotics, games, database systems, ...
- We may want to compare programs to decide which one to use
- We may want to determine whether a program will be ”fast enough”

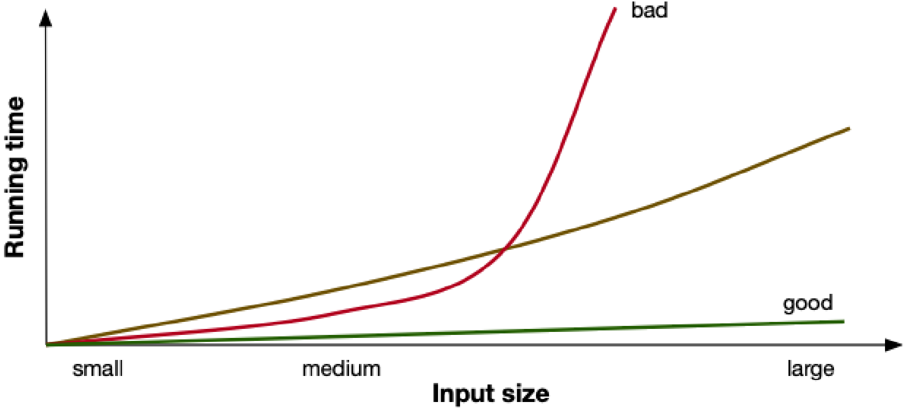
# Program Efficiency

What determines how fast a program runs?

- The operating system?
- Compilers?
- Hardware?
  - E.g., CPU, GPU, cache
- Load on the machine?
- Most important: the data structures and algorithms used

# Algorithm Efficiency

- The running time of an algorithm tends to be a function of input size
- Typically: larger input ⇒ longer running time
  - Small inputs: fast running time , regardless of algorithm
  - Larger inputs: slower, but how much slower?



# What to Analyse?

- Best-case performance
    - Not very useful
    - Usually only occurs for specific types of input
- Average-case performance
    - Difficult; need to know how the program is used
- Worst-case performance
    - Most important; determines how long the program could possibly run

# How to Analyse?

Efficiency of an algorithm can be investigated in two ways:

- Empirically: Measuring the time that a program implementing the algorithm takes to run
- Theoretically: Counting the number of basic operations performed by the algorithm as a function of input size

# Empirical Analysis

1. Write a program that implements the algorithm
2. Run the program with inputs of varying size and composition
3. Measure the runtime
4. Plot the results

# Empirical Analysis

## How to Measure Running Time

We can measure running time of a program using the `time` command.

The `time` command produces three times:

- real: total elapsed time
- user: CPU time spent executing program code
- sys: CPU time spent by the operating system on behalf of the program
    - e.g., opening a file

Example:

```
$ time ./prog
real    0m0.440s
user    0m0.380s
sys     0m0.000s
```

# Timing Execution

Absolute times will differ
between machines, between languages
...so we're not interested in absolute time.

We are interested in the *relative* change
as the input size increases

# Empirical Analysis

Let’s empirically analyse the following search algorithm:

```
// Returns the index of the given key in the array if it exists,
// or -1 otherwise
int linearSearch(int arr[], int size, int key) {
    for (int i = 0; i < size; i++) {
        if (arr[i] == key) {
            return i;
        }
    }
    return -1;
}
```

# Empirical Analysis

Sample results:

| Input Size | Runtime |
|---:|---:|
| 1,000,000 | 0.005 |
| 10,000,000 | 0.028 |
| 100,000,000 | 0.246 |
| 1,000,000,000 | 2.437 |

Conclusion: The worst-case runtime of linear search appears to grow linearly as input size increases.

# Limitations of Empirical Analysis

- Requires implementation of algorithm, which may be difficult
- Different choice of input data ⇒ different results
    - Choosing good inputs is extremely important
- Timing results affected by runtime environment
    - E.g., load on the machine
- In order to compare two algorithms...
    - Need ”comparable” implementation of each algorithm
    - Must use same inputs, same hardware, same O/S, same load

# Theoretical Analysis

- Uses high-level description of algorithm (pseudocode)
  - Can use the code if it is implemented already
- Characterises runtime as a function of input size
- Takes into account all possible inputs
- Allows us to evaluate the efficiency of the algorithm
  - Independent of the hardware/software environment

# Pseudocode

- Pseudocode is a plain language description of the steps in an algorithm
- Uses structural conventions of a regular programming language
    - if statements, loops
- Omits language-specific details
    - variable declarations

# Pseudocode

Pseudocode for linear search:

```
linearSearch(A, key):
    Input:  array A of n integers
    Output: index of key in A if it exists, otherwise -1

    for i from 0 up to n - 1 do
        if A[i] = key then
            return i
        end if
    end for

    return -1
```

# Primitive Operations

Every algorithm uses a core set of basic operations.

Examples:

- Assignment
- Indexing into an array
- Calling/returning from a function
- Evaluating an expression
- Increment/decrement

We call these operations **primitive** operations.

Assume that primitive operations take the same constant amount of time.

# Counting Primitive Operations

By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm as a function of the input size.

```
linearSearch(A, key):
    Input:  array A of n integers
    Output: index of key in A if it exists, otherwise -1

    for i from 0 up to n - 1 do      1 + (n + 1) + n
        if A[i] = key then           2n
            return i
        end if
    end for

    return -1                        1
                                     ------
                                     4n + 3
```

Note: Assuming that the for loop is implemented as
`for (int i = 0; i < n; i++)`
There is 1 assignment, n increments, and (n + 1) checks of the condition.

# Estimating Running Times

Linear search requires $4n + 3$ primitive operations in the worst case

If the time taken by a primitive operation is $c$, then the worst-case running time of linear search is $c(4n + 3)$.

Hence, the worst-case running time of linear search is linear in $n$.

# Estimating Running Times

## Lower-Order Terms

We are interested in how the running time of the algorithm changes as the input size is scaled:

- E.g., if we double the input size, how does the running time change?

As the input size increases, lower-order terms become less significant.

- For example, suppose the running time of an algorithm is $4n + 3$.
- As $n$ increases, the lower-order term (i.e., $3$) becomes less significant (i.e., becomes a smaller proportion of the running time)

# Estimating Running Times

## Constant Factors

Growth rate is not affected by constant factors.

Example: Suppose the running time $T(n)$ of an algorithm is $n^2$.

- What happens when we double the input size?

$$\begin{aligned} T(2n) &= (2n)^2 \\ &= 4n^2 \\ &= 4\,T(n) \end{aligned}$$

When we double the input size, the running time quadruples.

# Estimating Running Times

## Constant Factors

Example: Now suppose the running time $T(n)$ of an algorithm is $10n^2$.

- Now what happens when we double the input size?

$$\begin{aligned} T(2n) &= 10 \times (2n)^2 \\ &= 10 \times 4n^2 \\ &= 4 \times 10n^2 \\ &= 4T(n) \end{aligned}$$

When we double the input size, the running time also quadruples!

# Estimating Running Times

To summarise:

- Lower-order terms become insignificant as $n$ increases
- Growth rate is unaffected by constant factors

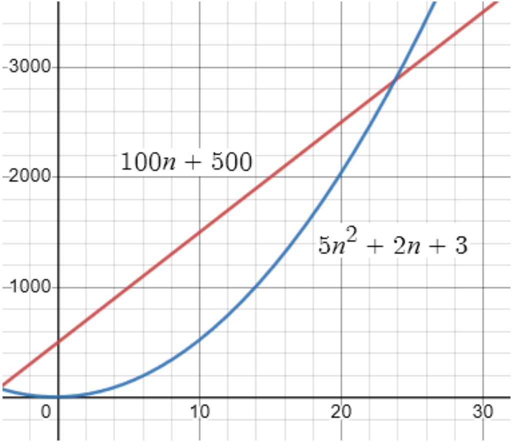
This means we can ignore lower-order terms and constant factors when characterising the growth rate of the running time of an algorithm.

Examples:

- If $T(n) = 100n + 500$, ignoring lower-order terms and constant factors gives $n$
- If $T(n) = 5n^2 + 2n + 3$, ignoring lower-order terms and constant factors gives $n^2$

# Estimating Running Times

This also means that for sufficiently large inputs, the algorithm that has the running time with the highest-order term will always take longer.



# Estimating Running Times

Discarding lower-order terms and constant factors...

- Allows us to easily compare the efficiency of algorithms
  - For example, if after discarding lower-order terms and constant factors, algorithm A has a running time of $n$ and algorithm B has a running time of $n^2$, then we can say that for sufficiently large inputs, algorithm A will perform better.

# Estimating Running Times

Since growth rate is not affected by constant factors, instead of counting primitive operations, we can simply count line executions.

This is because each line of code contains only a constant number of primitive operations.

```
linearSearch(A, key):
    Input:  array A of n integers
    Output: index of key in A if it exists, otherwise -1

    for i from 0 up to n - 1 do      n
        if A[i] = key then           n
            return i
        end if
    end for

    return -1                        1
                                     ------
                                     2n + 1
```

# Big-Oh Notation

Big-Oh is a notation used to describe the asymptotic relationship between functions.

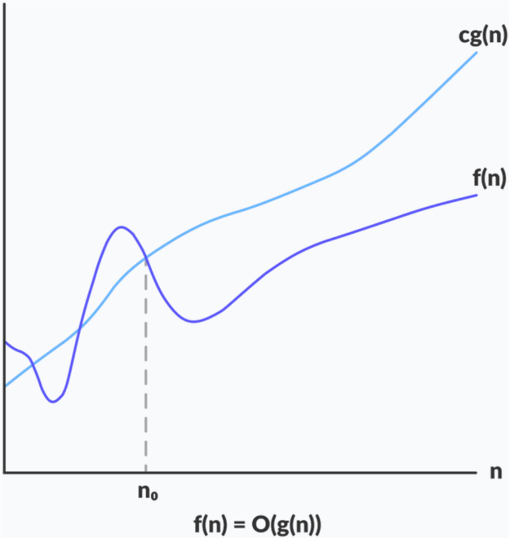
**Formally:**
Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if:

- There are positive constants $c$ and $n_0$ such that:
  - $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

**Informally:**
Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if for sufficiently large $n$, $f(n)$ is bounded above by some multiple of $g(n)$.

# Big-Oh Notation

## Example 1



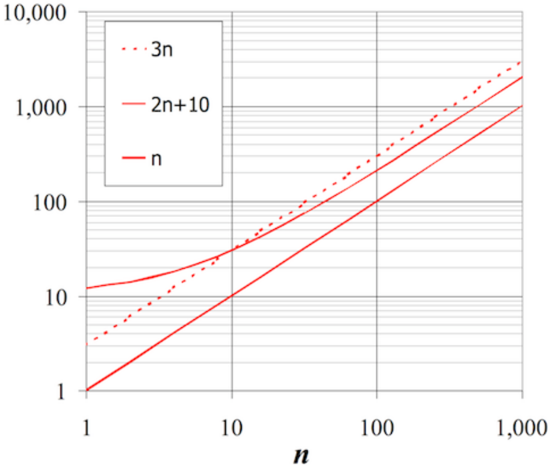
# Big-Oh Notation

## Example 2

Consider the functions $f(n) = 2n + 10$ and $g(n) = n$.
We want to show that $f(n)$ is $O(g(n))$, i.e., that $2n + 10$ is $O(n)$.

We need to find some $c$ such that for sufficiently large $n$, $2n + 10 \leq c \cdot n$.

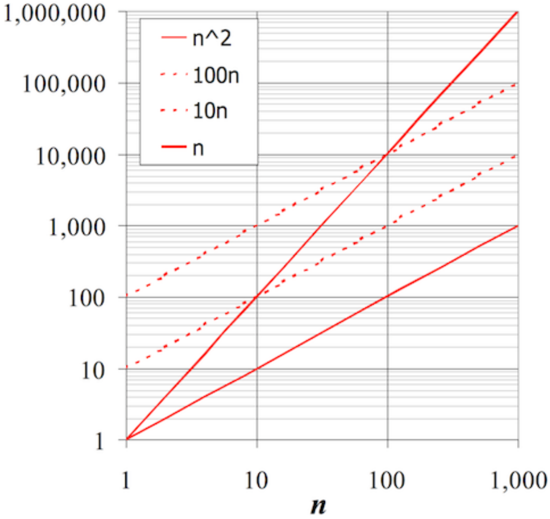
Yes! For $c = 3$, $2n + 10 \leq 3n$ when $n \geq 10$.



# Big-Oh Notation

## Example 3

Consider the functions $f(n) = n^2$ and $g(n) = n$.
We want to show that $f(n)$ is $O(g(n))$, i.e., that $n^2$ is $O(n)$.



We need to find some $c$ such that for sufficiently large $n$, $n^2 \leq c \cdot n$.

Impossible!

# Time Complexity

Time complexity is the amount of time taken by an algorithm to run, *as a function of the input*.

In this course, we usually express time complexity using big-Oh notation. For example, linear search is $O(n)$ in the worst case.

# Time Complexity

To determine the worst-case time complexity of an algorithm:

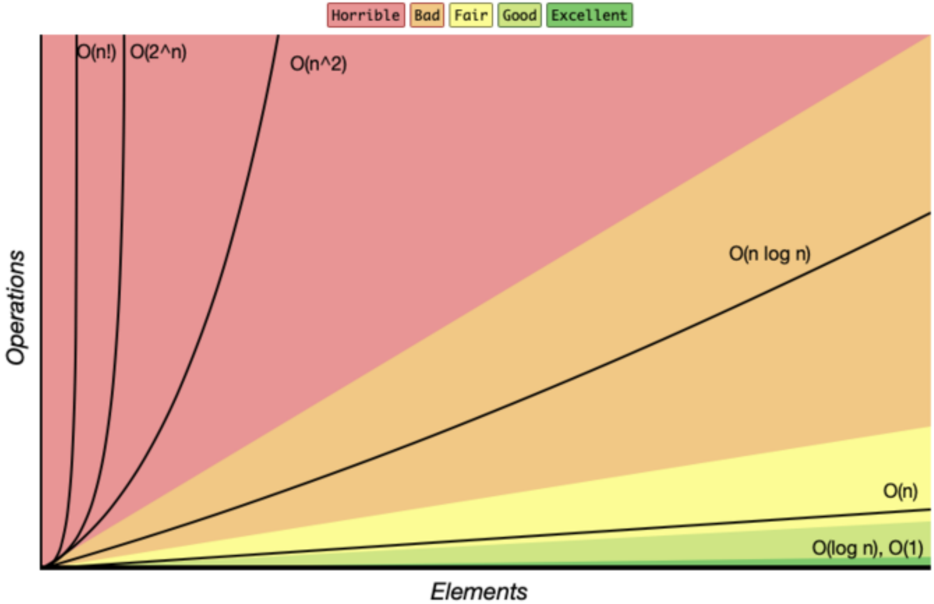
- Determine the number of primitive operations/line executions performed in the worst case in terms of the input size
- Discard lower-order terms and constant factors
- The worst-case time complexity is then the big-Oh of the term that remains

# Common Functions

Commonly encountered functions in algorithm analysis:

- Constant: $1$
- Logarithmic: $\log n$
- Linear: $n$
- N-Log-N: $n \log n$
- Quadratic: $n^2$
- Cubic: $n^3$
- Exponential: $2^n$
- Factorial: $n!$

# Common Functions



# Relatives of Big-Oh

## All The Mathematics!

$f(n)$ is $O(g(n))$
if $f(n)$ is asymptotically less than or equal to $g(n)$

$f(n)$ is $\Omega(g(n))$
if $f(n)$ is asymptotically greater than or equal to $g(n)$

$f(n)$ is $\Theta(g(n))$
if $f(n)$ is asymptotically equal to $g(n)$

Given $f(n)$ and $g(n)$, we say $f(n)$ is $O(g(n))$
if we have positive constants $c$ and $n_0$ such that

$$\forall n \geq n_0, f(n) \leq c \cdot g(n)$$

# Back to Linear Search

Linear search requires $4n + 3$ primitive operations in the worst case.

Therefore, linear search is $O(n)$ in the worst case.

# Searching in a Sorted Array

Is there a faster algorithm for searching an array?

Yes... if the array is sorted.

Let's start in the middle.

- If $key$ == $a[N/2]$, we found $key$; we're done!
- Otherwise, we split the array:
  ... if $key < a[N/2]$, we search the left half ($a[0]$ to $a[(N/2)-1)]$)
  ... if $key > a[N/2]$, we search the right half ($a[(N/2)+1)]$ to $a[N-1]$)

# Binary Search

Binary search is a more efficient search algorithm for **sorted arrays**:

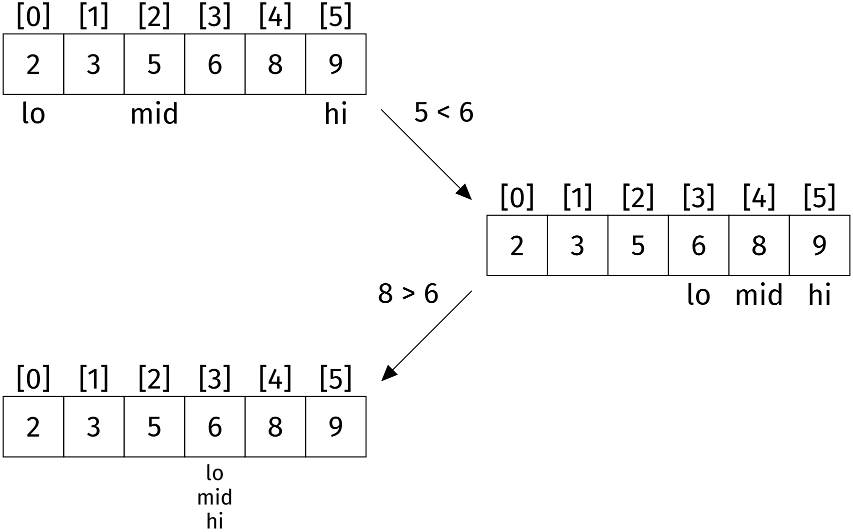
```c
int binarySearch(int arr[], int size, int key) {
    int lo = 0;
    int hi = size - 1;

    while (lo <= hi) {
        int mid = (lo + hi) / 2;

        if (key < arr[mid]) {
            hi = mid - 1;
        } else if (key > arr[mid]) {
            lo = mid + 1;
        } else {
            return mid;
        }
    }
    return -1;
}
```

# Binary Search

Successful search for 6:



| [0] | [1] | [2] | [3] | [4] | [5] |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 8 | 9 |
| lo | | mid | | | hi |

5 < 6

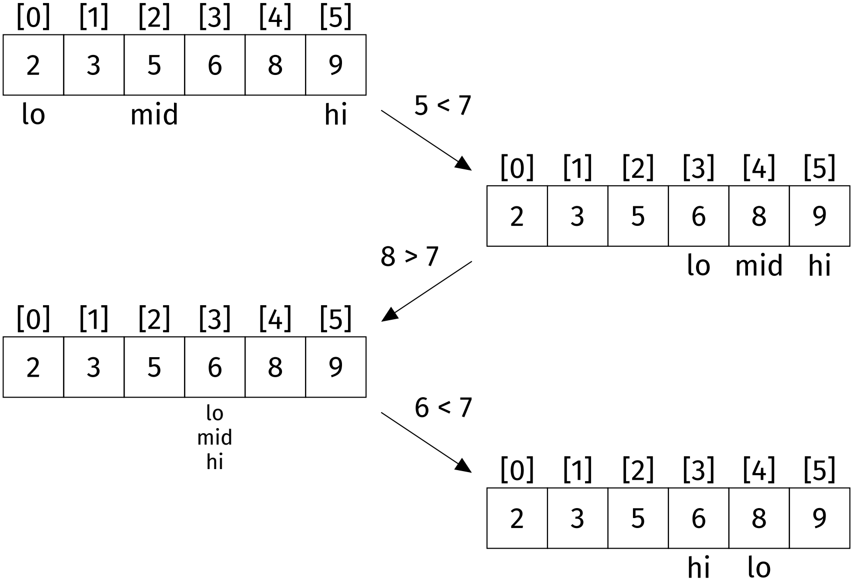
| [0] | [1] | [2] | [3] | [4] | [5] |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 8 | 9 |
| | | | lo | mid | hi |

8 > 6

| [0] | [1] | [2] | [3] | [4] | [5] |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 8 | 9 |
| | | | lo<br>mid<br>hi | | |

# Binary Search

Unsuccessful search for 7:



| [0] | [1] | [2] | [3] | [4] | [5] |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 8 | 9 |
| lo |  | mid |  |  | hi |

5 < 7

| [0] | [1] | [2] | [3] | [4] | [5] |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 8 | 9 |
|  |  |  | lo | mid | hi |

8 > 7

| [0] | [1] | [2] | [3] | [4] | [5] |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 8 | 9 |
|  |  |  | lo<br>mid<br>hi |  |  |

6 < 7

| [0] | [1] | [2] | [3] | [4] | [5] |
|---|---|---|---|---|---|
| 2 | 3 | 5 | 6 | 8 | 9 |
|  |  |  | hi | lo |  |

# Binary Search

## Analysis

How many iterations of the loop?

- Best case: 1 iteration
    - Item is found right away
- Worst case: $\log_2 n$ iterations
    - Item does not exist
    - Every iteration, the size of the subarray being searched is halved

Thus, binary search is $O(\log_2 n)$ or simply $O(\log n)$

# Predicting Time

If I know my algorithm is quadratic (i.e., $O(n^2)$),
and I know that for a dataset of 1000 items,
it takes 1.2 seconds to run …

- how long for 2000?
- how long for 10,000?
- how long for 100,000?
- how long for 1,000,000?

# Predicting Time

If I know my algorithm is quadratic (i.e., $O(n^2)$),
and I know that for a dataset of 1000 items,
it takes 1.2 seconds to run ...

- how long for 2000? 4.8 seconds
- how long for 10,000?
- how long for 100,000?
- how long for 1,000,000?

# Predicting Time

If I know my algorithm is quadratic (i.e., $O(n^2)$),
and I know that for a dataset of 1000 items,
it takes 1.2 seconds to run ...

- how long for 2000? 4.8 seconds
- how long for 10,000? 120 seconds (2 mins)
- how long for 100,000?
- how long for 1,000,000?

# Predicting Time

If I know my algorithm is quadratic (i.e., $O(n^2)$),
and I know that for a dataset of 1000 items,
it takes 1.2 seconds to run ...

- how long for 2000? 4.8 seconds
- how long for 10,000? 120 seconds (2 mins)
- how long for 100,000? 12000 seconds (3.3 hours)
- how long for 1,000,000?

# Predicting Time

If I know my algorithm is quadratic (i.e., $O(n^2)$),
and I know that for a dataset of 1000 items,
it takes 1.2 seconds to run ...

- how long for 2000? 4.8 seconds
- how long for 10,000? 120 seconds (2 mins)
- how long for 100,000? 12000 seconds (3.3 hours)
- how long for 1,000,000? 1200000 seconds (13.9 days)