COMP1917: Computing 1

7. Number Storage and Accuracy

Reading: Moffat, Section 13.2

Outline

- Binary Arithmetic
- Negative Numbers
- Overflow
- Floating Point
- Roundoff Errors
- Type Conversion

Decimal Arithmetic – Addition

carry

```
  1  1
  2 0 9 7   Addend
+       5   Augend
-----------
  2 1 0 2   Sum
```

- Important principle of “sum” and “carry”

Binary Arithmetic – Addition

- Similar idea: “sum” and “carry”
- Four cases to consider:

Addend	0	0	1	1
Augend	0	1	0	1
Sum	0	1	1	0
Carry	0	0	0	1

Binary Arithmetic – Addition

$10_{10} + 12_{10} = 22_{10}$

$$\begin{array}{cccccc} & 1 & 0 & 1 & 0 & + \\ & 1 & 1 & 0 & 0 & \\ \hline 1 & 0 & 1 & 1 & 0 & \end{array}$$

Unsigned Data Types in C

type	bytes	bits	range
`unsigned char`	1	8	0 ... 255
`unsigned short`	2	16	0 ... 65535
`unsigned int`	4	32	0 ... 4294967295
		n	0 ... $2^n - 1$

Note: these sizes are machine dependent. Some machines also provide an "`unsigned long`" type using a larger number of bytes. You can use the `sizeof()` function to determine the sizes on your machine.

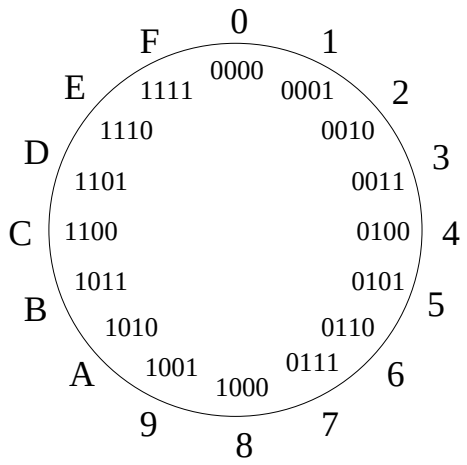
Overflow

Question: What will happen when this code is executed ?

```
unsigned char c = 250;
int i;

for( i=0; i < 10; i++ ) {
   printf("c = %3d\n", c );
   c++;
}
```


"Clock" Arithmetic



Representations for Negative Numbers

Hex	Binary	Unsigned	Sign-Mag	Excess-7	2's Complement
F	1111	15	−7	+8	−1
E	1110	14	−6	+7	−2
D	1101	13	−5	+6	−3
C	1100	12	−4	+5	−4
B	1011	11	−3	+4	−5
A	1010	10	−2	+3	−6
9	1001	9	−1	+2	−7
8	1000	8	−0	+1	−8
7	0111	7	+7	0	+7
6	0110	6	+6	−1	+6
5	0101	5	+5	−2	+5
4	0100	4	+4	−3	+4
3	0011	3	+3	−4	+3
2	0010	2	+2	−5	+2
1	0001	1	+1	−6	+1
0	0000	0	+0	−7	0

Comparison of Representations

- Signed-Magnitude is difficult to compute with, and has two zeros
- Excess-7 is useful for floating point exponents, but not for general integers
- 2's Complement is the most convenient, and most widely used

Justification for 2's Complement

Motivation:

We want to find a way of representing negative numbers which allows us to use the same hardware we already use for positive numbers.

Question 1:

What number should we use for "minus one" ?
i.e. what number, when one is added to it, becomes zero ?

"Minus One"

Answer 1:

The binary number 11111111 should be used for "minus one".

Check:

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} +$$

(Note: the final carry bit is ignored)

Question 2:

What number should be the negative of 10011100 ?

Two's Complement

Answer 2:

The negative of 10011100 should be

$$01100011 + 1 = 01100100$$

Check:

1	0	0	1	1	1	0	0	+
0	1	1	0	0	0	1	1	
1	1	1	1	1	1	1	1	+
							1	
0	0	0	0	0	0	0	0	

Computing Two's Complement

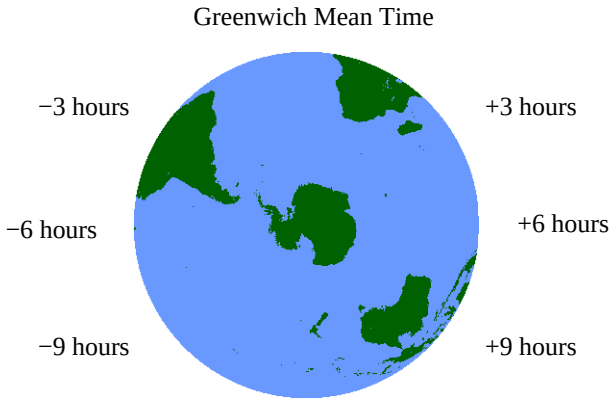
General rule for finding the negative of a binary number:

Step 1: replace every 1 with 0, every 0 with 1
Step 2: add 1

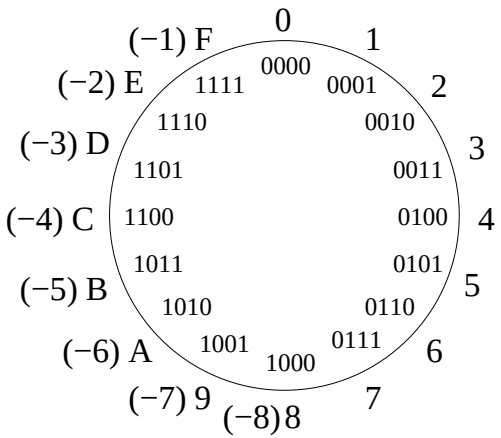
Example: find the 2's Complement representation for -58_{10}

0	0	1	1	1	0	1	0	
1	1	0	0	0	1	0	1	one's complement
							1	+1
1	1	0	0	0	1	1	0	two's complement

Time Zones



Signed Clock Arithmetic



Sign and MSB

You can tell the sign of the number from the Most Signifcant Bit (MSB)

- if MSB is `1`, number is negative
- if MSB is `0`, number is positive or zero

Note: this means there is always "one extra" negative number.

Signed Data Types in C

type	bytes	bits	range
`char`	1	8	-128 ... $+127$
`short`	2	16	-32768 ... $+32767$
`int`	4	32	-2147483648 ... $+2147483647$
		n	-2^{n-1} ... $+(2^{n-1}-1)$

Again, the exact sizes are machine dependent.
Some machines provide a "`long`" type using more bytes.

Signed Overflow

Question: What will happen when this code is executed ?

```
int i=0;

while ( i >= 0 ) {
   i += 1024;
}

printf( "%d  %d\n", i-1, i );
```


Overflow in Two's Complement

- In two's complement we can represent numbers in the range $-(2^{n-1})$ ... $+(2^{n-1}-1)$
- If we try to add two positive binary numbers x and y where $x+y > 2^{n-1}-1$, the sum will result in a negative number (MSB is 1)
  - positive overflow
- If we try to add two negative binary numbers $-x$ and $-y$ where $x+y > 2^{n-1}$, the sum will result in a positive number (the MSB is 0)
  - negative overflow
- Can use XOR gate in hardware to determine overflow condition

Positive Overflow in Two's Complement

- Addition of positive numbers without overflow

$$\begin{array}{cccc} 0 & x & x & x \\ 0 & x & x & x \; + \\ \hline 0 & x & x & x \end{array}$$

  - Carry into MSB must have been 0; carry out of MSB is 0
- Addition of positive numbers with overflow

$$\begin{array}{cccc} 0 & x & x & x \\ 0 & x & x & x \; + \\ \hline 1 & x & x & x \end{array}$$

  - Carry into MSB must have been 1; carry out of MSB is 0
- *carry in* $\neq$ *carry out* means overflow has occurred

Negative Overflow in Two's Complement

- Addition of negative numbers without overflow

$$\begin{array}{cccc} 1 & x & x & x \\ 1 & x & x & x \; + \\ \hline 1 & 1 & x & x & x \end{array}$$

  - Carry into MSB must have been 1; carry out of MSB is 1
- Addition of negative numbers with overflow

$$\begin{array}{cccc} 1 & x & x & x \\ 1 & x & x & x \; + \\ \hline 1 & 0 & x & x & x \end{array}$$

  - Carry into MSB must have been 0; carry out of MSB is 1
- *carry in* $\neq$ *carry out* means overflow has occurred

Decimal Floating Point Numbers

- We want to be able to represent very large and very small numbers, with adequate precision. This can be done with "scientific" or "exponential" notation
  - speed of light $= 1079252848.8 = 1.079 \times 10^9$ km/h
  - mass of proton $= 1.672 \times 10^{-27}$ kg
- This exponential form has 3 components:
  - sign ('+' or '-')
  - exponent (positive or negative integer)
  - fractional part

Binary Floating Point Numbers

Floating point numbers in the computer

- are in an "exponential" binary form
- are stored in a limited number of bits.

  For example, if 16 bits are available, we might allocate:
  - 1 bit for the sign (0 = '+', 1 = '-')
  - 4 bits for the binary exponent (in Excess-7 form)
  - 11 bits for fractional part, in binary notation

Floating Point Example

- How do we interpret this bit pattern as a floating-point number?

$$\underbrace{1}_{\texttt{Sign}}\ \underbrace{1100}_{\texttt{Exp}}\ \underbrace{01010110000}_{\texttt{Fraction}}$$

  - Because `Sign` $= 1$, the number is negative
  - Exponent is `1100` in Excess-7, which is $12 - 7 = 5$
  - Binary number is:

$$-1.0101011 \times 2^5 = -101010.11$$

  - Decimal equivalent is -42.75

Floating Point Details

- Highest exponent `1111` is reserved for `+infinity`, `-infinity` or `NaN`
- For exponents between `0001` (-6) and `1110` (+7), we assume a '1' in front of the fractional part (as in the previous example)
- Lowest exponent `0000` is treated as a special case, in order to represent very small numbers (including zero)
  - we assume '0' instead of '1' in front of the fractional part
  - to compensate, the exponent is increased by one (to -6)
  - for example:

$$\underbrace{0}_{\texttt{Sign}}\ \underbrace{0000}_{\texttt{Exp}}\ \underbrace{00101000000}_{\texttt{Fraction}} = 0.00101 \times 2^{-6}$$

Floating Point Types in C

type	bytes	bits	sign	Exponent	Fraction
`float`	4	32	1 bit	8 bits (Excess-127)	23 bits
`double`	8	64	1 bit	11 bits (Excess-1023)	52 bits

There are Web sites where you can type a number and see its representation as a `float` or `double`

`http://babbage.cs.qc.cuny.edu/IEEE-754`

Roundoff Errors

Question: What will happen when this code is executed?

```
float x = 0.0;

while( x < 1.0 ) {
   x = x + 0.02;
}

printf( "x = %1.10f\n", x );
```

Roundoff Example

Answer:

```
x = 1.0199996233
```

Why?

- the Binary expansion of 0.02 does not terminate; so it instead gets truncated, producing a small error.
- these small errors accumulate, causing the loop to execute one time too many.
- this problem can often be avoided by using an `int` rather than a `float` to test the loop condition

Big or Small Numbers First?

Which code will produce the more accurate result?

```
float x = 0.0;
int i;

for(i=0; i<100; i++) {
   x += 0.01;
}
x += 1000000.0;

printf("x=%1.2f\n",x);
```

```
float x = 0.0;
int i;

x += 1000000.0;
for(i=0; i<100; i++) {
   x += 0.01;
}

printf("x=%1.2f\n",x);
```


Type Conversions

In an expression where you have operands of different types, they are automatically converted to a common type such that:

- the operand with the "narrower" type is converted into a "wider" type. This is done only if there is no loss of information.

Warning: Expressions that might lose information, *e.g.* assigning a float to an integer, are permissible. If you are lucky, the compiler may generate a warning.

The best defense against loss of information in automatic type conversion is to be explicit in your type conversion, i.e. when in doubt, make the type conversion explicit.

Type Conversions *cont.*

What is the output of this code ?

```
float x = 22 / 7;

printf( "%1.2f\n",            x       );
printf( "%1.2f\n",          22 / 7.0  );
printf( "%1.2f\n", (float)  22 / 7    );
printf( "%1.2f\n", (float) ( 22 / 7 ) );
printf( "%1.2f\n",          22 / 7    );
```

Note: the output may be different from one machine to another!