Dealing with Incomplete Agents' Preferences and an Uncertain Agenda in Group Decision Making via Sequential Majority Voting

Maria Silvia Pini[*], **Francesca Rossi**[*], **Kristen Brent Venable**[*] **and Toby Walsh**[**]
[*] University of Padova, Italy, E-mail: {mpini,frossi,kvenable}@math.unipd.it
[**] NICTA and UNSW Sydney, Australia, E-mail: Toby.Walsh@nicta.com.au

Abstract

We consider multi-agent systems where agents' preferences are aggregated via sequential majority voting: each decision is taken by performing a sequence of pairwise comparisons where each comparison is a weighted majority vote among the agents. Incompleteness in the agents' preferences is common in many real-life settings due to privacy issues or an ongoing elicitation process. In addition, there may be uncertainty about how the preferences are aggregated. For example, the agenda (a tree whose leaves are labelled with the decisions being compared) may not yet be known or fixed. We therefore study how to determine collectively optimal decisions (also called winners) when preferences may be incomplete, and when the agenda may be uncertain. We show that it is computationally easy to determine if a candidate decision always wins, or may win, whatever the agenda. On the other hand, it is computationally hard to know whether a candidate decision wins in at least one agenda for at least one completion of the agents' preferences. These results hold even if the agenda must be balanced so that each candidate decision faces the same number of majority votes. Such results are useful for reasoning about preference elicitation. They help understand the complexity of tasks such as determining if a decision can be taken collectively, as well as knowing if the winner can be manipulated by appropriately ordering the agenda.

Introduction

A general method for aggregating preferences in multi-agent systems, in order to take a collective decision, is running an election among the different options using a voting rule. Unfortunately, eliciting preferences from agents to be able to run such an election is a difficult, time-consuming and costly process. Agents may also be unwilling to reveal all their preferences for privacy reasons. Fortunately, we can often determine the outcome before all the preferences have been revealed (Conitzer & Sandholm 2002b). For example, it may be that one option has so many votes that it will win whatever happens with the remaining votes. We can then stop eliciting preferences.

In addition to uncertainty about the agents' preferences, we may have uncertainty about how the voting rule will be applied. For instance, in sequential majority voting (sometimes called the "Cup" or "tournament" rule), which has been extensively studied in Social Choice Theory (Moulin 1991; Laslier 1997), preferences are aggregated by a sequence of pairwise comparisons. The order of these comparisons (which is often called the "agenda") may not be fixed or known. Nevertheless, we may still be able to determine information about the outcome. For example, it may be that one option cannot win however the voting rule is applied. This is useful, for example, if we want to know if the chair can control the election to make his favored option win.

In this paper we study the computational complexity of determining the possible and Condorcet winners in sequential majority voting when preferences may be incomplete and/or we may not know the agenda. We argue that the notions of possible and Condorcet winners considered here are more reasonable than the earlier notions in (Lang *et al.* 2007) as the new notions are based on incomplete profiles as opposed to incomplete majority graphs which potentially throw away some information and may suggest candidates can win when they cannot. The old notions in (Lang *et al.* 2007) are upper or lower approximations of the new notions.

We show that determining if an option always wins, or may win, in every agenda is polynomial. On the other hand, determining if an option wins in at least one completion of the preferences and at least one agenda is NP-complete. All these results hold even if the agenda is required to be balanced. Because the choice of the agenda may be under the control of the chair, our results can be interpreted in terms of difficulty of manipulation by the chair (as in, e.g., (Bartholdi, Tovey, & Trick 1989)).

Background

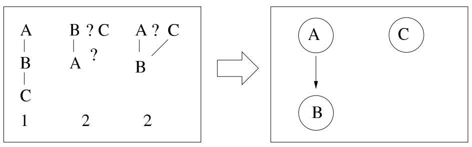
Preferences. We assume that each agent's preferences are specified by a total order (TO) (that is, by an asymmetric, irreflexive and transitive order) over a set of candidates (denoted by Ω). The candidates represent the possible options over which agents will vote. However, an agent may choose to reveal only partially his total order. More precisely, given two candidates, say $A, B \in \Omega$, an agent specifies exactly one of the following: $A < B$ (meaning A is worse than B), $A > B$, or $A?B$, where $A?B$ means that the relation between A and B has not yet been revealed. We assume that an agent's preferences are transitively closed. That is, if they declare $A > B$, and $B > C$ then they also have $A > C$.

Example 1 *Given candidates A, B, and C, an agent may state preferences such as* $A > B$, $B > C$, *and* $A > C$, *or* $A > B$, $B?C$ *and* $A?C$. *However, an agent cannot state preferences such as* $A > B$, $B > C$, $C > A$ *as this is not transitive and thus not a total order.*

Profiles. A *weighted profile* is a sequence of total orders describing the preferences for n agents, each of which has a given weight. A weighted profile is *incomplete* if one or more of the preference relations is incomplete. For simplicity, we assume that the sum of the weights of the agents is odd. An (incomplete) *unweighted profile*, also called *egalitarian profile*, is one in which each agent has weight 1. Given a weighted profile P, its *corresponding unweighted profile* $U(P)$ is the profile obtained from P by replacing every ordering, say O, expressed by an agent with weight k_i by k_i agents with weight 1 all expressing O.

Majority graphs. Given an (incomplete) weighted profile P, the *majority graph* $M(P)$ induced by P is the directed graph whose set of vertices is Ω, and where an edge from A to B (denoted by $A >_m B$) denotes a strict weighted majority of voters who prefer A to B. The assumption to have an odd sum of weights ensures that there is never a tied result. This simplification is not essential. We can have an even sum of weights, but in this case we have to specify how we deal with tied results. Thus, for simplicity we assume that the sum of weights is odd. A majority graph is said to be complete if, for any two vertices, there is a directed edge between them. Notice that, if P is incomplete, $M(P)$ may be incomplete as well. Also, if $M(P)$ is incomplete, the set of all complete majority graphs extending $M(P)$ corresponds to a (possibly proper) superset of the set of complete majority graphs induced by all possible completions of P.

Example 2 *Consider the incomplete weighted profile P in Figure 1 (a). There are three agents* a_1, a_2 *and* a_3 *with weights resp.* 1, 2, *and* 2 *that express the following preferences:* a_1 *states* $A > B > C$, a_2 *states* $B > A, A?C, B?C$ *and* a_3 *states* $A > B, A?C, C > B$. *The majority graph induced by P, called* $M(P)$, *shown in Figure 1 (b), has three nodes A, B and C and one edge from A to B, since there is a weighted majority of agents that prefer A to B. There are no edges between A and C and between B and C since there are no weighted majorities that prefer one candidate to the other.*

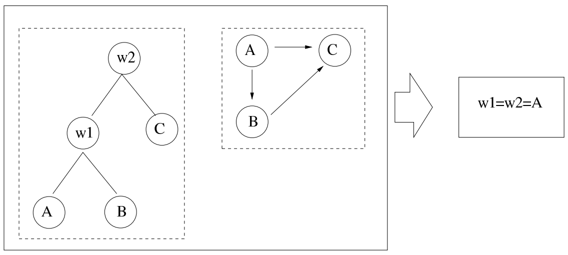


(a) (b)

Figure 1: An incomplete weighted profile and its majority graph.

Sequential majority voting. Given a set of candidates, the sequential majority voting rule is defined by a binary tree (also called an *agenda*) with one candidate per leaf. Each internal node represents the candidate that wins the pairwise election between the node's children. The winner of every pairwise election is computed by the weighted majority rule, where A beats B iff there is a weighted majority of votes stating $A > B$. The candidate at the root of the agenda is the overall winner. Given a complete profile, candidates which win whatever the agenda are called *Condorcet winners*.

Example 3 *Assume to have three candidates A, B and C. Consider the agenda T shown in Figure 2 (a). According to this agenda, A must first play against B, and then the winner, called* w_1, *must play against C. The winner, called* w_2, *is the overall winner. If we have the majority graph M shown in Figure 2 (b),* $w_1 = A$ *and* $w_2 = A$. *Note that A is a Condorcet winner, since it is the overall winner in all possible agendas.*



(a) (b) (c)

Figure 2: How sequential majority voting works.

Winners from majority graphs. Four types of potential winners have been defined (Lang *et al.* 2007) for sequential majority voting. Given an incomplete majority graph G induced by an incomplete profile P, consider a candidate A,

- A is a *weak Condorcet winner*[1] for G (i.e., $A \in WC(G)$) iff there is a completion of G such that A wins in every agenda;
- A is a *strong Condorcet winner* for G (i.e., $A \in SC(G)$) iff for every completion of G, A wins in every agenda;
- A is a *weak possible winner* for G (i.e., $A \in WP(G)$) iff there exists a completion of G and an agenda for which A wins;
- A is a *strong possible winner* for G (i.e., $A \in SP(G)$) iff for every completion of G there is an agenda for which A wins.

When the majority graph is complete, strong and weak Condorcet winners coincide (that is, $SC(G) = WC(G)$). Similarly, strong and weak possible winners coincide in this case (that is, $SP(G) = WP(G)$). In (Lang *et al.* 2007), it

[1] In (Lang *et al.* 2007) a Condorcet winner is called a necessary winner.

is proved that $WP(G)$, $SP(G)$, $WC(G)$, and $SC(G)$ can all be computed in polynomial time.

Profiles, majority graphs, and weights

These notions of possible and Condorcet winner are based on an incomplete majority graph. It is, however, often more useful and meaningful to start directly from the incomplete profile inducing the majority graph. Given an incomplete profile, there can be more completions of its induced majority graphs than majority graphs induced by completing the profile. An incomplete majority graph throws away information about how individual agents have voted. For example, we lose information about correlations between votes. Such correlations may prevent a candidate from being able to win.

Example 4 *Consider an incomplete profile P with just one agent and three candidates (A, B, and C), where the agent declare only $A > B$. The induced majority graph $M(P)$ has only one arc from A to B. In this situation, B is a weak possible winner (that is, $B \in WP(M(P))$), since there is a completion of the majority graph (with arcs from B to C and from C to A) and an agenda where B wins (we first compare A with C, C wins, and then C with B, and B wins). However, there is no way to complete profile P and set up the agenda so B wins. In fact, the possible completions of P are $A > B > C$, $A > C > B$, and $C > A > B$, and in all these cases B is always beaten at least by A. Hence, there is no agenda where B wins. Note that the completion of the majority graph that allows us to conclude that $B \in WP(M(P))$ cannot be obtained in any possible completion of the agent's preferences of P, since it violates transitivity. Since B cannot win in any completion of P, it is rather misleading to consider B as a potential winner.*

Hence, unlike (Lang *et al.* 2007), we will define possible and Condorcet winners starting directly from profiles, rather than the induced majority graphs.

As in (Conitzer & Sandholm 2002a), we consider weighted votes. Weighted voting systems are used in a number of real-world settings like shareholder meetings and elected assemblies. Weights are useful in multiagent systems where we have different types of agents. Weights are also interesting from a computational perspective. Computing the weak/strong possible/Condorcet winners with unweighted votes is always polynomial. If there is a bounded number of candidates, there are only a polynomial number of different ways to complete an incomplete profile. Similarly, if there is a bounded number of candidates, there are only a polynomial number of different ways to complete the missing links in an incomplete majority graph. There are also only a polynomial number of different agendas. All the possibilities can therefore be tested in polynomial time. On the other hand, adding weights to the votes may introduce computational complexity. For example, as we will show later, computing weak possible winners becomes NP-hard when we add weights. Finally, the weighted case informs us about the unweighted case in the presence of uncertainty about the votes. For instance, if constructive coalitional manipulation with weighted votes is intractable, then it is hard to compute the probability of winning in the unweighted case when there is uncertainty about how the votes have been cast (Conitzer & Sandholm 2002a). Reasoning about weighted votes is thus closely related to reasoning about unweighted votes where we have probabilities on the distribution of votes.

Possible and Condorcet winners from profiles

We consider the following new notions of possible and Condorcet winner:

Definition 1 *Let P be an incomplete weighted profile and A a candidate.*

- *A is a* weak Condorcet winner *for P (i.e., $A \in WC(P)$) iff there is a completion of P such that A is a winner for all agendas;*
- *A is a* strong Condorcet winner *for P (i.e., $A \in SC(P)$) iff for every completion of P, and for every agenda, A is a winner;*
- *A is a* weak possible winner *for P (i.e., $A \in WP(P)$) iff there exists a completion of P and an agenda for which A wins;*
- *A is a* strong possible winner *for P (i.e.,$A \in SP(P)$) iff for every completion of P there is an agenda for which A wins.*

It is easy to see that, when the profile is complete, strong and weak Condorcet winners coincide. The same holds also for strong and weak possible winners.

Example 5 *Consider the profile P given in Example 2. We have that $SC(P) = SP(P) = \emptyset$, $WC(P) = \{A, C\}$, and $WP(P) = \{A, B, C\}$. More precisely, A and C are weak Condorcet winners, since there are completions of P where they win in all the agendas. In fact, A wins in all the agendas in the completion of P where a_1 states $A > B > C$, a_2 states $C > B > A$ and a_3 states $A > C > B$, while C wins in all the agendas in the completion of P where a_1 states $A > B > C$, a_2 states $C > B > A$ and a_3 states $C > A > B$. The outcome B is not a weak Condorcet winner, since there are no completions where it wins in every agenda. However, B is a weak possible winner, since there is a completion of P and an agenda where B wins (e.g. a_1 states $A > B > C$, a_2 states $B > C > A$ and a_3 states $C > A > B$, and A first competes with C and then the winner competes with B). Notice that in this example the weak and strong possible and Condorcet winners obtained considering the completions of P coincide with those obtained from considering the completions of the majority graph induced by P. However, as shown in Example 4, this is not true in general.*

These four notions are related to interesting issues in voting theory:

- Weak Condorcet winners are related to *destructive control*. A chair may try to build an agenda in which some candidate loses however the votes are completed. If a candidate is not in WC(P), then the chair can choose an agenda such that it must lose. Thus, the complexity of computing WC(P) is related to the difficulty of destructive control.

- Strong Condorcet winners are related to the *possibility of controlling/manipulating* the election. If SC(P) is non-empty, then neither the chair nor any of the voters can do anything to change the result. Thus, the complexity of computing SC(P) is related to the difficulty of manipulation/control.
- Weak possible winners are related to *participation incentives*. If a candidate is not in WP(P), it has no chance of winning. If it is easy for a candidate to know whether they are not in WP(P), he may drop out of the election. It is therefore desirable that computing WP(P) is difficult.
- Strong possible winners are related to *constructive control*. If a candidate is in SP(P), the chair can make the candidate win by choosing an appropriate agenda. Thus, it is desirable that computing SP(P) is difficult.

Comparing the notions of winners

We now compare the notions of winners defined in (Lang *et al.* 2007) and those defined here. Since in (Lang *et al.* 2007) weights were not considered, we first consider unweighted profiles.

Unweighted profiles

Given an incomplete unweighted profile P and the incomplete majority graph G induced by P, that is, $G = M(P)$, we already observed that the completions of G are a (possibly proper) superset of the set of complete majority graphs induced by all possible completions of P. This observation leads to the following results.

Theorem 1 *Given an incomplete unweighted profile P,*

1. $WP(M(P)) \supseteq WP(P)$*;*
2. $SP(M(P)) \subseteq SP(P)$*;*
3. $WC(M(P)) = WC(P)$*;*
4. $SC(M(P)) = SC(P)$*.*

Proof: Let us consider the four items separately.

1. $WP(M(P)) \supseteq WP(P)$.
   If a candidate A belongs to $WP(P)$, there is a completion of P, say P', and an agenda, such that A wins. Thus $A \in WP(G')$ where G' is the complete majority graph induced by P'. Since G' is one of all the possible completions of $M(P)$), then $A \in WP(M(P)$.
2. $SP(M(P)) \subseteq SP(P)$.
   If a candidate is a possible winner for every completion of G, it is also a possible winner for the majority graphs induced by the completions of P, since they are a subset of the set of all the completions of $M(P)$.
3. $WC(M(P)) = WC(P)$.
   Similar reasoning to the first item can be used to show that $WC(M(P)) \supseteq WC(P)$. We can also prove that $WC(M(P)) \subseteq WC(P)$. In fact, if a candidate A belongs to $WC(M(P))$, then there must be one or more completions of the majority graph where A has only outgoing edges. Among such completions, there is at least one which derives from a completion of the profile in which all $A?C$ become $A > C$ (for all C). Thus, setting this is sufficient to make A a weak Condorcet winner without contradicting transitivity of the profile.
4. $SC(M(P)) = SC(P)$.
   Similar reasoning to the second item can be used to show that $SC(M(P)) \subseteq SC(P)$. We can also prove that $SC(M(P)) \supseteq SC(P)$. In fact, if a candidate belongs to $SC(P)$, then it is a Condorcet winner, i.e., it beats every other candidate, for every completion of P. Thus it must beat every other candidate in the part without uncertainty. Hence, in the (possibly incomplete) majority graph $M(P)$ induced by P, there are outgoing edges from this candidate to every other candidate, and so this candidate must belong to $SC(M(P))$. □

Notice that there are cases in which the subset relation $WP(M(P)) \supseteq WP(P)$ is strict. In fact, a candidate can be a possible winner for a completion of $M(P)$ which is not induced by any completion of P, as shown previously in Example 4.

Weighted profiles

We next consider weighted profiles. Although weighted profiles were not considered in (Lang *et al.* 2007), the same notions defined there can be given for majority graphs induced by weighted profiles. The analogous results to Theorem 1 hold in this more general setting. To prove this, we first show that, given an incomplete weighted profile P and its corresponding unweighted profile $U(P)$, $SC(P) = SC(U(P))$ (resp., $WC(P) = WC(U(P))$). That is, the set of strong (resp., weak) Condorcet winners for P coincides with the set of strong (resp., weak) Condorcet winners for the unweighted profile corresponding to P. We also show that $M(P) = M(U(P))$. That is, the majority graphs of P and $U(P)$ coincide.

Theorem 2 *Given an incomplete weighted profile P,*

1. $M(P) = M(U(P))$*;*
2. $SC(P) = SC(U(P))$*;*
3. $WC(P) = WC(U(P))$*.*

Proof:

1. $M(P) = M(U(P))$.
   The statement can be easily proven since $U(P)$ is a profile obtained from P by replacing each agent with weight k_i and with preference ordering O by k_i agents with weight 1 all with preference ordering O.
2. $SC(P) = SC(U(P))$.
   ($\supseteq$) This follows from the completions of $U(P)$ being a superset of the completions of P.
   ($\subseteq$) Assume that $A \notin SC(U(P))$. Then A does not have $m-1$ outgoing edges (where $m = |\Omega|$) in $M(U(P))$ (Lang *et al.* 2007). Hence, since $M(P) = M(U(P))$, A does not have $m-1$ outgoing edges in $M(P)$. Hence, there is a candidate B s.t. $B >_m A$ or $B ?_m A$ in $M(P)$. If $B >_m A$ in $M(P)$, then for every completion of P we have $B > A$, and thus A cannot win in every agenda. If $B ?_m A$ in $M(P)$, then there exists a completion of P where we replace every $A?B$ with $B > A$, where A may

not win. Hence A does not win in every completion and agenda.

3. $WC(P) = WC(U(P))$.
($\subseteq$) This follows from the completions of $U(P)$ being a superset of the completions of P.
($\supseteq$) Assume that $A \in WC(U(P))$.Then A has no ingoing edges in $M(U(P))$ (Lang *et al.* 2007). Hence, since $M(U(P)) = M(P)$, A has no ingoing edges in $M(U(P))$. Thus, if we replace, for every B, $A?B$ in P with $A > B$, we obtain a completion of P where A wins in every agenda. Hence $A \in WC(P)$. □

We can now compare the notions of winners in the weighted case.

Theorem 3 *Given an incomplete weighted profile P,*

1. *$WP(M(P)) \supseteq WP(P)$, that is, the set of the weak possible winners for the majority graph induced by P contains or is equal to the set of the weak possible winners for P;*
2. *$SP(M(P)) \subseteq SP(P)$, that is, the set of the strong possible winners for the majority graph induced by P is contained or is equal to the set of the strong possible winners for P;*
3. *$SC(M(P)) = SC(P)$, that is, the set of the strong Condorcet winners for the majority graph induced by P is equal to the set of the strong Condorcet winners for P;*
4. *$WC(M(P)) = WC(P)$, that is, the set of the weak Condorcet winners for the majority graph induced by P is equal to the set of the weak Condorcet winners for P.*

Proof: Let $U(P)$ be the unweighted profile obtained from P.

- 1st and 2nd item:
Since the completions of P are a subset of the completions of $U(P)$, $WP(P) \subseteq WP(U(P))$ and $SP(P) \supseteq SP(U(P))$. Now, since $M(P) = M(U(P))$ by Theorem 2, and since $SP(G)$ and $WP(G)$ depend only on the majority graph G under consideration, $WP(M(P)) \supseteq WP(P)$ and $SP(M(P)) \subseteq SP(P)$.
- 3rd and 4th item:
To prove that $SC(P) = SC(M(P))$, we may notice that $SC(P) = SC(U(P))$ by Theorem 2, $SC(U(P)) = SC(M(U(P)))$ by Theorem 1, and $SC(M(U(P)) = SC(M(P))$ by Theorem 2 and by the fact that $SC(G)$ depends only on the majority graph G considered. The same reasoning allows us to conclude that $WC(P) = WC(M(P))$. □

Note that Theorems 1 and 3 show that the same relationships hold with or without weights. It is perhaps interesting to observe that a stronger relationship cannot be shown to hold in the more specific case of unweighted votes.

Complexity of determining winners

We now turn our attention to the complexity of determining possible and Condorcet winners from profiles. We start by showing that computing the weak or strong Condorcet winners is polynomial in the number of agents and candidates.

Theorem 4 *Given an incomplete weighted profile P, the sets $WC(P)$ and $SC(P)$ are polynomial to compute.*

Proof: By Theorem 3, $WC(P) = WC(M(P))$ and $SC(P) = SC(M(P))$. Moreover, by Theorem 2 we know that $M(P) = M(U(P))$, where $U(P)$ is the corresponding unweighted profile obtained from P. Thus, we have that $WC(P) = WC(M(U(P)))$ and $SC(P) = SC(M(U(P)))$. In (Lang *et al.* 2007) the authors show that, given any majority graph G obtained from an unweighted profile, it is polynomial to compute $WC(G)$ and $SC(G)$. Hence, it is polynomial to compute $WC(M(U(P)))$ and $SC(M(U(P)))$. □

Since, as noted above, WC(P) is related to destructive control and SC(P) is related to the possibility of control or manipulation, this means that:

- It is easy for a chair to control destructively the election. That is, given a candidate A, it is easy for the chair to know whether, no matter how votes will be completed, there is an agenda where A does not win.
- It is also easy for a chair or a voter to know whether control/manipulation is possible.

We next show that computing weak possible winners is intractable in general.

Theorem 5 *Given an incomplete weighted profile P with 3 or more candidates, deciding if a candidate is in WP(P) is NP-complete.*

Proof: Clearly the problem is in NP as a polynomial witness is a completion and an agenda in which the candidate wins. To show it is NP-complete, we give a reduction from the number partitioning problem. The reduction is based around constructing a Condorcet cycle and is similar to those used in (Conitzer & Sandholm 2002a) to show that manipulation is computationally hard even with a small number of candidates when votes are weighted. The reduction is, however, different to that in Theorem 8 in (Conitzer & Sandholm 2002a) as the reduction there concerns the randomized Cup rule and requires 7 or more candidate.

We have a bag of integers, k_i with sum $2k$ and we wish to decide if they can be partitioned into two bags, each with sum k. We want to show that a candidate B is a weak possible winner if and only if such a partition exists. We construct an incomplete profile over three candidates (A, B, and C) as follows. We have 1 vote for $B > C > A$ of weight 1, 1 vote $B > A > C$ of weight $2k-1$, and 1 vote $C > B > A$ of weight $2k-1$. At this point, the total weight of votes with $B > A$ exceeds that of $A > B$ by $4k-1$, the total weight of votes with $B > C$ exceeds that of $C > B$ by 1, and the total weight of votes with $C > A$ exceeds that of $A > C$ by 1.

We also have, for each k_i, a partially specified vote of weight $2k_i$ in which we know just that $A > B$. As the total weight of these partially specified votes is $4k$, we are sure A beats B in the final result by 1 vote. The only agenda in which B can win is the one in which A plays C and the winner then plays B. In addition, for B to win, the partially specified votes need to be completed so that B beats C, and

C beats A in the final result. We show that this is possible iff there is a partition of size k. Suppose there is such a partition. Then let the votes in one bag be $A > B > C$ and the votes in the other be $C > A > B$. Then, A beats B, B beats C and C beats A, all by 1 vote in the final result. On the other hand, suppose there is a way to cast the votes to give the result A beats B, B beats C and C beats A. All the uncast votes rank A above B. In addition, at least half the weight of votes must rank B above C, and at least half the weight of votes must rank C above A. Since A is above B, C cannot be both above A and below B. Thus precisely half the weight of votes ranks C above A and half ranks B above C. Hence we have a partition of equal weight. We therefore conclude that B can win iff there is a partition of size k. That is, deciding if B is a weak possible winner is NP-complete. We can extend the reduction to more than 3 candidates by placing any additional candidate at the bottom of every voters' preference ordering (it does not matter how). $\square$

Note that computing weak possible winners from an incomplete majority graph is polynomial (Lang *et al.* 2007). Thus, adding weights to the votes and computing weak possible winners from the incomplete profile instead of the majority graph makes the problem intractable. On the other hand, adding weights to the votes did not make weak and strong Condorcet winners harder to compute.

We recall that $WP(P)$ is related to participation incentives. Thus Theorem 5 tells us that it is difficult for a candidate to know whether they have chance to win. This makes it less probable that they drop out.

Also, $WP(P) \subseteq WP(M(P))$ (see Theorem 3). Thus, while $WP(P)$ is difficult to compute, it is easy to compute a superset of it, that is, $WP(M(P))$.

The complexity of determining strong possible winners from an incomplete profile (that is, the set $SP(P)$) remains an open problem. However, we know that computing strong possible winners from incomplete majority graphs (that is, $SP(M(P))$) without weights is easy (Lang *et al.* 2007). This gives us an easy way to compute a subset of $SP(P)$: if $SP(M(P))$ is not empty, we can easily compute it and find at least some of the candidates in $SP(P)$.

Fair possible and Condorcet winners

All the notions of winners defined so far consider agendas of any shape. Agendas that are unbalanced may not be considered "fair". Such agendas may allow weak candidates, that can beat only a small number of candidates, to end up winning the election. We therefore consider, as in (Lang *et al.* 2007), agendas that are balanced binary trees.

Given a complete profile P, a candidate A is said to be a *fair possible winner* for P iff there is a balanced agenda in which A wins. A balanced agenda is a binary tree in which the difference between the maximum and the minimum depth among the leaves is less than or equal to 1. Testing whether a candidate is a fair possible winner over weighted majority graphs is NP-hard (Lang *et al.* 2007).

Example 6 *Figure 3 shows two balanced agendas.*

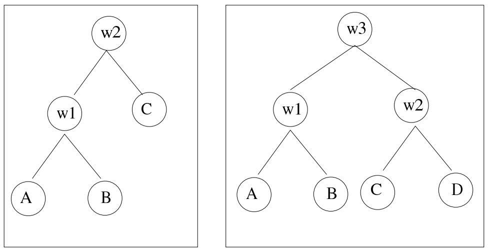


Figure 3: Two balanced agendas.

We now apply this notion of fairness to our definition of winners based on incomplete profiles. Thus, given an incomplete weighted profile P, we define *fair strong Condorcet* (FSC(P)), *fair weak Condorcet* (FWC(P)), *fair strong possible* (FSP(P)), and *fair weak possible* (FWP(P)) winners in an analogous way to Definition 1 but limited to fair agendas. For example, a candidate is in FSC(P) iff they win in all completions of profile P and in all *balanced* agendas.

We now show that it is easy to compute fair weak Condorcet or fair strong Condorcet winners based on the observation that fairness does not change these sets.

Theorem 6 *Given an incomplete weighted profile P,*

- *$FSC(P) = SC(P)$ and $FWC(P) = WC(P)$;*
- *$FWC(P)$ and $FSC(P)$ are polynomial to compute.*

Proof: We first show that $FSC(P) = SC(P)$ (resp., $FWC(P) = SW(P)$).
($\supseteq$) If $A \in SC(P)$ (resp., $WC(P)$), for every completion (resp., for some completion) of P, A wins in every agenda. Thus A wins also in balanced agendas. Hence, $A \in FSC(P)$ (resp., $A \in FWC(P)$).
($\subseteq$) If $A \in FSC(P)$ (resp., $FWC(P)$), for every completion (resp., for some completion) of P, A wins in every balanced agenda. In every balanced agenda, A must win against at least a candidate (the one in A's first match). If A wins in every balanced agenda, it therefore means that A must win against every candidate. Thus A wins in every agenda. Thus, $A \in SC(P)$ (resp., $A \in WC(P)$).

Since $FSC(P) = SC(P)$ and $FWC(P) = SW(P)$, and since, by Theorem 4, $SC(P)$ and $WC(P)$ are polynomial to compute, also $FSC(P)$ and $FWC(P)$ are polynomial to compute. $\square$

Thus, the test for destructive control (related to WC) and for the possibility of control/manipulation (related to SC) are easy even when we consider only fair agendas.

Let us now consider fair weak possible winners. Since every balanced agenda is also an agenda, we have that $FWP(P) \subseteq WP(P)$. We already know from Theorem 5 that determining WP(P) is difficult. We will now show that this remains so for FWP(P).

Theorem 7 *Given an incomplete weighted profile P with 3 or more candidates, deciding if a candidate is in FWP(P) is NP-complete.*

Proof: We use the same construction as in the proof of Theorem 5. Given the profile constructed there, the only possible fair agendas in which B wins are those in which A plays C, and (at some later point) B then plays the winner. All the additional candidates will be defeated by A, B and C so can be placed anywhere in the fair agenda. □

Since the notion of weak possible winner is related to the concept of losers (losers are those not in WP), this means that it is difficult to know whether a candidate is a loser (or alternatively still has a chance to win). This difficulty remains so even if we consider only balanced agendas.

The computational complexity of determining fair strong possible winners remains an open question, just as is the complexity of computing strong possible winners. We only know that, since every balanced agenda is also an agenda, $FSP(P) \subseteq SP(P)$.

Related work

There has been much research on the computational complexity of determining winners of various kinds for several voting rules, and of the relationship with the complexity of problems found in preference elicitation and manipulation. Our results follow this same line of work while focusing on sequential majority voting.

The most related work is (Lang *et al.* 2007) Like our paper, this considers the computational complexity of determining winners for sequential majority voting. However, they start from an incomplete majority graph which throws away information about individual votes, whilst we start from an incomplete profile.

Conitzer and Sandholm also consider sequential majority voting (Conitzer & Sandholm 2002a), but they assume a complete profile and a fixed agenda. They show that, if the agenda is fixed and balanced, determining the candidates that win in at least one completion of the profile is polynomial, but randomizing the agenda makes deciding the probability that a candidate wins (and thus manipulation) NP-hard. They also prove that constructive manipulation is intractable for the Borda, Copeland, Maximin and STV rules using weighted votes even with a small number of candidates. However, all of these rules are polynomial to manipulate destructively except STV.

Conitzer and Sandholm also prove that deciding if preference elicitation is over (that is, determining if the remaining votes can be cast so a given candidate does not win) is NP-hard for the STV rule (Conitzer & Sandholm 2002b). For other common voting rules like plurality and Borda, they show that it is polynomial to decide if preference elicitation is over.

The notions of possible and necessary winners are not new. They were introduced by Konczak and Lang in (Konczak & Lang 2005) in the context of positional scoring voting rules with incomplete profiles. A possible winner in (Konczak & Lang 2005) is a candidate that can win in at least a completion of profile, while a necessary winner is a candidate that wins in every completion of the profile. We have adapted these notions to the context of sequential majority voting with complete profiles, where the unknown part is the agenda. Hence, we have defined possible winners as those candidates that may win in at least an agenda and necessary winners (called here Condorcet winners) as those candidates that win in every agenda. We have also considered the presence of incomplete profiles and in this case we have defined new notions of winners: weak (resp., strong) possible and necessary winners, that are those candidates that are possible and necessary winners in some (resp., in all) completions of the profile. We have also analyzed the complexity of determining weak and strong possible and necessary winners from incomplete profiles for sequential majority rule, and we have shown that determining weak possible winners is NP-hard, whilst determining the other kinds of winners is polynomial. Konczak and Lang proved that it is polynomial to compute both possible and necessary winners for positional scoring voting rules like the Borda and plurality rule, as well as for a non-positional rule like Condorcet (Konczak & Lang 2005).

Pini *et al.* prove that computing the possible and necessary winners for the STV rule is NP-hard (Pini *et al.* 2007). They show it is NP-hard even to approximate these sets within some constant factor in size. They also give a preference elicitation procedure which focuses just on the set of possible winners.

Finally, Brandt *et al.* consider different notions of winners starting from incomplete majority graphs (Brandt, Fischer, & Harrenstein 2007). We plan to investigate these kinds of winners in our framework.

Conclusions

We have considered multi-agent settings where agents' preferences may be incomplete and are aggregated using weighted sequential majority voting. For this setting, we have shown that it is easy to determine weak and strong Condorcet winners, i.e., to determine the candidates that win whatever the agenda, while it is hard to know whether a candidate is a weak possible winner, i.e., if the candidate wins in at least one agenda for at least one completion of the agents' preferences. This is hard even if we require that the agenda be a balanced tree. These results show that, for weighted sequential majority voting, it is

- computationally easy to test if destructive control is possible, even if we consider only fair agendas;
- computationally easy to test if there is a guaranteed winner, even for fair agendas;
- computationally difficult to test if a candidate is a loser, even for fair agendas.

Our results are thus useful to understand the complexity of both manipulation and preference elicitation.

The computational complexity of testing whether constructive control is possible (that is, of finding strong possible winners) with fair or unfair agendas remains open and needs to be studied further. Another interesting direction for future work is deciding which candidates are most likely to win, which is related to probabilistic approaches to voting theory. We also plan to study other forms of uncertainty in the application of the voting rule, such as uncertain weights in a scoring rule, or a chair who can choose between different voting rules. We intend to analyze the presence of ties in agents' preferences. Adding ties requires adding a tie-breaking rule to be able to declare a winner in each pairwise comparison. We believe similar results can be derived for such weak (as opposed to total) orders. The analysis will have to be more complex to deal with the extra cases. However, the set of completions of the majority graph remains a superset of the set of completions of the profile. Thus all the results based on this fact still hold.

Acknowledgments

This work has been partially supported by Italian MIUR PRIN project "Constraints and Preferences" (n. 2005015491). The last author is funded by the Department of Broadband, Communications and the Digital Economy, and the Australian Research Council. We would like to thank Jerome Lang for his valuable comments.

References

Bartholdi, J.; Tovey, C.; and Trick, M. 1989. The computational difficulty of manipulating an election. *Social Choice and Welfare* 6(3):227–241.

Brandt, F.; Fischer, F.; and Harrenstein, P. 2007. The computational complexity of choice sets. In *Proc. TARK 2007*, 82–91.

Conitzer, V., and Sandholm, T. 2002a. Complexity of manipulating an election with few candidates. In *Proc. AAAI-02*, 314–319.

Conitzer, V., and Sandholm, T. 2002b. Vote elicitation: complexity and strategy-proofness. In *Proc. AAAI-02*, 392–397.

Konczak, K., and Lang, J. 2005. Voting procedures with incomplete preferences. In *Proc. IJCAI-05 Multidisciplinary Workshop on Advances in Preference Handling*.

Lang, J.; Pini, M. S.; Rossi, F.; Venable, K. B.; and Walsh, T. 2007. Winner determination in sequential majority voting. In *Proc. IJCAI 2007*, 1372–1377. AAAI Press.

Laslier, J.-F. 1997. *Tournament solutions and majority voting*. Springer.

Moulin, H. 1991. *Axioms of Cooperative Decision Making*. Cambridge University Press.

Pini, M. S.; Rossi, F.; Venable, K. B.; and Walsh, T. 2007. Incompleteness and incomparability in preference aggregation. In *Proc. IJCAI 2007*, 1464–1469. AAAI Press.