Aggregating partially ordered preferences

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Abstract. Preferences are not always expressible via complete linear orders: sometimes it is more natural to allow for the presence of incomparable outcomes. This may hold both in the agents' preference ordering and in the social order. In this paper we consider this scenario and we study what properties it may have. In particular, we show that, despite the added expressivity and ability to resolve conflicts provided by incomparability, classical impossibility results (such as Arrow's theorem, Muller-Satterthwaite's theorem, and Gibbard-Satterthwaite's theorem) still hold. We also prove some possibility results, generalizing Sen's theorem for majority voting. To prove these results, we define new notions of unanimity, monotonicity, dictator, triple-wise value-restriction, and strategy-proofness, which are suitable and natural generalizations of the classical ones for complete orders.

1 Introduction

Many problems require the aggregation of preferences of different agents. For example, when planning a wedding, we must combine the preferences of the bride, the groom and possibly some or all of the in-laws. Incomparability is an useful mechanism to resolve conflict when aggregating such preferences. If half of the agents prefer a to b and the other half prefer b to a, then it may be best to say that a and b are incomparable. In addition, an agent's preferences are not necessarily total. As Kelly noticed in page 58 of [13], "completeness is not an innocuous assumption" since one could be asked "[...] to judge between two alternatives so unlike one another he just can't compare them". For example, while it is easy and reasonable to compare two apartments, it may be difficult to compare an apartment and a house. We may wish simply to declare them incomparable. Moreover, an agent may have several possibly conflicting preference criteria he wants to follow, and their combination can naturally lead to an incomplete order. For example, one may want a cheap but large apartment, so an 80 square metre apartment which costs 100.000 euros is incomparable to a 50 square metre apartment which costs 60.000 euros. We study the situation here that both the preferences of an agent and the result of preference aggregation can be an incomplete order (that is, a reflexive and transitive binary relation).

In this context, it is natural to ask if we can combine preferences satisfying some natural and desirable properties. For example, it is certainly reasonable to require that if all agents expressing their preferences agree that candidate A is better than candidate B, this should also be true in the result of the preference aggregation. This property in commonly known as *unanimity*. Moreover, it is also desirable that the order between two agents in the result of the preference aggregation should be independent of how other candidates are ranked. This is an intuitive description of what is known in the literature as *independence to irrelevant alternatives*. Finally, any reasonable preference aggregation rule should not allow for the existence of an agent that, regardless what other agents vote, is able to dictate (more or less strictly) his own ordering in the result. The absence of a such an agent has been denoted in the literature as *non-dictatorship*.

For complete orders, Arrow's theorem shows that it is impossible for such properties to coexist [1]. We show that this result can be generalized to incomplete orders under certain conditions. This is both disappointing and a little surprising. By moving from complete orders to incomplete orders, we enrich greatly our ability to combine outcomes. As in the example above, we can use incomparability to resolve conflict and thereby not contradict agents. Nevertheless, under the conditions identified here, we still do not escape the reach of Arrow's theorem.

These results assume that one is interested in obtaining a (possibly incomplete) order over the different outcomes as the result of preference aggregation. One may wonder if the situation is easier when we are only interested in the most preferred outcomes in the aggregated preferences. However, we show that even in this case (that is, when considering social choice functions over incomplete orders) the impossibility result continues to hold. This is a generalization of the Muller-Satterthwaite theorem [18] for strict complete orders.

We also identify two cases of social welfare functions over strict incomplete orders where it is possible to satisfy at the same time independence to irrelevant alternatives, unanimity and non-dictatorship. One of such cases corresponds to the generalization of Sen's theorem for majority voting over strict complete orders [23]. The second case concerns profiles with no chain of ordered pairs. We then consider the notion of strategy-proofness, which denotes the non-manipulability of a social choice function. The Gibbard-Satterthwaite result [10] tells us that it is not possible for a social choice function defined over strict complete orders to be at same time non-manipulable and have no dictators. We prove that this result holds also when the social choice function is defined over strict incomplete orders. To do this, we first generalize the notion of strategy-proofness to partially ordered preferences.

The paper is organized as follows. In Section 2 we give the basic notions, as well as, the main results on social welfare and social choice functions. Section 3 then defines social welfare functions on partially ordered profiles and introduces the considered properties, while Sections 4 and 5 give our possibility and impossibility results with incomplete orders. Section 6 shows cases in which the majority rule can simultaneously satisfy all the properties. Then, Section 7 defines social choice functions over strict incomplete orders and their properties, and Section 8 proves the corresponding impossibility result. Section 9 defines the notion of strategy-proofness for social choice functions over incomplete orders and shows that the Gibbard-Satterthwaite result holds also in the case of strict incomplete orders. Finally, Section 10 describes related work, and Section 11 summarizes the results of the paper and gives directions for future work.

All the proofs of the results in this paper are contained in the Appendix. This paper is an extended and improved version of [20, 19].

2 Background

We consider the aggregation of the preferences of a set of n agents, N, over a set of outcomes Ω. The preferences of each agent over the outcomes are modelled by a binary relation (that is, a subset of $\Omega \times \Omega$). We use the symbol $\succsim$ to denote such a binary relation. Intuitively, when we write $a \succsim b$ we mean that "a is at least as good as b".

2.1 Basic notions

An **incomplete order (IO)** is a binary relation which is reflexive and transitive. That is,

- $\forall a \in \Omega$, $a \succsim a$;
- $\forall a, b, c \in \Omega$, with $a \neq b \neq c$, $a \succsim b$ and $b \succsim c$ implies $a \succsim c$.

Given an IO $\succsim$, we write:

- $a \bowtie b$, read "a is incomparable with b", iff $a \not\succsim b$ and $b \not\succsim a$;
- $a \succ b$, read "a is preferred to b", iff $a \succsim b$ and $b \not\succsim a$;
- $a \sim b$, read "a is indifferent to b", iff $a \succsim b$ and $b \succsim a$.

Thus, a and b are incomparable when they are not ordered. They are instead indifferent when they are ordered in both directions. Notice that indifference is different from equality. Notice also that, while incomparability ($\bowtie$) is not transitive in general, indifference ($\sim$) and strict preference ($\succ$) are transitive.

A **complete order (CO)** is an IO where, $\forall a, b \in \Omega$ with $a \neq b$, $a \succsim b$ or $b \succsim a$.

An IO (resp., CO) is **strict** (written SIO, resp., SCO) when $\forall a, b \in \Omega$ with $a \neq b$, we have $a \not\sim b$.

Given an outcome $t \in \Omega$, t is said to be **undominated** iff $\forall a \in \Omega$ with $a \neq t$, $a \not\succ t$ (that is, either $t \succ a$, or $t \sim a$, or $t \bowtie a$).

Both incomplete and complete orders can have more than one undominated outcome. Given an IO or CO o, we write

- **top(o)**= $\{t \in \Omega | \forall a \in \Omega$ with $a \neq t, a \not\succ t\}$ and
- **bottom(o)**= $\{b \in \Omega | \forall a \in \Omega$ with $a \neq b, b \not\succ a\}$.

In the case of COs, all undominated outcomes are indifferent, while in IOs any two undominated outcomes can be either indifferent or incomparable.

To summarize, in a CO, two elements can be either ordered or indifferent. On the other hand, in an IO, two elements can be either ordered, indifferent, or incomparable.

In this paper we will sometimes need to consider IOs with some restrictions. In particular, a **restricted IO (rIO)** is an IO o where

- $\forall t, t' \in top(o)$, $t \sim t'$, or
- $\forall b, b' \in bottom(o)$, $b \sim b'$.

Thus, in a rIO, the top elements are all indifferent, or the bottom elements are all indifferent. Notice that the "or" is not an exclusive or, thus it is possible that both the top elements and the bottom elements are all indifferent.

The intuition behind the notion of an rIO is to move from COs towards IOs by allowing incomparability everywhere except in one extreme (top or bottom) of the ordering. In other words, in an rIO there is a preference level which is highest (or lowest) than all others.

2.2 Social welfare functions

We will now give the main notions of social welfare [2, 21, 12], that will be useful in the rest of the paper.

Assume that each agent specifies his preferences via a CO over the possible outcomes. A **profile** p over COs (resp., SCOs) is a sequence of n COs (resp., SCOs)$p_1, \ldots, p_n$ over outcomes, one for each agent $i \in N = \{1, \ldots, n\}$. Given an agent i, we denote the collection of COs of all the other agents by p_{-i}. We will denote the set of all profiles over COs with $\mathcal{P}^*$.

A **social welfare function** f over COs is a function from $\mathcal{P}^*$ to the set of COs over the outcomes.

Given a profile p over COs, we denote the CO of agent i in p by p_i.

We denote with $f(p)$ the CO obtained by applying f to p. Moreover, given $a, b \in \Omega$, we will write $p_i(a, b)$ (resp., $f(p)(a, b)$) to indicate the restriction of ordering p_i (resp., $f(p)$) on the two outcomes a and b.

There is a number of desirable properties for a social welfare function over COs. In particular, given a social welfare function f over COs, we consider the following properties:

- **Freeness**: f is surjective, that is, for every CO o, there is a profile p over COs such that $f(p) = o$.

- **Unanimity**: for every profile p over COs, for all $a, b \in \Omega$, if $a \succ_{p_i} b$ for all $i \in N$, then $a \succ_{f(p)} b$. In words, if all agents agree that a is preferable to b, then the resulting order must agree as well.
- **Independence to irrelevant alternatives** (IIA): for every pair of profiles p, p' over COs, for all $a, b \in \Omega$, if $p_i(a, b) = p'_i(a, b)$ for all $i \in N$, then $f(p)(a, b) = f(p')(a, b)$. In words, the ordering between a and b in the result depends only on the relation between a and b given by the agents.
- **Monotonicity**: for every pair of profiles p, p' over COs, for all agents $i \in N$, for all $a, b \in \Omega$, if $a \succ_{f(p)} b$ and, whenever $a \succ_{p_i} b$, then $a \succ_{p'_i} b$, then $a \succ_{f(p')} b$. In words, if a is preferable to b in the result, and we move to a profile where, if a was preferable to b, it continues to be preferable to b, then a is still preferable to b in the new result.
- **Absence of a dictator**: a **dictator** is an agent i such that, in every profile p over COs, if $a \succ_{p_i} b$ then $a \succ_{f(p)} b$. If there is no dictator, f is said to be non-dictatorial.

One of the most influential results in social welfare theory states the impossibility of the coexistence of IIA, unanimity, and absence of a dictator for every social welfare function.

Theorem 1 (Arrow's theorem [1]). *Given a social welfare function f over COs, assume that there are at least two agents and three outcomes. If f is unanimous and IIA, then f is dictatorial.*

It is possible to prove that freeness, monotonicity and IIA imply unanimity (see Chapter 3 of [12]). On the other hand, there are social welfare functions which are free, unanimous and IIA but not monotonic [23]. Thus, unanimity and IIA does not imply freeness and monotonicity. Therefore a weaker version of Arrow's result on complete orders states that, with at least two voters and three outcomes, it is not possible to have at the same time freeness, monotonicity, IIA, and non-dictatorship [12].

A very reasonable voting function which is often used in elections is the majority rule. In this voting function, for each pair of outcomes, the order between them in the result is what the majority says on this pair. This function, however, can produce an ordering which is cyclic and thus it is not a social welfare function. A sufficient condition on profiles that avoids generating cycles is the **triple-wise value-restriction** [23]: for every triple of outcomes x_1, x_2, x_3, there exists $x_i \in \{x_1, x_2, x_3\}$ and $r \in \{1, 2, 3\}$ such that no agent ranks x_i as his r-th preference among $x1$, $x2$, $x3$.

Given such a restriction, it has been shown that the majority rule is indeed a social welfare function that satisfies unanimity, IIA and has no dictators.

such a restriction is sufficient also to have the three above properties (IIA, unanimity, and absence of a dictator) coexist for majority voting.

Theorem 2 (Sen's theorem [23]). *The majority rule f over profiles (over SCOs) satisfying the triple-wise value-restriction is a unanimous, IIA and non-dictatorial social welfare function.*

2.3 Social choice functions

When we are aggregating the preferences of a number of agents, we might not need to order all outcomes. It might be enough to know the "most preferred" outcome.

A **social choice function** on SCOs [2, 10, 21] is a mapping from a profile over SCOs to an outcome, also called the winner. Given a social choice function f and a profile p over SCOs, we denote the winner with $f(p)$.

We now give some desirable properties that social choice functions should have. Let a, b be two outcomes in Ω. We say that a social choice function f over SCOs is:

- **onto** iff, for any outcome a, there is a profile p over SCOs such that $f(p) = a$.
- **unanimous** iff, for every profile p over SCOs, $a = top(p_i)$ for every agent i implies $f(p) = a$. In words, a social choice function is unanimous if, when an outcome is the most preferred element of all the agents' orderings, then it is the winner.
- **monotonic** iff, for every pair of profiles p and p' over SCOs, if $f(p) = a$, and for every other outcome b and for every agent i, $a \succ_{p_i} b$ implies $a \succ_{p'_i} b$, then $f(p') = a$. In words, a social choice function over SCOs is monotonic if, whenever an agent moves up the position of the winner in his ordering, then such an outcome continues to be the winner.
- **non-dictatorial**, iff there is no dictator. A **dictator** is an agent i such that, for every profile p over SCOs, $f(p) = top(p_i)$. In words, a dictator is an agent such that the winner is its most preferred outcome.
- **strategy-proof** iff, for every agent i, for every pair of profiles p, p' over SCOs, which are different only for the ranking of agent i, (i.e., $p' = (p_{-i}, p'_i)$) $f(p) \neq f(p')$ implies that $f(p) \succ_{p_i} f(p')$, i.e., $f(p)$ is better than $f(p')$ according to the ranking given by agent i in profile p.

The notion of strategy-proofness has also been considered in the more general case of multi-valued social choice functions, which return sets of winners instead of a single winner. In particular, [27] proposed the following generalization of the above definition of strategy-proofness for multi-valued social choice functions over SCOs: a multi-valued social choice function f over SCOs is **strategy-proof** iff for every agent i, for every pair of profiles p and p' over SCOs, which are different only for the ranking of agent i, (i.e., $p' = (p_{-i}, p'_i)$),

– $\forall a \in f(p) \setminus f(p'), \forall b \in f(p'), a \succ_{p_i} b$;
– $\forall a \in f(p') \setminus f(p), \forall b \in f(p), a \prec_{p_i} b$.

Results similar to Arrow's theorem have been proved in the context of social choice functions.

Theorem 3 (Muller-Satterthwaite's theorem [18]). *Given a social choice function f over SCOs, assume that there are at least two agents and three outcomes. If f is unanimous and monotonic, then f is dictatorial.*

Proposition 1 ([18]). *Given a social choice function f over SCOs, if f is strategy-proof and onto, then f is unanimous and monotonic.*

Theorem 4 (Gibbard-Satterthwaite's theorem [10, 22]). *Given a social choice function f over SCOs, assume that there are at least two agents and three outcomes. If f is onto and strategy-proof, then f is dictatorial.*

3 Social welfare functions over IOs

In this paper we consider profiles over IOs. This means that both the agents' ordering and the social ordering may have incomparable outcomes.

The definitions of profiles, social welfare functions, as well as the properties of freeness, unanimity and IIA are the same as in Section 2, except that they apply to IOs instead of COs. What is significantly different are the notions of monotonicity and dictator.

Monotonicity: for every pair of profiles p, p' over IOs, for every agent $i \in N$, for all $a, b \in \Omega$, if $a \succ_{f(p)} b$ and

– whenever $a \succ_{p_i} b$, then $a \succ_{p'_i} b$,
– whenever $a \sim_{p_i} b$ or $a \bowtie_{p_i} b$, then $a \not\prec_{p'_i} b$,

then $a \succ_{f(p')} b$. In words, if a is preferable to b in the result, and we move to a profile where a is not worsened w.r.t. b, then a is still preferable to b in the new result.

When we may have incomparable candidates, different notions of a dictator can be considered. In this paper we define the following three notions:

Strong dictator: an agent i such that, for every profile p over IOs, $f(p) = p_i$, that is, his ordering is the result;
Dictator: an agent i such that, for every profile p, if $a \succ_{p_i} b$ then $a \succ_{f(p)} b$.
Weak dictator: an agent i such that, for every profile p, if $a \succ_{p_i} b$, then $a \succ_{f(p)} b$ or $a \bowtie_{f(p)} b$.

Notice that, for the notions of dictator and weak dictator, nothing is said about the relationship between a and b in $f(p)$ if a is incomparable or indifferent to b in p_i. Notice also that the definitions of dictator and weak dictator given above, when restricted to complete orders, coincide with the one considered by Arrow [2].

Clearly, a strong dictator is a dictator, and a dictator is a weak dictator. Note also that, whilst there can only be one strong dictator or dictator, there can be any number of weak dictators. For example, the social welfare function which, given any profile, puts every outcome incomparable to every other outcome in the result, is such that all agents are weak dictators. Instead, nobody is a strong dictator or a dictator.

4 Dictators and strong dictators

Arrow's impossibility theorem [1, 12] shows that, with at least two agents and three outcomes, a social welfare function on complete orders cannot be at the same time unanimous, IIA, and non-dictatorial.

We will now consider the counterpart to Arrow's theorem for our context of partially ordered preferences.

Proposition 2. *A social welfare function over IOs with at least two agents and at least two outcomes, can at the same time be IIA, unanimous, and have no dictator and, thus, no strong dictator.*

For example, the *Pareto rule*, according to which two outcomes are ordered if every agent agrees, but are incomparable otherwise, is IIA, unanimous, and has no dictators. In fact, no agent i stating $a \succ_{p_i} b$ can dictate $a \succ_{Pareto(p)} b$: it is enough that another agent j states that $a \not\succ_{p_j} b$ to have $a \bowtie_{Pareto(p)} b$.

Note that a social welfare function that has no dictators has no strong dictators either. Actually, requiring that an IIA and unanimous function has no strong dictator is a very weak property to demand. Even voting rules which appear very "unfair" may not have a strong dictator.

For example, suppose the agents are ordered in some fixed way and in the result two outcomes are in the same relation as for the first agent (in the fixed ordering on the agents) who declares them not indifferent. This social welfare function (which we will call the *lex rule*) is IIA and unanimous, and has a dictator (the first agent). However the first agent is not a strong dictator, since the result may differ from his own ordering (only) on the pairs that he considers indifferent.

5 Generalizing Arrow's theorem

We now show that, under certain conditions, it is impossible for a social welfare function over incomplete orders to be IIA, unanimous, and have no weak dictator.

The conditions involve the shape of the incomplete orders. In particular, we assume the incomplete orders of the agents to be general (that is, IOs), but the resulting incomplete order to be restricted (that is, rIOs). We recall that a rIO has all top or all bottom elements indifferent.

Theorem 5. *Given a social welfare function f over IOs, assume the result is a rIO, and that there are at least two agents and three outcomes. Then it is impossible that f is unanimous, IIA, and has no weak dictators.*

Proof. In the Appendix.

Consider, for example, the Pareto rule defined in the previous section. This rule is unanimous and IIA. Moreover, every agent is a weak dictator since no agent can ever be contradicted. If an agent i says that $a \succ_{p_i} b$, then the result cannot be $b \succ_{Pareto(p)} a$. Note that we could consider a social welfare function which modifies the Pareto rule by applying the rule only to a strict subset of the agents, and ignoring the rest of the agents. Then the agents in the subset would all be weak dictators.

As with complete orders, we can also prove a weaker result in which we replace unanimity by monotonicity and freeness.

Corollary 1. *Given a social welfare function f over IOs, assume the result is a rIO and there are at least two agents and three outcomes. Then it is impossible that f is free, monotonic, IIA and has no weak dictators.*

Proof. In the Appendix.

A number of other results follow from the above theorem and corollary. If we denote the class of all social welfare functions mapping profiles composed of orderings of type A to an ordering of type B by $A^n \rightsquigarrow B$, then we have proved that, for functions in $IO^n \rightsquigarrow rIO$, IIA and unanimity imply the existence of a weak dictator. The first result concerns the restriction of the codomain of the social welfare functions.

Theorem 6. *If, for all functions in $A^n \rightsquigarrow B$, IIA and unanimity imply the existence of a weak dictator, then this is true also for all functions in $A^n \rightsquigarrow B'$, where B' is a subtype of B.*

Proof. In the Appendix.

This theorem implies, for example, that functions in $IO^n \rightsquigarrow O$, where O is any subtype of rIO, that are IIA and unanimous have at least a weak dictator. For example, this holds for functions in $IO^n \rightsquigarrow CO$, since CO is a subtype of rIO.

Consider now the restriction of the domain of the functions, that is, let us pass from $A^n \rightsquigarrow B$ to $A'^n \rightsquigarrow B$ where A' is a subtype of A.

We are interested in understanding whether IIA and unanimity imply the existence of a weak dictator also when performing such a restriction.

Theorem 7. *If, for all functions in $A^n \rightsquigarrow B$, IIA and unanimity imply the existence of a weak dictator, then this is true also for all functions in $A'^n \rightsquigarrow B$, where A' is a subtype of A.*

Proof. In the Appendix.

Summarizing, we have proved the same impossibility results for all functions with the following types:

- $IO^n \rightsquigarrow rIO$ (by Theorem 5);
- $IO^n \rightsquigarrow CO$ (by Theorem 5 and 6);
- $CO^n \rightsquigarrow rIO$ (by Theorem 5 and 7);
- $CO^n \rightsquigarrow CO$, that is, Arrow's theorem (by the result for $CO^n \rightsquigarrow rIO$ and Theorem 6);
- $CO^n \rightsquigarrow SCO$, (by Arrow's theorem and Theorem 6);
- $SCO^n \rightsquigarrow CO$, (by Arrow's theorem and Theorem 7);
- $SCO^n \rightsquigarrow SCO$ (by the result for $SCO^n \rightsquigarrow CO$ and Theorem 6).

We can arrange these seven results in a lattice where the ordering is given by either domain or codomain subset, as it can be seen in Figure 1. The lattice ordering can be defined as follows: $A \rightsquigarrow B \leq A' \rightsquigarrow B'$ iff $A \subseteq A'$ or $B \subseteq B'$.

$$\begin{tikzcd}
 & IO^n \rightsquigarrow rIO \arrow[dl] \arrow[dr] & \\
IO^n \rightsquigarrow CO \arrow[dr] & & CO^n \rightsquigarrow rIO \arrow[dl] \\
 & CO^n \rightsquigarrow CO \arrow[dl] \arrow[dr] & \\
CO^n \rightsquigarrow SCO \arrow[dr] & & SCO^n \rightsquigarrow CO \arrow[dl] \\
 & SCO^n \rightsquigarrow SCO &
\end{tikzcd}$$

Fig. 1. Lattice of impossibility results.

6 Generalizing Sen's theorem

We now consider ways of assuring that a social welfare function is at the same time IIA, unanimous and without a weak dictator. In fact, we will identify situations where the well known majority rule is transitive, which will allow us to conclude, since it has all the other properties.

The **majority rule**, that we will denote with maj, maps profiles over SIOs into SIOs. Given a profile p over SIOs, and two outcomes $a, b \in \Omega$, we have that

$a \succ_{maj(p)} b$ iff the cardinality of the set $\{i \in N | a \succ_{p_i} b\}$ is greater than the cardinality of the set $\{i \in N | b \succ_{p_i} a$ or $b \bowtie_{p_i} a\}$. If $a \not\succ_{maj(p)} b$ and $b \not\succ_{maj(p)} a$, then $a \bowtie_{maj(p)} b$.

In words, a is preferred to b in the result iff the number of agents which say that $a \succ b$ is greater than the number of agents which say that $b \succ a$ plus the number of those that say that a and b are incomparable.

We focus on the condition that Sen has proved sufficient in the case of COs, namely **triple-wise value-restriction** [23], as defined in Section 2. To apply Sen's theorem to this context, we will consider linearizations of our profiles.

Note that, as we have SIOs instead of SCOs, to assure transitivity in the resulting order, we must avoid both cycles (as in the complete order case) and incomparability in the wrong places. More precisely, if the result has $a \succ b \succ c$, we cannot have $c \succ a$, which would create a cycle, nor even $a \bowtie c$, since in both cases transitivity would not hold. We will say that a profile over SIOs p satisfies the **generalized triple-wise value-restriction** (G-TVR) if all the profiles obtained from p by linearizing any SIO to a SCO have the triple-wise value-restriction property.

Given a SIO, a linearization is a SCO represented by a binary relation which includes the relation of the IO.

Theorem 8. *The majority rule over profiles (over SIOs) satisfying the generalized triple-wise value-restriction is an IIA and unanimous social welfare function with no weak dictators.*

Proof. In the Appendix.

We have therefore generalized Sen's theorem for SCOs to SIOs. This result is useful when the profiles have few incomparable pairs and, within each profile, the agents have similar preference orderings.

On the other extreme, we will now give another possibility result which can be applied to profiles which order few outcomes. This result assures transitivity of the resulting ordering by a very simple approach: it just avoids the presence of chains in the result. That is, for any triple a_1, a_2, a_3 of outcomes, it makes sure that the result cannot contain $a_i \succ a_j \succ a_k$ where i, j, k is any permutation of $\{1, 2, 3\}$. This is done by restricting the classes of orderings allowed for the agents.

A profile p over SIOs is **non-chaining** iff one of the following occurs:

- for every triple of distinct outcomes, for every agent $i \in N$,
  - the outcomes are all incomparable in p_i, or
  - only two of them are ordered in p_i, or
  - two of them are incomparable and the other one is *more preferred* than both of them in p_i;
- for every triple of distinct outcomes, for every agent $i \in N$,
  - the outcomes are all incomparable in p_i;

- only two of them are ordered in p_i, or
- two of them are incomparable and the other one is *less preferred* than both of them in p_i;

Theorem 9. *The majority rule over non-chaining profiles over SIOs is an IIA and unanimous social welfare function with no weak dictators.*

Proof. In the Appendix.

7 Social choice functions over strict IOs

When we are aggregating preferences expressed using IOs, as in the case of COs, it might be enough to know the "most preferred" outcomes. For example, when aggregating the preferences of two people who want to buy an apartment, we don't need to know whether they prefer an 80 square metre apartment at the ground floor or a 50 square metre apartment at the 2nd floor, if they both prefer a 100 square metre apartment at the 3rd floor. They would just buy the 3rd floor apartment without trying to order the other two apartments. Social choice functions identify such most preferred outcomes, and do not care about the ordering on the other outcomes.

We consider here a generalization of social choice functions on SIOs. The definition of ontoness coincided with the one given for social choice functions over SCOs in Section 2. All other notions need to be extended. Let a, b be two outcomes in Ω and $A \subseteq \Omega$. We say that a social choice function f is:

- **unanimous** iff, for every profile p over SIOs,
  - $a \in top(p_i)$ for every agent i implies $a \in f(p)$;
  - $top(p_i) = A$ for every agent i implies $f(p) = A$.

  In words, a social choice function is unanimous if, when an outcome is among the most preferred elements of all the agents' orderings, then it is a winner. Also, when a set of outcomes is the set of most preferred elements of all the agents, then it coincides with the set of winners.
- **monotonic** iff, for every pair of profiles p and p' over SIOs,
  - if $a \in f(p)$ and for every other outcome b and for every agent i
    - $a \succ_{p_i} b$ implies $a \succ_{p'_i} b$ and
    - $a \bowtie_{p_i} b$ implies $a \bowtie_{p'_i} b$ or $a \succ_{p'_i} b$,

    then $a \in f(p')$;
  - if $f(p) = A$ for every outcome a in A, and for every outcome b in $\Omega \setminus A$, and for every agent i,
    - $a \succ_{p_i} b$ implies $a \succ_{p'_i} b$, and
    - $a \bowtie_{p_i} b$ implies $a \bowtie_{p'_i} b$ or $a \succ_{p'_i} b$,

then $f(p') = A$.

In words, a social choice function is monotonic if, whenever an agent moves up the position of a winner in his ordering, then such an outcome is not removed from the set of winners. Also, whenever an agent moves up the position of all the winners in his ordering, then the set of winners does not change.

It is important to notice that the above definitions generalize the usual ones given for SCOs described in Section 2.

As for social welfare functions, we define three notions of dictators:

- a **strong dictator** is an agent i such that, for every profile p over SIOs, $f(p) = top(p_i)$. In words, a strong dictator is an agent such that his most preferred outcomes are the winners.
- a **dictator** is an agent i such that, for every profile p over SIOs, $f(p) \subseteq top(p_i)$. In words, a dictator is an agent such that the winners are all among its most preferred outcomes.
- a **weak dictator** is an agent i such that, for every p over SIOs, $f(p) \cap top(p_i) \neq \emptyset$. In words, a weak dictator is an agent such that some of his most preferred outcomes are winners in the result. However, there could be winners which are not among his most preferred outcomes.

Notice that, in every profile p over SIOs, if a is the unique top of a weak dictator i, then $a \in f(p)$. However, this is not true if a is not the unique top of i.

Notice also that these three notions are consistent with the corresponding ones for social welfare functions. More precisely, a dictator (resp., weak, strong) for a social welfare function f over SIOs is also a dictator (resp. weak, strong) for the social choice function f' over SIOs obtained from f by $f'(p) = top(f(p))$ for any profile p over SIOs.

8 Generalizing Muller-Satterthwaite's theorem

We now show that it is possible for a social choice function over strict incomplete orders to be IIA, unanimous, and have no dictators.

Proposition 3. *A social choice function over SIOs can be at the same time unanimous, monotonic, and have no dictators.*

For example, the social choice function corresponding to the Pareto rule is unanimous, monotonic, and has no dictators. Another example is the social choice function which returns $\bigcup_i top(p_i)$, which again is unanimous, monotonic, and has no dictators. However, in both these rules, all the agents are weak dictators. On the other hand, the social choice function corresponding to the lex rule over SIOs, in contrast to that over IOs, has a strong dictator (which is thus also a dictator), that is the most important agent.

In the following theorem we generalize the Muller-Satterthwaite's theorem [18] to social choice functions over SIOs for weak dictators.

Theorem 10. *If we have at least two agents and at least three outcomes, and the social choice function over SIOs is unanimous and monotonic, then there is at least one weak dictator.*

Proof. In the Appendix.

This means that, even if we are only interested in obtaining a set of winners, rather than a whole preference ordering over all the outcomes, it is impossible for the desirable properties to coexist.

9 Generalizing Gibbard-Satterthwaite's theorem

In this section we consider another well known concept in the context of social choice functions, that is, strategy-proofness. Intuitively, a social choice function is strategy-proof if it is best for each agent to order outcomes as they prefer and not to try to vote tactically. We now propose a definition of strategy-proofness in the case of SIOs. Given such a definition, we prove a generalization of Gibbard-Satterthwaite's theorem.

A social choice function f on SIOs is **strategy-proof** iff, for every agent $i \in N$, for every pair of profiles $p, p' \in \mathcal{P}$, where $p' = (p_{-i}, p'_i)$ (i.e., p' is like p except that we substitute p'_i for p_i), $f(p) \neq f(p')$ implies that:

1. $\forall a \in f(p) \setminus f(p')$, $\forall b \in f(p) \cap f(p')$, $a \succ_{p_i} b$ or $a \bowtie_{p_i} b$;
2. $\forall a \in f(p') \setminus f(p)$,
   (a) $\forall b \in f(p)$, $a \prec_{p_i} b$ or $a \bowtie_{p_i} b$, and
   (b) $\exists b \in f(p)$, $a \prec_{p_i} b$.

In words, a function is strategy-proof if, passing from a profile p to a profile $p' = (p_{-i}, p'_i)$, every winner which is eliminated is better (in p_i) than or incomparable to any winner which is kept. Also, any new winner is worse than or incomparable to all old winners, and it is worse than at least one of them.

It can be shown that such a definition of strategy-proofness generalizes the one given in [10] for strict complete orders and social choice functions returning a single winner and the one given in [27] for strict complete orders and multi-valued social choice functions (see Section 2). For example, the definition in [10] is just a specific case of item 2 of our definition. Therefore, if a social choice function over SIOs is strategy-proof according to our definition, then it is strategy-proof also over SCOs for the definitions in [10, 27]. Hence, if a social choice function over SCOs is not strategy-proof for the definitions in [10, 27], then it is not strategy-proof even over SIOs. The implication does not hold in the reverse direction. In fact, there could be a function which is strategy-proof if we look only at profiles over SCOs, while it is manipulable if we look also at profiles over SIOs. The reason is that in the larger domain there could be ways for an agent to lie and profit.

To show that the above definition of strategy-proofness is reasonable, we now provide some examples of situations which, according to this definition, are not considered manipulations. In all examples we will assume that f is a social choice function and p, p' two profiles over SIOs such that $p' = (p_{-i}, p'_i)$.

- $\Omega = \{b, c\}$, $b \bowtie_{p_i} c$, $f(p) = \{b, c\}$ and $f(p') = \{c\}$. That is, we have two outcomes, judged incomparable by p_i, that are both winners in p. However, only c is the winner in p'. In this case, item 1 is satisfied, since the only old winner which is eliminated, b, is incomparable to the one which is kept, c. Item 2 is trivially satisfied since there are no new winners.
- $\Omega = \{a, b, c, d\}$, $a \succ_{p_i} b$, $c \succ_{p_i} d$, and for any other pair of different outcomes $\omega, \omega' \in \Omega$ we have $\omega \bowtie_{p_i} \omega'$. The sets of winners are: $f(p) = \{a, c\}$ and $f(p') = \{b, d\}$. Item 1 is trivially satisfied since no old winners are kept. Item 2 is satisfied as well, since $b \prec_{p_i} a$, $b \bowtie_{p_i} c$, $d \prec_{p_i} c$, and $d \bowtie_{p_i} a$.
- The outcomes and profiles are the same as before. That is, $\Omega = \{a, b, c, d\}$, $a \succ_{p_i} b$, $c \succ_{p_i} d$, and for any other pair of different outcomes $\omega, \omega' \in \Omega$ we have $\omega \bowtie_{p_i} \omega'$. However, now the set of winners are: $f(p) = \{a, b, c\}$ and $f(p') = \{c, d\}$. Item 1 is satisfied since the eliminated winners, a and b, are both incomparable to the one which is kept, that is, c. Moreover, the new winner d is worse than c and incomparable to a and b (item 2).
- $\Omega = \{a, b, c, d\}$, $a \succ_{p_i} b$, $a \succ_{p_i} c \succ_{p_i} d$, and for any other pair of different outcomes $\omega, \omega' \in \Omega$ we have $\omega \bowtie_{p_i} \omega'$. The sets of winners are: $f(p) = \{a\}$ and $f(p') = \{b, c, d\}$. Here no old winner is kept. Thus, item 1 trivially holds. All the new winners, that is, b, c and d, are all worse than the only old winner a. Thus item 2 is satisfied.
- $\Omega = \{a, b, c, d\}$, $a \succ_{p_i} b$ and for any other pair of different outcomes $\omega, \omega' \in \Omega$ we have $\omega \bowtie_{p_i} \omega'$. The sets of winners are: $f(p) = \{a, b, c, d\}$ and $f(p') = \{b, c, d\}$. Here only item 1 applies and it is satisfied since the only old winner which is eliminated, that is, a, is better than b and incomparable to c and d, which are kept.
- $\Omega = \{a, b, c, d, e\}$, $a \succ_{p_i} b$, $c \succ_{p_i} d$, and for any other pair of different outcomes $\omega, \omega' \in \Omega$ we have $\omega \bowtie_{p_i} \omega'$. The sets of winners are $f(p) = \{a, b, c, d, e\}$ and $f(p') = \{c, d, e\}$. Again, only item 1 applies, and it is satisfied since both a and

b are eliminated and they are incomparable to all the other winners which are kept.

We now present some examples of situations which are instead considered manipulations according to our definition of strategy-proofness:

- $\Omega = \{b, c\}$, $b \bowtie_{p_i} c$, $f(p) = \{c\}$ and $f(p') = \{b, c\}$. In this case, the new winner c is not worse than any other old winner. Thus item 2(b) is violated.
- $\Omega = \{a, b, c\}$, $c \succ_{p_i} b \succ_{p_i} a$, $f(p) = \{a, c\}$ and $f(p') = \{b\}$. In this example, the old winners are the two extremes of an ordered chain in p_i and the new winner is the outcome in the middle. This has been recognized to be an ambiguous case also in [3]. For us it is a manipulation (by the violation of item 2(a)), since the new winner b is better than the old winner a which has been eliminated. It is also considered a manipulation by [27].
- $\Omega = \{a, b, c\}$, $a \succ_{p_i} b$, and for any other pair of different outcomes $\omega, \omega' \in \Omega$, we have $\omega \bowtie_{p_i} \omega'$. Also, $f(p) = \{a\}$ and $f(p') = \{b, c\}$. In this case, the new winner c is not worse than any old winner. Thus item 2(b) is not satisfied.
- $\Omega = \{a, b, c, d, e\}$, $a \succ_{p_i} b$ and $c \succ_{p_i} d$, and for any other pair of different outcomes $\omega, \omega' \in \Omega$ we have $\omega \bowtie_{p_i} \omega'$. Also, $f(p) = \{a, b, c, d, e\}$ and $f(p') = \{a, c\}$. Old winners d and b have been removed even if they are worse than, respectively, c and a, which are kept. This violates item 1.
- $\Omega = \{a, b, c, d\}$, $a \succ_{p_i} b$, $c \succ_{p_i} d$, and for any other pair of different outcomes $\omega, \omega' \in \Omega$ we have $\omega \bowtie_{p_i} \omega'$. Also, $f(p) = \{a\}$ and $f(p') = \{b, c, d\}$. In this example there is a new winner, c, which is not worse than any old winner. Item 2(b) is therefore violated.

We now use our definition of strategy-proofness to generalize Gibbard-Satterthwaite's theorem [10, 22] to SIOs. In particular, we show that, if a social choice function is strategy-proof and onto (that is, surjective), then there is at least a weak dictator. To do this, we first prove that, if a social choice function is strategy-proof and onto, then it is unanimous and monotonic. Then we prove the final result by using Theorem 10.

Theorem 11. *If a social choice function f over SIOs is strategy-proof and onto, then it is unanimous and monotonic.*

Proof. In the Appendix.

Theorem 12. *If there are at least two agents and at least three outcomes, and the social choice function over SIOs is strategy-proof and onto, then there is at least one weak dictator.*

Proof. In the Appendix.

When we apply this theorem to a scenario with SCOs rather than SIOs, we get exactly the classical theorem of Gibbard-Satterthwaite [10]. In fact, with SCOs, weak dictators coincide with dictators, and our definition of strategy-proofness coincides with the classical one.

10 Related work

Our results relax the domain and the codomain of social welfare and choice functions. In fact, for both kinds of function, we consider profiles with possibly incomplete orders. Moreover, for social welfare functions, the codomain is again an incomplete order, although restricted. On the other hand, for social choice functions, we consider sets of winners rather than just one.

Since the original theorem by Arrow [1], many efforts have been made to weaken its conditions. In fact, both the domain and the codomain of a social welfare function have been the subject of more relaxed assumptions in several Arrow-like impossibility theorems. Here we review some of them and we relate them to our work.

In [17], Mas-Colell and Sonnenschein define a weak dictator as an agent i such that, for every pair of outcomes a, b if $a \succ_{p_i} b$, then $b \not\succ_{f(p)} a$, i.e., $a \succ_{f(p)} b$ or $a \sim_{f(p)} b$, while, according to our definition, it is an an agent i such that, for every pair of outcomes a, b if $a \succ_{p_i} b$, then $b \not\succsim_{f(p)} a$, i.e., $a \succ_{f(p)} b$ or $a \bowtie_{f(p)} b$. Thus in both definitions weak dictators cannot be contradicted completely.

The Arrovian impossibility result in [17] relaxes transitivity in the codomain. In particular, the domain of the functions is the set of profiles composed of complete orders, while the codomain is the set of complete quasi-transitive binary relations. Quasi-transitivity is a weaker assumption than transitivity. In contrast, we require transitivity both in the agents orderings and in the result. However, we do not require completeness in either.

In [8], Duggan and Schwartz permit the codomain to be an IO, and profiles to be strict weak orders, which are negatively transitive and asymmetric. This structure is more general than complete orders but less general than incomplete orders, since, for example, it does not allow situations where $a \succ_{p_i} b$ and c is incomparable to both a and b. As we do, the author redefines the classical notions of freeness, unanimity and IIA in terms of the orderings used. Moreover, the notions of dictator and vetoer which are introduced correspond, respectively, to our notions of dictator and weak dictator.

In [4], Barthelemy considers social orders to be incomplete, and agents to vote using an incomplete order. However, the set of profiles must be regular, meaning that, for any three alternatives, any configuration of their orders must be present in a profile. Thus we are more general in the profiles and less general in the final ordering. It should be noticed that, in contrast to our setting, the restriction imposed in [4] on the profiles is not compatible with total orders, and thus the classical Arrow's theorem cannot be obtained as a consequence.

In [26], Weymark considers agents who vote using complete orders, while we allow IOs. Moreover, the social order can be an IO, while we consider restricted IOs. In [26] it is shown how the relaxation on codomains considered leads to the existence of groups of voters which are characterized by collective dictatorship properties. Such groups are called α-oligarchies and β-oligarchies depending on which property they satisfy. It is easy to see that a β-oligarch is a weak dictator according to our definition, but not vice-versa.

A similar setting is considered by Dubois, Fargier and Perny in [6]. This work addresses multi-criteria decision making, rather than aggregating preferences of different agents. This means that, what is an agent in our context, is instead a criteria, i.e., a different aspect of the decision problem, in [6]. The orderings on the single criteria are assumed to be complete. This is in contrast with what we propose since we allow agents to vote with incomplete orders. The resulting ordering is instead a quasi-ordering (that is, an IO) and is thus more general than an rIO. However, the result in the paper, which states the existence of an oligarchy, defined in a similar way as in [26], requires several additional assumptions. For example, one assumption is the discrimination axiom, which requires that each agent orders strictly at least one triple of candidates.

In [5], Doyle and Wellman model agents' preferences using a non-monotonic logic, giving a quasi-ordering (that is, an IO) on the outcomes. An additional hypothesis is required, called conflict resolution, which states that if a pair is ordered by any agent, then it must be ordered also in the social order. Conflict resolution is a very strong property to require. For example, the Pareto rule does not respect it.

With respect to all these approaches, our profiles are more general, since in our results a profile can be any set of incomplete orders. However, the resulting order of a social welfare function is required to be a restricted incomplete order, that is, an incomplete order with all indifferent top elements, or all indifferent bottom elements. Thus our result is incomparable to these previous results. In addition, our possibility theorems for the majority rule is, to our knowledge, the first result of this kind for partially ordered profiles in social welfare functions. This applies also to the extension of the Muller-Satterthwaite theorem. However, weak dictators have been defined before for social choice functions by Kelly in [13]. In particular, in [13] a weak dictator is defined as an agent i such that, if $a \succ_{p_i} b$, $b \in f(p)$ implies $a \in f(p)$. It is easy to see that a weak dictator according to [13] is also a weak dictator according to our definition, since the condition above implies that $top(p_i) \cap f(p) \neq \emptyset$. However the converse is not true in general. In fact, assuming $\Omega = \{a, b, c\}$, and given a profile p where $a \succ_{p_i} b \succ_{p_i} c$ and $f(p) = \{a, c\}$, i is a weak dictator according to our definition but not according to [13].

A crucial issue in our definition of strategy-proofness is how to order subsets of outcomes with respect to the preference of a single agent. This problem has been addressed before as the problem of strategy-proofness without resoluteness [3, 7, 27, 21, 11]. As mentioned in Section 9, our definition generalizes the one in [27] to the case of strict incomplete orders. Since that definition is stronger than that of [7], our theorem is not a generalization of the Duggan-Schwartz theorem [7]. However, it is also not an instance of it.

In addition, efforts have been made to weaken the conditions on the profiles. In particular, both the Gibbard-Satterthwaite and the Duggan-Schwartz theorems have been shown to hold also when indifference is allowed in the profiles [24, 25]. We show that the Gibbard-Satterthwaite result continues to hold when we allow simultaneously incomparability (but not indifference) in the profiles and multiple winners.

11 Conclusions

We have proved that social welfare functions on incomplete orders cannot be at the same time unanimous, IIA, and with no weak dictator if the result is an incomplete order with indifference between all top or all bottom elements. This result generalizes Arrow's impossibility theorem for combining complete orders. We have then extended Muller-Satterthwaite's theorem for social choice functions by proving that social choice functions over incomplete orders cannot be at the same time unanimous, monotonic, and without weak dictators. These results show that the use of incomplete orders, both for each agent and in the result of the preference aggregation, does not change many of the fundamental impossibility properties connected to social welfare and social choice functions.

On the possibility side, we have proved that the majority rule can be an IIA and unanimous social welfare function with no weak dictators if we put some restrictions on the shape of the orderings of the agents. In particular, this happens when all the linearizations of the incomplete orders of the agents have the triple-wise value-restriction, and also when they do not contain any chain of ordered pairs.

We have also generalized the notion of strategy-proofness to the context of social choice functions over partially ordered preferences, by extending naturally the one in [27], and we have shown that the well known Gibbard-Satterthwaite result, which states that strategy-proofness and freeness imply dictators, holds also in this more general context.

12 Future work

In this paper, incompleteness in the preference ordering is interpreted as incomparability. However, it is sometimes useful to interpret it as lack of knowledge, for example in contexts where the agents don't want to reveal all their preferences, or when we are eliciting preferences and agents do not reveal their preferences all at once. Moreover, the two interpretations of incompleteness in the preference ordering can be combined, since agents may want to express that some objects are really incomparable, while they may want to hide the actual relationship among other objects. In this more general context, the notion of winners can be generalized to the notions of *possible* and *necessary* winners [14, 16, 15]. We plan to study possibility and impossibility results on the coexistence of IIA, unanimity and non-dictatorship, as well as non-manipulability, also in this more general scenario.

We have proved impossibility results for aggregating partially ordered preferences. It is likely that there are ways around these negative results. We plan for example to investigate certain social choice functions on partial orders which may be computationally hard to manipulate. As another example, we intend to find certain restrictions on the way agents vote (such as single-peakedness) which may guarantee strategy-proofness.

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Appendix: Proofs of main results

Theorem 5. *Given a social welfare function f over IOs, assume the result is a rIO, and that there are at least two agents and three outcomes. Then it is impossible that f is unanimous, IIA, and has no weak dictators.*

Proof. The structure of the proof is similar to the one of Arrow's theorem presented in [9]. However, each step is adapted to the context of incomplete orders. As usual, we denote with N the set of n voting agents, with Ω the set of outcomes and with $\mathcal{P}$ the set of profiles. We assume the resulting ordering is a rIO over Ω with all bottom elements indifferent. The proof can then be repeated very similarly for the other case in which the resulting ordering is assumed to be a rIO with all top elements indifferent.

1. First we prove that, given a profile $p \in \mathcal{P}$ and an outcome $b \in \Omega$ such that, $\forall i \in N$, $\{b\} = top(p_i)$ or $\{b\} = bottom(p_i)$, then $b \in top(f(p))$ or $b \in bottom(f(p))$. Moreover, if $b \in top(f(p))$ (resp., $b \in bottom(f(p))$), then, for every $a \in top(f(p))$ (resp., $a \in bottom(f(p))$), it must be $b \bowtie a$.
   In words, if b is the unique top element or the unique bottom element for all agents, then b must be a top or bottom element in the resulting order $f(p)$. Moreover, only elements incomparable with b can be at the top (or at the bottom) of $f(p)$ together with b.
   We will prove it by contradiction.
   Assume that $b \notin top(f(p)) \cup bottom(f(p))$, that is, that b is not a top or bottom element in the result, or that $b \in top(f(p))$ or $b \in bottom(f(p))$ together with some indifferent elements. Then there must be two other distinct elements $a, c \in \Omega$ such that $a \succeq_{f(p)} b \succeq_{f(p)} c$.
   Let us consider a profile p' obtained from p by setting, $\forall i \in N$, $p'_i(a, c)$ equal to $c \succ_{p'_i} a$, and leaving b in its position. This is always possible since b is either a top or a bottom element. Thus, $\forall i \in N$, $p_i(a, b) = p'_i(a, b)$ and $p_i(c, b) = p'_i(c, b)$, that is, the ordering relation between a and b and between c and b in p' is the same as in p for all agents.
   By unanimity, we must have $c \succ_{f(p')} a$. By IIA, we have $a \succeq_{f(p')} b$ and $b \succeq_{f(p')} c$, as we had in p. By transitivity, we have $a \succeq_{f(p')} c$, which is a contradiction with $c \succ_{f(p')} a$.
   Notice that, since $f(p)$ is an rIO, if $b \in bottom(f(p))$, then it must be $b = boottom(f(p))$.
2. We now prove that there is a pivotal agent n^* such that, when he moves b from its bottom to its top, then b becomes one of the top elements in the result.
   Let us start by considering a profile π such that $\forall i \in N$, $\{b\} = bottom(\pi_i)$, that is, such that all agents have b as the unique bottom. Then, $\{b\} = bottom(f(\pi))$ by unanimity. We now consider a scenario in which each agent in turn changes from having b as its unique bottom to having b as its unique top. If we denote with π' a profile obtained after any number of such changes, by the first item in this proof, it must be that $b = bottom(f(\pi'))$ or $b \in top(f(\pi'))$ with possibly some elements incomparable with it.
   Let n^* be the first agent such that, when n^* changes b from being its unique bottom to being its unique top, b moves from being the bottom element of the result to the set of top elements, all incomparable with b, of the result. Note that n^* must exist, because when all agents have made b their unique top element, by unanimity b must be the unique top element in the result.
3. We continue by proving that n^* is a weak dictator for pairs of elements not involving b. For an agent i to be a weak dictator on a pair, say (a, c), it means that, in all profiles p, if $a \succ_{p_i} c$, then $a \succ_{f(p)} c$ or $a \bowtie_{f(p)} c$.

Let us consider the following profiles obtained from profile π defined in the previous item of the proof, in the context of moving b from the bottom to the top of each agent's incomplete order.

- π_1**:** $\forall i < n^*$, $\{b\} = top(\pi_{1_i})$ and $\forall j \geq n^*$, $\{b\} = bottom(\pi_{1_j})$. By the definition of n^* and the fact that $f(\pi_1)$ is an rIO (and thus all the bottom elements are indifferent), $b = bottom(f(\pi_1))$. Thus, $\forall d \in \Omega$, $d \succ_{f(\pi_1)} b$.
- π_2**:** $\forall i \leq n^*$, $\{b\} = top(\pi_{2_i})$ and $\forall j > n^*$, $\{b\} = bottom(\pi_{2_j})$. By the definition of n^*, $b \in top(f(\pi_2))$, and thus, $\forall d \in \Omega$, $b \succ_{f(\pi_2)} d$ or $b \bowtie_{f(\pi_2)} d$.
- π_3**:** As π_2, except for $a \succ_{\pi_{3_{n^*}}} b$ and for all $i \neq n^*$, $\pi_{3_i}(a, c)$ can be any relation. In words, in π_3, n^* has now moved a above b (and thus also above c), and all other agents move freely a and c leaving b in the top or bottom position.

By IIA, since passing from π_1 to π_3 the relation between a and b has not been changed, and since $a \succ_{f(\pi_1)} b$, then $a \succ_{f(\pi_3)} b$. Also, $b \succ_{f(\pi_3)} c$ or $b \bowtie_{f(\pi_3)} c$, since, $\forall i$, $\pi_{3_i}(b, c) = \pi_{2_i}(b, c)$ and ($b \succ_{f(\pi_2)} d$ or $b \bowtie_{f(\pi_2)} d$) for all $d \in \Omega$ and thus, in particular, for c. By transitivity, it cannot be $c \succ_{f(\pi_3)} a$ or $c \sim_{f(\pi_3)} a$, since it would imply $c \succ_{f(\pi_3)} b$ which is contradictory with the assumption that $b \succ_{f(\pi_3)} c$ or $b \bowtie_{f(\pi_3)} c$. Thus n^* is a weak dictator for pairs not involving b.

4. We now prove that there exists an agent n' which is a weak dictator for pairs not involving outcome c, where c is any outcome different from b. To do this, it is enough to use the same construction as above but with c in place of b. Thus n' is the first agent, in the ordering of the agents used in step 2, that, by moving c from the unique bottom to the unique top, makes c move from the bottom to the top of the result.
5. We show now that $n^* = n'$. We will consider two cases: either $n' < n^*$ or $n' > n^*$. Assume $n' < n^*$. Then consider again profile π_1. In this profile, n' has b at its unique top. Thus $b \succ_{n'} a$ for every a. If n' is a weak dictator for pairs not involving c, then it must be $b \succ_{f(\pi_1)} a$ or $b \bowtie_{f(\pi_1)} a$, which is in contradiction with the fact that b is the unique bottom of $f(\pi_1)$.
   Assume now $n' > n^*$. Let us consider profile q where, $\forall i \in N$, $\{b\} = bottom(q_i)$ and $\{c\} = top(q_i)$. That is, we start with all agents having b at the bottom and c at the top. Then we swap b and c in each ordering, going through the agents in the same order as in the previous constructions.
   Consider now profile q', in which $\forall i \leq n^*$, $\{b\} = top(q'_i)$ and $\{c\} = bottom(q'_i)$, and $\forall j > n^*$, $\{b\} = bottom(q'_j)$ and $\{c\} = top(q_j)$. By the previous part of the proof, we have that $b \in top(f(q'))$, since $\{b\} = top(q'_{n^*})$. Moreover, since c is the top or the bottom element of all agents' ordering in q', by item 1 of the proof we have that $c \in top(f(q'))$ or $c \in bottom(f(q'))$.
   If $c \in bottom(f(q'))$, then we can repeat the same construction as in points 1,2,3 starting with c at the top instead of b at the bottom, and by moving c from being the unique top to being the unique bottom of the agents. Then n^* makes

c move from top to bottom in the result when it moves c from unique top to unique bottom in its ordering. Thus, by step 3, n^* is a weak dictator for pairs not involving c. This contradicts the hypothesis that n' is the first weak dictator for pairs not involving c in the ordering of the agents. Thus we can conclude that $n^* = n'$.
Otherwise, $c \in top(f(q'))$. Thus we have both b and c in $top(f(q'))$. Let us now take any other candidate a. It must be $a \bowtie_{f(q')} b$ because n' dictates over pairs not involving c, and also $a \bowtie_{f(q')} c$ because n^* dictates over pairs not involving b. Let us now assume that, given any two candidates a and a' different from both b and c, n^* and n' contradict each other (that is, one says $a > a'$ and the other one says $a' > a$). Since both n^* and n' dictate on such pairs, in the result $f(q')$ we must have $a \bowtie a'$. As of the relationship between b and c in $f(q')$, we can only have $b \bowtie c$ or $b \sim c$. Whatever it is, $f(q')$ is not an rIO. Thus we reach a contradiction with the hypothesis.

6. Since we have shown that $n^* = n'$, n^* is a weak dictator over all pairs except the pair (b, c). We will now show that this agent also dictates on the pair (b, c). To do this, we repeat steps 1,2,3 of the proof with a at the top or bottom, finding an agent n" which dictates on pairs not involving a. Then, by step 5 we prove that $n^* = n$". Thus n^* dictates on all pairs. □

Corollary 1 *Given a social welfare function f over IOs, assume the result is a rIO and there are at least two agents and three outcomes. Then it is impossible that f is free, monotonic, IIA and has no weak dictators.*

Proof. Suppose f is free, monotonic, and IIA. We will prove that it is also unanimous. Consider a profile p such that $a \succsim_{p_i} b$ for every $i \in N$. Consider now profile p' obtained from p by setting $\{a\} = top(p'_i)$ $\forall i \in N$. By IIA, we have $f(p)(a, b) = f(p')(a, b)$. Suppose that $a \prec_{f(p')} b$ or $a \bowtie_{f(p')} b$. Since $a \succ_{p'_i} b$ for every agent $i \in N$, by monotonicity, there is no profile p'' such that $a \succsim_{f(p'')} b$. This is in contradiction with the assumption that f is free. Thus it must be $a \succsim_{f(p')} b$. Thus we also have $a \succsim_{f(p)} b$, since $f(p)(a, b) = f(p')(a, b)$. Thus f is unanimous.

We can now apply Theorem 5, which tells us that it is impossible to have at the same time unanimity, IIA, and no weak dictators. Thus we can conclude that it is impossible to have freeness, monotonicity, IIA, and no weak dictators. □

Theorem 6. *If, for all functions in $A^n \rightsquigarrow B$, IIA and unanimity imply the existence of a weak dictator, then this is true also for all functions in $A^n \rightsquigarrow B'$, where B' is a subtype of B.*

Proof. It follows trivially, since by restricting the co-domain we are just considering a subset of the social welfare functions. □

Theorem 7. *If, for all functions in $A^n \rightsquigarrow B$, IIA and unanimity imply the existence of a weak dictator, then this is true also for all functions in $A'^n \rightsquigarrow B$, where A' is a subtype of A.*

Proof. Let us assume that for all functions in $A^n \rightsquigarrow B$ IIA and unanimity implies the existence of a weak dictator. This means that if a function in $A^n \rightsquigarrow B$ is IIA and unanimous then there is an agent i such that, in any profile $p \in A^n$, if $a \succ_{p_i} b$, then either $a \succ_{f(p)} b$ or $a \bowtie_{f(p)} b$. Since A' is a subtype of A we have that $A'^n \subset A^n$, and thus i is a weak dictator also for all profiles in A'^n. We conclude by noticing that all functions in $A'^n \rightsquigarrow B$ are restrictions of functions in $A^n \rightsquigarrow B$. □

Theorem 8. *The majority rule over profiles (over SIOs) satisfying the generalized triple-wise value-restriction is an IIA and unanimous social welfare function with no weak dictators.*

Proof. Given a profile p over SIOs, we say that profile p' is a linearization of p if, $\forall i \in N$, p'_i is a SCO that linearizes the IO p_i. If p satisfies the generalized triple-wise value-restriction then any of its linearizations, say p', satisfies it as well. Thus the majority rule applied to p' produces an ordering without cycles, by Sen's theorem. Since p' is a linearization of p, p has a smaller or equal set of ordered pairs w.r.t. p'. Therefore, if the majority rule has not produced any cycle starting from p', it cannot produce any cycle if it starts from p. In fact, the majority rule counts the number of agents that order a pair, so if the profile has less ordered pairs, a smaller or equal number of pairs are ordered in the result as well.

Assume now to have $a \succ_{maj(p)} b \succ_{maj(p)} c$ and $a \bowtie_{maj(p)} c$ in the result. We will now show that, if this is the case, then there is a linearization which doesn't satisfy the triple-wise value-restriction property.

In fact, since $a \succ_{maj(p)} b$ in the result, we know that the cardinality of the set $\{i \in N | a \succ_{p_i} b\}$ is greater than the cardinality of the set $\{i \in N | b \succ_{p_i} a \text{ or } b \bowtie_{p_i} a\}$. Similarly, the cardinality of the set $\{i \in N | b \succ_{p_i} c\}$ is greater than the cardinality of the set $\{i \in N | c \succ_{p_i} b \text{ or } c \bowtie_{p_i} b\}$. Since we are assuming that $a \bowtie_{maj(p)} c$, then the cardinality of the set $\{i \in N | a \succ_{p_i} c\}$ is smaller than or equal to the cardinality of the set $\{i \in N | c \succ_{p_i} a \text{ or } c \bowtie_{p_i} a\}$. We know that a majority of agents says that $a \succ_{p_i} b$. Let us now assume that no agent says that a is incomparable to b and that no agent says that b is incomparable to c. Let us consider the agents that say $a \succ_{p_i} b$. Then each such agent, i, must have one of the following orderings: (1) $a \succ_{p_i} b \succ_{p_i} c$; (2) $a \succ_{p_i} b \wedge c \succ_{p_i} b$; (3) $a \succ_{p_i} c \succ_{p_i} b$; (4) $c \succ_{p_i} a \succ_{p_i} b$. We want to prove that there is at least one agent that says (1) and that there is at least one agent that says either (2) or (4). In fact, it is not possible that all the agents which have said $a \succ_{p_i} b$ have all ordering (1) or (3), since that would mean that there is also a majority that says $a \succ_{p_i} c$, while $a \bowtie_{maj(p)} c$ by hypothesis. Moreover, it is not possible that all the agents that say $a \succ_{p_i} b$ vote as (2) or (3) or (4) but none using (1), since this would imply that $c \succ_{p(i)} b$. Thus we can conclude that at least one agent must vote as in (1) and at least one agent must vote either as in (2) or as in (4).

Now let us consider the agents in the majority that vote $b \succ_{p_i} c$. Each of them can have one of the following orderings: (i) $a \succ_{p_i} b \succ_{p_i} c$; (ii) $b \succ_{p_i} c \wedge b \succ_{p_i} a$; (iii) $b \succ_{p_i} c \succ_{p_i} a$; (iv) $b \succ_{p_i} a \succ_{p_i} c$. As before, they cannot all vote (i), otherwise using the majority rule we would have $a \succ_{maj(p)} c$, and not $a \bowtie_{maj(p)} c$ as assumed. However, at least one must vote (i), since (ii), (iii) and (iv) order b above a, while we know there is a majority saying $a \succ_{p_i} b$. Moreover, it is also not possible that all voters that say $b \succ_{p_i} c$ vote as in (i) or as in (iv), since again that would imply $a \succ_{maj(p)} c$. Thus, there is at least an agent that votes as in (ii) or as in (iii).

To summarize, we have an agent, say i, that says (1) (or (i)), that is, $a \succ_{p_i} b \succ_{p_i} c$ and we have an agent, say j, that says either (2) or (4). Notice that, if j votes as in (2), we can linearize (2) into (4) by adding $c \succ_{p_j} a$. Thus, for agent j we have $c \succ_{p_j} a \succ_{p_j} b$. Finally, we have an agent k that votes either (ii) or (iii). Again, (ii) can be linearized into (iii) by adding $c \succ_{p_k} a$. Thus for agent k we have $b \succ_{p_k} c \succ_{p_k} a$. This is a linearization which violates the triple-wise value-restriction property.

Notice that, if we allow the agents to express incomparability between a and b, or between b and c, this means that agents that voted (3) or (4) now could vote (2) and that agents that voted (iii) or (iv) now could vote (ii) or give even more incomparability. However, this does not prevent the possibility to linearize the orderings as above. □

Theorem 9. *The majority rule over non-chaining profiles over SIOs is an IIA and unanimous social welfare function with no weak dictators.*

Proof. We just need to show that the majority rule is transitive since it has all other properties of fairness. In this proof we denote with S_{aRb} the set $\{i \in N | aR_{p_i}b\}$ where $R \in \{\prec, \succ, \bowtie\}$. Moreover, we say that a profile p over SIOs is of type α if, for any triple of distinct outcomes, $\forall i \in N$, they are all incomparable in p_i, or only two of them are ordered in p_i, or two of them are incomparable and the other one is more preferred than both in p_i.

Similarly, we say that p is of type β if for any triple of distinct outcomes, $\forall i \in N$, they are all incomparable in p_i, or only two of them are ordered in p_i, or two of them are incomparable and the other one is less preferred than both in p_i. Let us first consider the case in which the profile is of type α. We will prove that for every triple of outcomes a, b, and c it cannot be $a \succ_{maj(p)} b \succ_{maj(p)} c$, that is, there are no transitive chains in the ordering. Let us assume that $a \succ_{maj(p)} b \succ_{maj(p)} c$. Since $a \succ_{maj(p)} b$ then:

– (1) $|S_{a \succ b}| > |S_{b \succ a}| + |S_{a \bowtie b}|$

Also, from $b \succ_{maj(p)} c$ we have that:

– (2) $|S_{b\succ c}| > |S_{c\succ b}| + |S_{b\bowtie c}|$

Since p is of type α, we have that:

– (3) $|S_{a\succ b}| \leq |S_{b\bowtie c}|$
– (4) $|S_{b\succ c}| \leq |S_{b\succ a}| + |S_{b\bowtie a}|$

The reason for inequality (3) is that all the voters that say $a \succ_{p_i} b$ cannot order b and c. The reason for inequality (4) is that all the voters that put b above c must put a either below b or incomparable to b.

From the above inequalities we get the following inconsistency:

$$|S_{a\succ b}| \overset{(3)}{\leq} |S_{b\bowtie c}| \overset{(2)}{<} |S_{b\succ c}| \overset{(4)}{\leq} |S_{b\succ a}| + |S_{b\bowtie a}| \overset{(1)}{<} |S_{a\succ b}|.$$

Let us now consider the case in which profile p over SIOs is of type β, that is, for any agent i and any triple of distinct outcomes, they are all incomparable or only two of them are ordered, or there is one of them, which is less preferred than the other two.

In situation β, (1) and (2) still hold while we have, from the fact that p is of type β:

– (3) $|S_{b\succ c}| \leq |S_{a\bowtie b}|$
– (4) $|S_{a\succ b}| \leq |S_{c\succ b}| + |S_{c\bowtie b}|$.

The reason for inequality (3) is that all the voters that say $b \succ_{p_i} c$ cannot order a and b. The reason for inequality (4) is that all the voters that put a above b must either put b below c or incomparable to c.

¿From the above inequalities we get the following inconsistency:

$$|S_{b\succ c}| \overset{(3)}{\leq} |S_{a\bowtie b}| \overset{(1)}{<} |S_{a\succ b}| \overset{(4)}{\leq} |S_{c\succ b}| + |S_{c\bowtie b}| \overset{(2)}{<} |S_{b\succ c}|.$$

□

Theorem 10. *If we have at least two agents and at least three outcomes, and the social choice function over SIOs is unanimous and monotonic, then there is at least one weak dictator.*

Proof. The proof follows the scheme of the proof of the Muller-Satterthwaite theorem that can be found in [21].

Consider three outcomes a, b, and c, and a profile π_0 s.t. $\forall \pi_{0_i}$ we have that $a \bowtie_{\pi_{0_i}} c$, $top(\pi_{0_i}) = \{a, c\}$ and $bottom(\pi_{0_i}) = \{b\}$. By unanimity, $f(\pi_0) = \{a, c\}$.

Profile π_1						
b	$\dots$	b	$a \bowtie c$	$a \bowtie c$	$\dots$	$a \bowtie c$
$a \bowtie c$	$\dots$	$a \bowtie c$	b	.	$\dots$	.
.		.	.	.		.
.		.	.	.		.
.		.	.	.		.
.	$\dots$	.	.	b	$\dots$	b
1	$\dots$	$i-1$	i	$i+1$	$\dots$	n

Profile π_2						
b	$\dots$	b	b	$a \bowtie c$	$\dots$	$a \bowtie c$
$a \bowtie c$	$\dots$	$a \bowtie c$	$a \bowtie c$	.	$\dots$	.
.		.	.	.		.
.		.	.	.		.
.		.	.	.		.
.	$\dots$	.	.	b	$\dots$	b
1	$\dots$	$i-1$	i	$i+1$	$\dots$	n

Fig. 2. Profiles π_1 and π_2.

Let us now rise b one position at a time in agent 1's ordering. By monotonicity, the set of winners remains $\{a, c\}$ as long as b remains below a and below c. In fact, no other outcome $d \in \Omega \setminus \{a, c, b\}$ can become a winner since, by monotonicity, it would have been a winner in π_0, where b is below it. When b is risen above a and c, by monotonicity the set of winners may contain b. If we continue this process with the other agents in the order, at the end we must have b as the only winner by unanimity. Thus at some point b must appear in the set of winners. Let i be the agent such that, when he moves b above a and c, b appears in the set of winners.

Step 1. Consider profiles π_1 and π_2 in Figure 2: π_1 is the last profile where the set of winners, that is, $f(\pi_1)$, is still $\{a, c\}$, whereas π_2 is the first profile such that the set of winners contains b.

Consider now an outcome $d \in \Omega \setminus \{a, c, b\}$. If $d \notin f(\pi_1)$, then $d \notin f(\pi_2)$. In fact, assume $d \in f(\pi_2)$. Then, by monotonicity on π_2 and π_1, $d \in f(\pi_1)$ as well. Therefore, since $f(\pi_1) = \{a, c\}$, then $f(\pi_2)$ can be $\{b\}$, $\{a, b\}$, $\{c, b\}$ or $\{a, c, b\}$.

Step 2. Consider the new profiles π_1' and π_2' in Figure 3. In particular, π_1' (resp., π_2') is obtained from π_1 (resp., π_2) by making the bottom for all the agents before i equal to $\{a, c\}$ and placing a and c just above the bottom element b for agents after i.

Notice that $f(\pi_2')$ must contain b, by monotonicity on π_2 and π_2'. Let us call a set an *excluding set* (ES) if it is a non-empty subset of $(\Omega \setminus \{a, b, c\})$. Then, if D is an ES, $f(\pi_2')$ can be $\{b\}$, $\{b\} \cup D$, $\{a, b\}$, $\{a, b\} \cup D$, $\{c, b\}$, $\{c, b\} \cup D$, $\{a, c, b\}$, or $\{a, c, b\} \cup D$.

Let us show that none of the above cases can be ruled out.

If $c \notin f(\pi_2)$, then $c \notin f(\pi_2')$. In fact, assume $c \in f(\pi_2')$, then monotonicity on π_2' and π_2 implies $c \in f(\pi_2)$. Analogously, if $a \notin f(\pi_2)$, then $a \notin f(\pi_2')$. In particular,

Profile π_1'						
b	...	b	$a \bowtie c$	.	...	.
.	...	.	b	.	...	.
.		.	.	.		.
.		.	.	.		.
.		.	.	$a \bowtie c$		$a \bowtie c$
$a \bowtie c$	...	$a \bowtie c$	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

Profile π_2'						
b	...	b	b	.	...	.
.	...	.	$a \bowtie c$	.	...	.
.		.	.	.		.
.		.	.	.		.
.		.	.	$a \bowtie c$		$a \bowtie c$
$a \bowtie c$	...	$a \bowtie c$	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

Fig. 3. Profiles π_1' and π_2'.

if $f(\pi_2) = \{b\}$, then by monotonicity on π_2 and π_2', $f(\pi_2') = \{b\}$. Moreover, if $f(\pi_2) \neq \{b\}$, then $f(\pi_2') \neq \{b\}$, since $f(\pi_2') = \{b\}$ would imply $f(\pi_2) = \{b\}$ by monotonicity on π_2' and π_2.

Summarizing, if D is any ES,

- if $f(\pi_2) = \{b\}$, then $f(\pi_2') = \{b\}$;
- if $f(\pi_2) = \{a, b\}$, then $f(\pi_2') = \{b\} \cup D$, $\{a, b\}$, $\{a, b\} \cup D$;
- if $f(\pi_2) = \{c, b\}$, then $f(\pi_2') = \{b\} \cup D$, $\{c, b\}$, $\{c, b\} \cup D$;
- if $f(\pi_2) = \{a, b, c\}$, then $f(\pi_2') = \{b\} \cup D$, $\{a, b\}$, $\{a, b\} \cup D$, $\{c, b\}$, $\{c, b\} \cup D$, $\{a, c, b\}$ or $\{a, c, b\} \cup D$.

Let us now consider the possible results for π_1'.

Notice that $f(\pi_1')$ does not contain b. In fact, if we assume $b \in f(\pi_1')$, then by monotonicity on π_1' and π_1, also $f(\pi_1)$ should contain b.

Moreover, $f(\pi_1')$ is not an ES. In fact, if $f(\pi_1') = D$, and D is an ES, then by monotonicity on π_1' and π_2', we have $f(\pi_2') = D$, that is not one of the possible cases for $f(\pi_2')$. Hence, $f(\pi_1')$ can be $\{a\}$, $\{c\}$, $\{a, c\}$, $\{a\} \cup D$, $\{c\} \cup D$, $\{a, c\} \cup D$.

If $D \subset f(\pi_2')$ then $D \subset f(\pi_1')$, for monotonicity on profiles π_2' and π_1'.

Moreover $\forall d \in (\Omega \setminus \{a, b, c\})$, if $d \notin f(\pi_2')$, then $d \notin f(\pi_1')$. In fact, if we assume $d \in f(\pi_1')$, then monotonicity on π_1' and π_2' implies $d \in f(\pi_2')$.

Moreover, if $a \in f(\pi_2')$, then by monotonicity on profiles π_2' and π_1', $a \in f(\pi_1')$ and, for the same reason, if $c \in f(\pi_2')$ then $c \in f(\pi_1')$.

Summarizing,

- if $f(\pi_2') = \{b\}$, then $f(\pi_1')$ can be $\{a\}$, $\{c\}$ or $\{a, c\}$;
- if $f(\pi_2') = \{a, b\}$, then $f(\pi_1')$ can be $\{a\}$ or $\{a, c\}$;
- if $f(\pi_2') = \{c, b\}$, then $f(\pi_1')$ can be $\{c\}$ or $\{a, c\}$;
- if $f(\pi_2') = \{a, c, b\}$, then $f(\pi_1') = \{a, c\}$;
- if $f(\pi_2') = \{b\} \cup D$, then $f(\pi_1')$ can be $\{a\} \cup D$, $\{c\} \cup D$ or $\{a, c\} \cup D$;
- if $f(\pi_2') = \{a, b\} \cup D$, then $f(\pi_1')$ can be $\{a\} \cup D$ or $\{a, c\} \cup D$;

Profile π_3

.	...	.	$a \bowtie c$	.	...	.
.	...	.	e	.	...	.
.		.	b	.		.
e		e	.	e		e
b		b	.	$a \bowtie c$		$a \bowtie c$
$a \bowtie c$	...	$a \bowtie c$	.	b	...	b
1	...	$i-1$	i	$i+1$	...	n

Profile π_4

.	...	.	$a \bowtie c$	.	...	.
.	...	.	e	.	...	.
.		.	b	.		.
e		e	.	e		e
b		b	.	b		b
$a \bowtie c$	...	$a \bowtie c$	.	$a \bowtie c$	...	$a \bowtie c$
1	...	$i-1$	i	$i+1$	...	n

Fig. 4. Profiles π_3 and π_4.

– if $f(\pi'_2) = \{c, b\} \cup D$, then $f(\pi'_1)$ can be $\{c\} \cup D$ or $\{a, c\} \cup D$;
– if $f(\pi'_2) = \{a, c, b\} \cup D$, then $f(\pi'_1) = \{a, c\} \cup D$;

Step 3. Consider an outcome e, distinct from a, c, and b, and a profile π_3 in Figure 4, which shares with π'_1 the ranking of a and c versus any other outcome in all agents' rankings, and in which b is just above a and c (which are at the bottom) for any agent j with $j < i$, and e is just above b for any agent j with $j \leq i$ and just above a and c for any agent j with $j > i$.

Notice that $f(\pi_3)$ cannot contain b. In fact, if $b \in f(\pi_3)$, then, by monotonicity on π_3 and π_1, $b \in f(\pi_1)$.

Hence $f(\pi_3)$ can be $\{a\}$, $\{c\}$, D, $\{a, c\}$, $\{a\} \cup D$, $\{c\} \cup D$, $\{a, c\} \cup D$, where D is any ES.

By monotonicity on profiles π'_1 and π_3, if $a \in f(\pi'_1)$ then $a \in f(\pi_3)$ and if $c \in f(\pi'_1)$ then $c \in f(\pi_3)$. Thus, since $f(\pi'_1)$ always contains either a or c, it is not possible that $f(\pi_3) = D$ for any ES D.

Moreover, if $a \notin f(\pi'_1)$ then $a \notin f(\pi_3)$. In fact, if we assume $a \in f(\pi_3)$, then by monotonicity on profiles π_3 and π'_1 we have that $a \in f(\pi'_1)$. Analogously, if $c \notin f(\pi'_1)$ then $c \notin f(\pi_3)$.

By monotonicity on profiles π'_1 and π_3, if $f(\pi'_1) = \{a\}$ then $f(\pi_3) = \{a\}$, if $f(\pi'_1) = \{c\}$ then $f(\pi_3) = \{c\}$ and if $f(\pi'_1) = \{a, c\}$ then $f(\pi_3) = \{a, c\}$. In particular, if $f(\pi'_1) \neq \{a\}$ then $f(\pi_3) \neq \{a\}$. In fact, if $f(\pi_3) = \{a\}$, then by monotonicity on π_3 and π'_1, $f(\pi'_1)$ must be $\{a\}$. Analogously, if $f(\pi'_1) \neq \{c\}$ then $f(\pi_3) \neq \{c\}$, and if $f(\pi'_1) \neq \{a, c\}$ then $f(\pi_3) \neq \{a, c\}$.

By Step 2 we know that $f(\pi'_1)$ can be $\{a\}$, $\{c\}$, $\{a, c\}$, $\{a\} \cup D$, $\{c\} \cup D$ or $\{a, c\} \cup D$. Therefore, applying the reasoning above, we have that $f(\pi'_1)$ is of the same type of $f(\pi_3)$.

Step 4. Consider profile π_4 in Figure 4 derived from profile π_3 by swapping the ranking of outcomes a and b for every agent j with $j > i$, and profile π'_4 obtained from π_4 by setting outcome e to be the unique top of every agent's ranking. By unanimity, $f(\pi'_4) = \{e\}$.

Note that $f(\pi_4)$ cannot contain b. In fact, if $b \in f(\pi_4)$, by monotonicity on profiles π_4 and π'_4 we have $b \in f(\pi'_4)$, that is a contradiction since $f(\pi'_4) = \{e\}$.

Also, $\forall d \in (\Omega \setminus \{a, b, c\})$, if $d \notin f(\pi_3)$, then $d \notin f(\pi_4)$, and if $d \in f(\pi_3)$, then $d \in f(\pi_4)$.

Moreover, if $c \notin f(\pi_3)$, then $c \notin f(\pi_4)$ and analogously if $a \notin f(\pi_3)$, then $a \notin f(\pi_4)$. In fact, if $a \in f(\pi_4)$, by monotonicity on π_4 and π_3, then $a \in f(\pi_3)$.

Notice that if $f(\pi_3) \neq \{a\}$, then $f(\pi_4) \neq \{a\}$. In fact, if we assume $f(\pi_4) = \{a\}$, then by monotonicity on profiles π_4 and π_3, $f(\pi_3)$ must be $\{a\}$.

Analogously, if $f(\pi_3) \neq \{c\}$ then $f(\pi_4) \neq \{c\}$, if $f(\pi_3) \neq \{a, c\}$ then $f(\pi_4) \neq \{a, c\}$, if $f(\pi_3) \neq \{a\} \cup D$ then $f(\pi_4) \neq \{a\} \cup D$, and if $f(\pi_3) \neq \{a, c\} \cup D$ then $f(\pi_4) \neq \{a, c\} \cup D$.

By Step 3, we know that $f(\pi_3)$ can be $\{a\}$, $\{c\}$, $\{a, c\}$, $\{a\} \cup D$, $\{c\} \cup D$ or $\{a, c\} \cup D$. Therefore, by the reasoning above, $f(\pi_3) = f(\pi_4)$.

Step 5. Consider an arbitrary profile π_5 where the ordering of agent i satisfies $a \bowtie_{\pi_{5_i}} c$, $top(\pi_{5_i}) = \{a, c\}$, and $bottom(\pi_{5_j}) = \{a, c\}$, $\forall j \neq i$. Profile π_5 can be obtained from profile π_4 without reducing the ranking of a and c versus any other outcome in any agent's ranking. Remember that, by step 4, $f(\pi_4)$ can be $\{a\}$, $\{c\}$, $\{a, c\}$, $\{a\} \cup D$, $\{c\} \cup D$ or $\{a, c\} \cup D$. By monotonicity on profiles π_4 and π_5, if $a \in f(\pi_4)$, then $a \in f(\pi_5)$, and if $c \in f(\pi_4)$, then $c \in f(\pi_5)$. Therefore, since $f(\pi_4)$ always contains a or c or both, then $f(\pi_5)$ contains a or c or both. Thus, the set of winners of an arbitrary profile where a and c are the top elements of the ordering of i must contain at least one (a or c) of the top elements of the agent i, since π_5 is a worst case scenario for a and c among such profiles. Thus agent i is a weak dictator.

It is easy to see that this proof can be easily generalized to the case of more than two top elements for agent i. Moreover, the case of just one top element for agent i can be proved via a simpler version of this proof. □

Theorem 11. *If a social choice function f over SIOs is strategy-proof and onto, then it is unanimous and monotonic.*

Proof. The proof consists of two parts. Part 1 shows that, if f is strategy-proof, then it is monotonic, while Part 2 shows that, if f is onto and monotonic, then it is unanimous.

Part 1. Consider profiles p and p', which differ only for the ranking of *agent* i, i.e., such that $p' = (p_{-i}, p'_i)$.

Assume that: (i) $a \in f(p)$ and (ii) for any other outcome $b \neq a$, $a \succ_{p_i} b$ implies $a \succ_{p'_i} b$, and $a \bowtie_{p_i} b$ implies $a \bowtie_{p'_i} b$ or $a \succ_{p'_i} b$. We want to show that $a \in f(p')$.

Assume that $a \notin f(p')$. If f is strategy-proof, then, by item 1 of definition of strategy-proofness, $\forall b \in f(p) \cap f(p')$, $a \succ_{p_i} b$ or $a \bowtie_{p_i} b$. Hence, by hypothesis (ii), we have $a \succ_{p'_i} b$ or $a \bowtie_{p'_i} b$. Moreover, by item 2 of definition of strategy-proofness, we have that $\forall b \in f(p') \setminus f(p)$, $a \succ_{p_i} b$ or $a \bowtie_{p_i} b$. Hence, by hypothesis (ii), $\forall b \in f(p') \setminus f(p)$, $a \succ_{p'_i} b$ or $a \bowtie_{p'_i} b$. Therefore, $\forall b \in f(p')$, $a \succ_{p'_i} b$ or $a \bowtie_{p'_i} b$. Consider now the move from p' to p. Then, by item 2 of definition of strategy-proofness, we have that $\forall a \in f(p) \setminus f(p')$, it must hold that $\forall b \in f(p')$, $a \prec_{p'_i} b$ or $a \bowtie_{p'_i} b$, and $\exists b \in f(p')$, $a \prec_{p'_i} b$. This is a contradiction, since we have shown before that $\exists a \in f(p) \setminus f(p')$ such that $\forall b \in f(p')$, $a \succ_{p'_i} b$ or $a \bowtie_{p'_i} b$.

Assume now that: (iii) $\forall a \in f(p)$ and for any other outcome $b \in \Omega \setminus f(p)$, $a \succ_{p_i} b$ implies $a \succ_{p'_i} b$, and $a \bowtie_{p_i} b$ implies $a \bowtie_{p_i} b$ or $a \succ_{p'_i} b$. We want to show that $f(p) = f(p')$. Assume that $f(p) \neq f(p')$, that is, $\exists\ a \in f(p) \setminus f(p')$ or that $\exists\ a \in f(p') \setminus f(p)$. If $\exists\ a \in f(p) \setminus f(p')$, then we can assume that f is strategy-proof and we can perform the same reasoning above which leads to a contradiction. If $\exists\ a \in f(p') \setminus f(p)$, if we assume that f is strategy-proof, then, passing from p' to p, the outcome a has been removed, hence by item 1 of definition of strategy-proofness, $\forall b \in f(p) \cap f(p')$, $a \succ_{p'_i} b$ or $a \bowtie_{p'_i} b$. Hence by hypothesis (iii), it must be that $\forall b \in f(p) \cap f(p')$, $a \succ_{p_i} b$ or $a \bowtie_{p_i} b$. Otherwise, if $\exists b \in f(p) \cap f(p')$ s.t. $b \succ_{p_i} a$, then, by hypothesis (iii), it must be that $b \succ_{p'_i} a$. Moreover, by item 2 of definition of strategy-proofness, $\forall b \in f(p) \setminus f(p')$, $a \succ_{p'_i} b$ or $a \bowtie_{p'_i} b$. Hence, again by hypothesis (iii), it must be that $a \succ_{p_i} b$ or $a \bowtie_{p_i} b$. Thus, for every $b \in f(p)$, $a \succ_{p_i} b$ or $a \bowtie_{p_i} b$. However, since f is strategy-proof, $\forall a \in f(p') \setminus f(p)$, then passing from profile p to profile p', by item 2 of definition of strategy-proofness, it must hold that $\forall b \in f(p)$, $b \succ_{p_i} a$ or $b \bowtie_{p_i} a$ and $\exists b \in f(p)$, $b \succ_{p_i} a$. This is a contradiction, since we have shown before that, given $a \in f(p') \setminus f(p)$, for every $b \in f(p)$, $a \succ_{p_i} b$ or $a \bowtie_{p_i} b$.

Consider a profile q such that $a \in f(q)$ and a profile q' such that *for every agent* i and for every outcome b, $a \succ_{q_i} b$ implies $a \succ_{q'_i} b$, and $a \bowtie_{q_i} b$ implies $a \bowtie_{q'_i} b$ or $a \succ_{q'_i} b$. We want to prove that $a \in f(q')$ (first part of definition of monotonicity). Since we can move from $q = (q_1, ...q_n)$ to $q' = (q'_1, ...q'_n)$, passing from $q = (q_1, q_2..., q_n)$ to $(q'_1, q_2..., q_n)$, and $(q'_1 q'_2, ..., q_n)$ to $(q'_1, q'_2, ..., q_n)$ and so on, and we have shown above that at each step a remains in the set of winners, $a \in f(q')$. The same reasoning holds for profiles q and q' such that $f(q) = A$ and, for every agent i, $\forall a \in A$, for every $b \in \Omega \setminus A$, we have that $a \succ_{q_i} b$ implies $a \succ_{q'_i} b$, and $a \bowtie_{q_i} b$ implies $a \bowtie_{q'_i} b$ or $a \succ_{q'_i} b$. In this case we conclude that $f(q') = A$ (the second part of the definition of monotonicity). We have thus shown that f is monotonic.

Part 2.

Given $a \in \Omega$, consider a set of outcomes S containing a. Then, by ontoness, there is a profile r such that $f(r) = S$. If we consider a profile r', obtained from profile r by putting $a \in top(r'_i)$, $\forall i$, then, by monotonicity on profiles r and r', $f(r')$ contains

a. Therefore, whenever a is one of the top elements of every agent's ordering in a profile, then such an outcome must be a winner in the result. Since the outcome a is arbitrary, then f satisfies item 1 of the definition of unanimity.

Given a set of outcomes A, consider now a profile p such that $f(p) = A$. If we consider profile p', obtained from profile p by putting $top(p'_i) = A$, $\forall i$, then by monotonicity on profiles p and p' we have $f(p') = A$. Therefore, whenever a set A is the the set of top elements of every agent's ranking in a profile, then such a set coincides with the set of winners. Thus f satisfies item 2 of the definition of unanimity. □

Theorem 12. *If there are at least two agents and at least three outcomes, and the social choice function over SIOs is strategy-proof and onto, then there is at least one weak dictator.*

Proof. By Theorem 11, if a social choice function is onto and strategy-proof, then it is monotonic and unanimous. By Theorem 10, if it is unanimous and monotonic, then there is at least one weak dictator. Thus, strategy-proofness and ontoness implies there must be a weak dictator. □