COMP1917: Computing 1

6. Binary and Hexadecimal

Reading: Moffat, Section 13.2

Outline

- Number Systems
- Binary Computation
- Converting between Binary and Decimal
- Octal and Hexadecimal
- Fractional Component

Number Systems

- Most cultures have developed number systems based on 5 or 10, but there are some exceptions (often based on counting the spaces between the fingers as well as the fingers):
  - Base 8 (Yuki of northern California)
  - Base 9 (Nenet language of Russia)
  - Base 12 (hours of the day)
  - Base 20 (Maya of Central America)
  - Base 60 (Sumerians)
- For digital computers, it is convenient to use Base 2 (Binary), Base 8 (Octal) or Base 16 (Hexadecimal).

Decimal Representation

- Can interpret decimal number 4705 as:

$$4 \times 10^3 + 7 \times 10^2 + 0 \times 10^1 + 5 \times 10^0$$

- The base or radix is 10
  Digits 0 – 9
- Place values:

$\cdots$	1000	100	10	1
$\cdots$	10^3	10^2	10^1	10^0

- Write number as 4705_{10}
  - Note use of subscript to denote base

Binary Computation

- In a similar way, can interpret binary number `1011` as:

$$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

- The base or radix is 2
  Digits `0` and `1`
- Place values:

$\cdots$	8	4	2	1
$\cdots$	2^3	2^2	2^1	2^0

- Write number as $\texttt{1011}_2$
  $(= \texttt{11}_{10})$

Binary Computation

- Binary representation is convenient because
  - easily represented using two voltage levels: low and high
  - simple arithmetic tables that can be implemented using logic gates

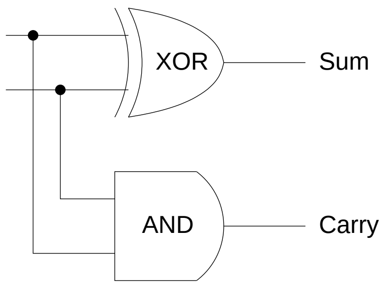
+	**0**	**1**
0	00	01
1	01	10

-

$\times$	**0**	**1**
0	0	0
1	0	1

Binary Representation

- Addition of 1-bit binary numbers can be achieved by half-adder circuit.



- General addition requires combination of half-adder circuits

Converting from base x to Decimal

- Consider 4-digit number in arbitrary base x : $\texttt{abcd}_x$
- Rewrite in polynomial form:

$$ax^3 + bx^2 + cx + d$$

- Or nested form:

$$((ax + b)x + c)x + d$$

- Perform necessary computations to achieve decimal conversion

Examples

- Convert $\texttt{1101}_2$ to base 10 (Decimal):
  - $\texttt{1101}_2 = 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0 = \texttt{13}_{10}$
  - $\texttt{1101}_2 = ((1\times 2+1)\times 2+0)\times 2+1 = \texttt{13}_{10}$
- Convert $\texttt{11001011}_2$ to base 10:
  - $\texttt{11001011}_2$
    $= 1.2^7 + 1.2^6 + 0.2^5 + 0.2^4 + 1.2^3 + 0.2^2 + 1.2^1 + 1.2^0 = \texttt{203}_{10}$
  - $\texttt{11001011}_2$
    $= ((((((1.2+1).2+0).2+0).2+1).2+0).2+1).2+1 = \texttt{203}_{10}$

Converting to base *x*

- a base x number in the form $((\mathtt{a}x+\mathtt{b})x+\mathtt{c})x+\mathtt{d}$ can be written:

$$\begin{aligned} \mathtt{abcd}_x &= \mathrm{C}x+\mathtt{d}\\ \mathrm{C} &= \mathrm{B}x+\mathtt{c}\\ \mathrm{B} &= \mathtt{a}x+\mathtt{b}\\ \mathtt{a} &= 0x+\mathtt{a} \end{aligned}$$

- Numbers $\mathtt{a,b,c,d}$ are all less than base x
- Equations above imply that $\mathtt{d,c,b,a}$ are remainders when the number $\mathtt{abcd}_x$ is repeatedly divided by x
- To convert an integer to base x
  - repeatedly divide quotient by x until quotient is 0
  - write the remainders in reverse

Examples

$$\begin{aligned} 25_{10}\div 2 &= 12 \text{ r } 1 \uparrow\\ 12_{10}\div 2 &= 6 \text{ r } 0 \uparrow\\ 6_{10}\div 2 &= 3 \text{ r } 0 \uparrow\\ 3_{10}\div 2 &= 1 \text{ r } 1 \uparrow\\ 1_{10}\div 2 &= 0 \text{ r } 1 \uparrow \end{aligned}$$

- Write remainders from left to right reading from bottom to top.
- $\texttt{25}_{10} = \overrightarrow{\texttt{11001}_2}$
- **ALWAYS check your answer:**
  $\texttt{11001}_2 = 1\times 2^4 + 1\times 2^3 + 0\times 2^2 + 0\times 2^1 + 1\times 2^0 = \texttt{25}_{10}$

Examples

- Convert $\texttt{11972}_{10}$ to base 8 (Octal)

$$\begin{aligned} 11972_{10}\div 8 &= 1496 \text{ r } 4\\ 1496\div 8 &= 187 \text{ r } 0\\ 187\div 8 &= 23 \text{ r } 3\\ 23\div 8 &= 2 \text{ r } 7\\ 2\div 8 &= 0 \text{ r } 2 \end{aligned}$$

- $\texttt{11972}_{10} = \texttt{27304}_8$
- $\texttt{27304}_8 = 2\times 8^4 + 7\times 8^3 + 3\times 8^2 + 0\times 8^1 + 4\times 8^0 = \texttt{11972}_{10}$

Decimal to Binary — With Division

- Repeated division of quotient by 2 as in previous example
- repeat until quotient is 0
- output remainders in the reverse of the order in which they were generated

Decimal to Binary — With Division

$$\begin{array}{rcrcl}
573_{10} \div 2 & = & 286 & r & 1 \\
286 \div 2 & = & 143 & r & 0 \\
143 \div 2 & = & 71 & r & 1 \\
71 \div 2 & = & 35 & r & 1 \\
35 \div 2 & = & 17 & r & 1 \\
17 \div 2 & = & 8 & r & 1 \\
8 \div 2 & = & 4 & r & 0 \\
4 \div 2 & = & 2 & r & 0 \\
2 \div 2 & = & 1 & r & 0 \\
1 \div 2 & = & 0 & r & 1
\end{array}$$

- $573_{10} = 1000111101_2$
- Check: $1.2^9 + 0.2^8 + 0.2^7 + 0.2^6 + 1.2^5 + 1.2^4 + 1.2^3 + 1.2^2 + 0.2^1 + 1.2^0 = 573_{10}$

Decimal to Binary — Without Division

- find largest power of 2 smaller than current number; subtract this from the number and repeat on the difference
- usually not practical for larger numbers
- powers of 2 are: 128, 64, 32, 16, 8, 4, 2, 1

Decimal to Binary — Without Division

- Convert 174_{10} to binary:

$$\begin{array}{lcll}
174 - 128 = 46 & \longrightarrow & 1 & \text{in } 2^7 \text{ place} \\
64 > 46 & \longrightarrow & 0 & \text{in } 2^6 \text{ place} \\
46 - 32 = 14 & \longrightarrow & 1 & \text{in } 2^5 \text{ place} \\
16 > 14 & \longrightarrow & 0 & \text{in } 2^4 \text{ place} \\
14 - 8 = 6 & \longrightarrow & 1 & \text{in } 2^3 \text{ place} \\
6 - 4 = 2 & \longrightarrow & 1 & \text{in } 2^2 \text{ place} \\
2 - 2 = 0 & \longrightarrow & 1 & \text{in } 2^1 \text{ place} \\
1 > 0 & \longrightarrow & 0 & \text{in } 2^0 \text{ place} \\
= 10101110_2 & & &
\end{array}$$

Exercises

- Convert the following to binary:
  - 5_{10}
  - 10_{10}
  - 12_{10}
  - 23_{10}

Octal Representation

- Can interpret octal number 7053_8 as:

$$7 \times 8^3 + 0 \times 8^2 + 5 \times 8^1 + 3 \times 8^0$$

- The base or radix is 8
  Digits 0,1,2,3,4,5,6,7
- Place values:

···	512	64	8	1
···	8^3	8^2	8^1	8^0

- Write number as 7053_8
  $(= 3627_{10})$

Binary to Octal

0	1	2	3	4	5	6	7
000	001	010	011	100	101	110	111

- Idea: Collect bits into groups of three starting from right to left
- “pad” out left-hand side with 0’s if necessary
- Convert each group of three bits into its equivalent octal representation (given in table above)
- Example: Convert 1011111000101001_2 to Octal:

001	011	111	000	101	001_2
1	3	7	0	5	1_8

Octal to Binary

- Reverse the previous process
- Convert each octal digit into equivalent 3-bit binary representation
- Example: Convert 27015_8 to Binary:

2	7	0	1	5_8
010	111	000	001	101_2

Hexadecimal Representation

- Can interpret hexadecimal number `3AF1` as:

$$3 \times 16^3 + 10 \times 16^2 + 15 \times 16^1 + 1 \times 16^0$$

- The base or radix is 16
  Digits `0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F`
- Place values:

⋯	4096	256	16	1
⋯	16^3	16^2	16^1	16^0

- Write number as $\texttt{3AF1}_{16}$
  $(= \texttt{15089}_{10})$

Binary to Hexadecimal

0	1	2	3	4	5	6	7
0000	0001	0010	0011	0100	0101	0110	0111
8	9	A	B	C	D	E	F
1000	1001	1010	1011	1100	1101	1110	1111

- Idea: Collect bits into groups of four starting from right to left
- “pad” out left-hand side with `0`’s if necessary
- Convert each group of four bits into its equivalent hexadecimal representation (given in table above)

Binary to Hexadecimal

- Example: Convert $\texttt{1011111000101001}_2$ to Hex:

1011	1110	0010	1001_2
B	E	2	9_{16}

- Example: Convert $\texttt{10111101011100}_2$ to Hex:

0010	1111	0101	1100
2	F	5	C_{16}

Hexadecimal to Binary

- Reverse the previous process
- Convert each hex digit into equivalent 4-bit binary representation
- Example: Convert $\texttt{AD5}_{16}$ to Binary:

A	D	5
1010	1101	0101_2

Exercises

- Convert to binary
  - $\texttt{53}_{10}$
  - $\texttt{5F3A}_{16}$
  - $\texttt{12D}_{16}$
  - $\texttt{3701}_{8}$
  - $\texttt{4232}_{8}$
- Convert to octal
  - $\texttt{10101111011}_{2}$
  - $\texttt{5F3A}_{16}$
- Convert to hexadecimal
  - $\texttt{10101111011}_{2}$
  - $\texttt{3701}_{8}$

Fractions

- In the same way that we use a decimal point (.) to represent fractional quantities for decimal numbers, we use the radix point to represent fractional quantities in any base
- For example, $\texttt{abcd.pqrs}_x$ (note: base x) represents

$$ax^3 + bx^2 + cx^1 + dx^0 + px^{-1} + qx^{-2} + rx^{-3} + sx^{-4}$$

- Numbers to the left of radix point represent integer component
- Numbers to the right of radix point represent fractional component
- Polynomial above can be evaluated to determine equivalent decimal representation.

Converting Fractions to Base x

- Convert integer component as usual
- To convert fractional component use “separate and multiply” technique
- Consider fractional component of $\texttt{abcd.pqrs}_x$
  - $px^{-1} + qx^{-2} + rx^{-3} + sx^{-4}$
  - Multiplying by x gives: $p + qx^{-1} + rx^{-2} + sx^{-3}$
  - Multiplying remaining fractional component by x gives: $q + rx^{-1} + sx^{-2}$
- Repeat until fractional part exhausted or you have sufficient digits (Note: process is not guaranteed to terminate)

Example

- Convert 23.3125_{10} to base 8
- Integer component

$$\begin{array}{rcccc} 23_{10} \div 8 & = & 2 & \text{r} & 7 \\ 2 \div 8 & = & 0 & \text{r} & 2 \end{array}$$

- Fractional component

$$\begin{array}{lcl} .3125_{10} \times 8 & = & 2.5 \\ .5_{10} \times 8 & = & 4.0 \end{array}$$

- Therefore: $23.3125_{10} = 27.24_8$

Example

- Convert 23.3125_{10} to base 2
- Integer component

$$
\begin{array}{rcrcl}
23_{10} \div 2 & = & 11 & \text{r} & 1 \\
11 \div 2 & = & 5 & \text{r} & 1 \\
5 \div 2 & = & 2 & \text{r} & 1 \\
2 \div 2 & = & 1 & \text{r} & 0 \\
1 \div 2 & = & 0 & \text{r} & 1
\end{array}
$$

Example *continued*

- Fractional component

$$
\begin{array}{lcl}
.3125_{10} \times 2 & = & 0.625 \\
.625 \times 2 & = & 1.25 \\
.25 \times 2 & = & 0.5 \\
.5 \times 2 & = & 1.0
\end{array}
$$

- Therefore, $23.3125_{10} = 10111.0101_2$