Floating Point Numbers

- C has three floating point types
  - float ... typically 32-bit (lower precision, narrower range)
  - double ... typically 64-bit (higher precision, wider range)
  - long double ... typically 128-bits (but maybe only 80 bits used)
- Floating point constants, e.g.: 3.14159 1.0e-9 are double
- Reminder: division of 2 ints in C yields an int.
  - but division of double and int in C yields a double.

Floating Point Number - Output

double d = 4/7.0;
// prints in decimal with (default) 6 decimal places
printf("%lf\n", d); // prints 0.571429
// prints in scientific notation
printf("%le\n", d); // prints 5.714286e-01
// picks best of decimal and scientific notation
printf("%lg\n", d); // prints 0.571429
// prints in decimal with 9 decimal places
printf("%.9lf\n", d); // prints 0.571428571
// prints in decimal with 1 decimal place and field width of 5
printf("%10.1lf\n", d); // prints 0.6
Floating Point Numbers

- If we represent floating point numbers with a fixed small number of bits
  - There are only a finite number of bit patterns
  - Can only represent a finite subset of reals
- Almost all real values will have no exact representation
- Value of arithmetic operations may be real with no exact representation
- We must use closest value which can be exactly represented
- This approximation introduces an error into our calculations
- Often, does not matter
- Sometimes ... can be disastrous

Fixed Point Representation

- Can have fractional numbers in other bases, e.g.: 110.101\(_2\) == 6.625\(_{10}\)
- Could represent fractional numbers similarly to integers by assuming decimal point is in fixed position
- For example with 32 bits:
  - 16 bits could be used for integer part
  - 16 bits could be used for the fraction
  - Equivalent to storing values as integers after multiplying (scaling) by \(2^{16}\)
  - Major limitation is only small range of values can be represented
    - Minimum \(2^{-16} \approx 0.000015\)
    - Maximum \(2^{15} \approx 32768\)
- Usable for some problems, but not ideal
- Used on small embedded processors without silicon floating point

floating_types.c - print characteristics of floating point types

```c
float f;
double d;
long double l;

printf("float %lu bytes min=%-12g max=%g\n", sizeof f, FLT_MIN, FLT_MAX);
printf("double %lu bytes min=%-12g max=%g\n", sizeof d, DBL_MIN, DBL_MAX);
printf("long double %lu bytes min=%-12Lg max=%Lg\n", sizeof l, LDBL_MIN, LDBL_MAX);
```

```
$ ./floating_types
float 4 bytes min=1.17549e-38 max=3.40282e+38
double 8 bytes min=2.22507e-308 max=1.79769e+308
long double 16 bytes min=3.3621e-4932 max=1.18973e+4932
```
IEEE 754 standard

- C floats almost always IEEE 754 single precision (binary32)
- C double almost always IEEE 754 double precision (binary64)
- C long double might be IEEE 754 (binary128)
- IEEE 754 representation has 3 parts: sign, fraction and exponent
- numbers have form $sign \times fraction \times 2^{exponent}$, where sign is +/- 
- fraction always has 1 digit before decimal point (normalized)
  - as a consequence only 1 representation for any value
- exponent is stored as positive number by adding constant value (bias)
- numbers close to zero have higher precision (more accurate)

Floating Point Numbers

Example of normalising the fraction part in binary:
- 1010.1011 is normalized as $1.0101011 \times 2^{011}$
- 1010.1011 = 10 + 11/16 = 10.6875
- $1.0101011 \times 2^{011} = (1 + 43/128) \times 2^{3} = 1.3359375 \times 8 = 10.6875$

The normalised fraction part always has 1 before the decimal point.

Example of determining the exponent in binary:
- if exponent is 8-bits, then the bias $= 2^{8-1} - 1 = 127$
- valid bit patterns for exponent: 00000001 .. 11111110
- correspond to $B$ exponent values -126 .. 127

Internal structure of floating point values

<table>
<thead>
<tr>
<th>31</th>
<th>30</th>
<th>23</th>
<th>22</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>exp</td>
<td>fraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sign</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

single precision

<table>
<thead>
<tr>
<th>63</th>
<th>62</th>
<th>52</th>
<th>51</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>exp</td>
<td>fraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sign</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

double precision
IEEE-754 Single Precision example: 0.15625

0.15625 is represented in IEEE-754 single-precision by these bits:
00111110000100000000000000000000

<table>
<thead>
<tr>
<th>sign</th>
<th>exponent</th>
<th>fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>01111100</td>
<td>010000000000000000000000</td>
</tr>
</tbody>
</table>

sign bit = 0
sign = +
raw exponent = 01111100 binary
= 124 decimal
actual exponent = 124 - exponent_bias
= 124 - 127
= -3

number = +1.01000000000000000000000 binary * 2**-3
= 1.25 decimal * 2**-3
= 1.25 * 0.125
= 0.15625

source code for explain_float_representation.c

$ ./explain_float_representation -0.125

-0.125 is represented as a float (IEEE-754 single-precision) by these bits:
10111110000000000000000000000000

<table>
<thead>
<tr>
<th>sign</th>
<th>exponent</th>
<th>fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>01111100</td>
<td>000000000000000000000000</td>
</tr>
</tbody>
</table>

sign bit = 1
sign = -
raw exponent = 01111100 binary
= 124 decimal
actual exponent = 124 - exponent_bias
= 124 - 127
= -3

number = -1.00000000000000000000000 binary * 2**-3
= -1 decimal * 2**-3
= -1 * 0.125
= -0.125

$ ./explain_float_representation 150.75

150.75 is represented in IEEE-754 single-precision by these bits:
01000011000101101100000000000000

<table>
<thead>
<tr>
<th>sign</th>
<th>exponent</th>
<th>fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10001101</td>
<td>001011011000000000000000</td>
</tr>
</tbody>
</table>

sign bit = 0
sign = +
raw exponent = 10001101 binary
= 134 decimal
actual exponent = 134 - exponent_bias
= 134 - 127
= 7

number = +1.0010110110000000000000000 binary * 2**7
= 1.17773 decimal * 2**7
= 1.17773 * 128
= 150.75
IEEE-754 Single Precision example: -96.125

$ ./explain_float_representation -96.125
-96.125 is represented in IEEE-754 single-precision by these bits:
11000010110000000100000000000000

<table>
<thead>
<tr>
<th>sign</th>
<th>exponent</th>
<th>fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10000101</td>
<td>10000000100000000000000</td>
</tr>
</tbody>
</table>

-96.125 is represented in IEEE-754 single-precision by these bits:
11000010110000000100000000000000

| sign bit = 1 |
| sign = - |
| raw exponent = 10000101 binary |
| = 133 decimal |
| actual exponent = 133 - exponent_bias |
| = 133 - 127 |
| = 6 |
| number = -1.10000000100000000000000 binary * 2**6 |
| = -1.50195 decimal * 2**6 |
| = -1.50195 * 64 |
| = -96.125 |

IEEE-754 Single Precision exploring bit patterns #1

$ ./explain_float_representation 00111101110011001100110011001101
sign bit = 0
sign = +
raw exponent = 0111011 binary
= 123 decimal
actual exponent = 123 - exponent_bias
= 123 - 127
= -4
number = +1.10011001100110011001101 binary * 2**-4
= 1.6 decimal * 2**-4
= 1.6 * 0.0625
= 0.1

infinity.c: exploring infinity

- IEEE 754 has a representation for +/- infinity
- propagates sensibly through calculations

```c
double x = 1.0/0.0;
printf("%lf\n", x); // prints inf
printf("%lf\n", -x); // prints -inf
printf("%lf\n", x - 1); // prints inf
printf("%lf\n", 2 * atan(x)); // prints 3.141593
printf("%d\n", 42 < x); // prints 1 (true)
printf("%d\n", x == INFINITY); // prints 1 (true)
```

source code for infinity.c
nan.c: handling errors robustly

- C (IEEE-754) has a representation for invalid results:
  - NaN (not a number)
- ensures errors propagates sensibly through calculations

```c
double x = 0.0/0.0;
printf("%lf\n", x); // prints nan
printf("%lf\n", x - 1); // prints nan
printf("%d\n", x == x); // prints 0 (false)
printf("%d\n", isnan(x)); // prints 1 (true)
```

source code for nan.c

IEEE-754 Single Precision example: inf

```bash
$ ./explain_float_representation inf
inf is represented in IEEE-754 single-precision by these bits:
01111111000000000000000000000000
sign | exponent | fraction
    0 | 1111111 | 000000000000000000000000
sign bit = 0
sign = +
raw exponent = 1111111 binary
             = 255 decimal
number = +inf
```

IEEE-754 Single Precision exploring bit patterns #2

```bash
$ ./explain_float_representation 01111111100000000000000000000000
sign bit = 0
sign = +
raw exponent = 1111111 binary
             = 255 decimal
number = NaN
```

source code for explain_float_representation.c
Consequences of most reals not having exact representations

double a, b;
a = 0.1;
b = 1 - (a + a + a + a + a + a + a + a + a + a);
if (b != 0) {
    // better would be fabs(b) > 0.000001
    printf("1 != 0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1+0.1\n");
}
printf("b = %g\n", b); // prints 1.11022e-16

- do not use == and != with floating point values
- instead check if values are close

Consequences of most reals not having exact representations

double x = 0.000000011;
double y = (1 - cos(x)) / (x * x);
// correct answer y = ~0.5
// prints y = 0.917540
printf("y = %lf\n", y);
// division of similar approximate value
// produces large error
// sometimes called catastrophic cancellation
printf("%g\n", 1 - cos(x)); // prints 1.11022e-16
printf("%g\n", x * x); // prints 1.21e-16

Another reason not to use == with floating point values

if (d == d) {
    printf("d == d is true\n");
} else {
    // will be executed if d is a NaN
    printf("d == d is not true\n");
}
if (d == d + 1) {
    // may be executed if d is large
    // because closest possible representation for d + 1
    // is also closest possible representation for d
    printf("d == d + 1 is true\n");
} else {
    printf("d == d + 1 is false\n");
}
Another reason not to use == with floating point values

```bash
$ dcc double_not_always.c -o double_not_always
$ ./double_not_always 42.3
  d = 42.3
  d == d is true
  d == d + 1 is false
$ ./double_not_always 4200000000000000000
  d = 4.2e+18
  d == d is true
  d == d + 1 is true
$ ./double_not_always NaN
  d = nan
  d == d is not true
  d == d + 1 is false
```

because closest possible representation for d + 1 is also closest possible representation for d

Consequences of most reals not having exact representations

```c
double d = 9007199254740992;
// loop never terminates
while (d < 9007199254740999) {
    printf("%lf\n", d); // always prints 9007199254740992.000000
    // 9007199254740993 can not be represented as a double
    // closest double is 9007199254740992.0
    // so 9007199254740992.0 + 1 = 9007199254740992.0
    d = d + 1;
}
```

- 9007199254740993 is $2^{53} + 1$
- it is smallest integer which can not be represented exactly as a double
- The closest double to 9007199254740993 is 9007199254740992.0
- aside: 9007199254740993 can not be represented by a int32_t
  - it can be represented by int64_t

Exercise: Floating point → Decimal

Convert the following floating point numbers to decimal.

Assume that they are in IEEE 754 single-precision format.

- 0 10000000 11000000000000000000000
- 1 01111110 10000000000000000000000