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Finding Pure Strategy Nash Equilibria

In two player games:

- for each strategy of opponent, underline own best reply
- a cell with both entries underlined represents a (pure-strategy) Nash Equilibrium

E.g., original Prisoner's Dilemma – Flood (1950)

		Player 2	
		Loyal	**Fink**
Player 1	**Loyal**	(-1, -1)	$(-3, \underline{0})$
	Fink	$(\underline{0}, -3)$	$(\underline{-2}, \underline{-2})$

Fink Fink is a (pure-strategy) Nash Equilibrium

(**Note:** We can also remove dominated strategies in nonzero-sum games)

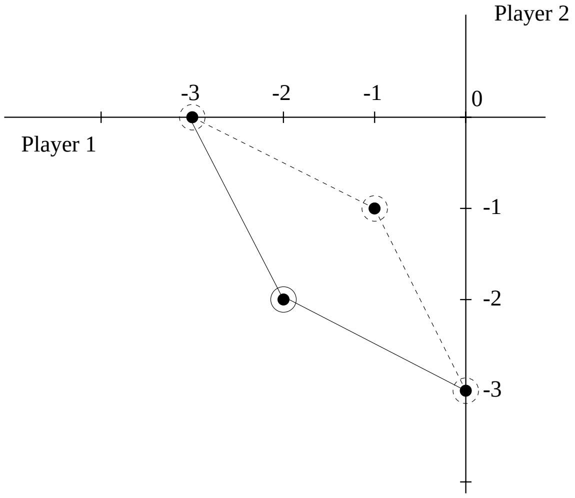
Pareto Optimality

Definition: An outcome is *Pareto optimal* if there is no other outcome which would give both players a higher payoff or would give one player the same payoff and the other player a higher payoff.

		Player 2	
		Loyal	**Fink**
Player 1	**Loyal**	$(-1, -1)^*$	$(-3, \underline{0})^*$
	Fink	$(\underline{0}, -3)^*$	$(\underline{-2}, \underline{-2})$

Pareto optimal points are on the north east boundary of the payoff polygon

Payoff Polygon



Exercise 4 (from last week)

Find the pure-strategy Nash equilibria and Pareto optimal outcomes

Player 2

Player 1		**A**	**B**	**C**
	1	(0, 4)	(4, 0)	(5, 3)
	2	(4, 0)	(0, 4)	(5, 3)
	3	(3, 5)	(3, 5)	(6, 6)

John

Jane		**Opera**	**Fight**
	Opera	(2, 1)	(0, 0)
	Fight	(0, 0)	(1, 2)

Mixed-Strategy Nash Equilibria

As with zero-sum games there may be no pure-strategy Nash equilibria in nonzero-sum games

How do we find mixed-strategy Nash equilibria in nonzero-sum games?

Each player considers their opponent's "half" of the game and determines a mixed-strategy just as in the zero-sum case

Slide 7

Calculating Mixed-Strategy Nash Equilibria

Player 2

		A	**B**
Player 1	**1**	(4, 8)	(2, 0)
	2	(6, 2)	(0, 8)

Slide 8

Player 1 considers Player 2's "half" of the game and determines their mixed-strategy

Player 2

		A	**B**
Player 1	**1**	8	0
	2	2	8

A: $8x + 2(1 - x) = 6x + 2$
B: $0x + 8(1 - x) = 8 - 8x$
$6x + 2 = 8 - 8x$
Therefore, $x = \frac{3}{7}$

Player 1 should play **1**: $\frac{3}{7}$, **2**: $\frac{4}{7}$ (Player 1's *equalising strategy* — optimal mixed-strategy)

Value to Player 2 = $4\frac{4}{7}$

Slide 9

Player 2 considers Player 1's "half" of the game and determines their mixed-strategy

Player 2

		A	**B**
Player 1	**1**	4	2
	2	6	0

1: $4x + 2(1-x) = 2x + 2$

2: $6x + 0(1-x) = 6x$

$2x + 2 = 6x$

Therefore, $x = \frac{1}{2}$

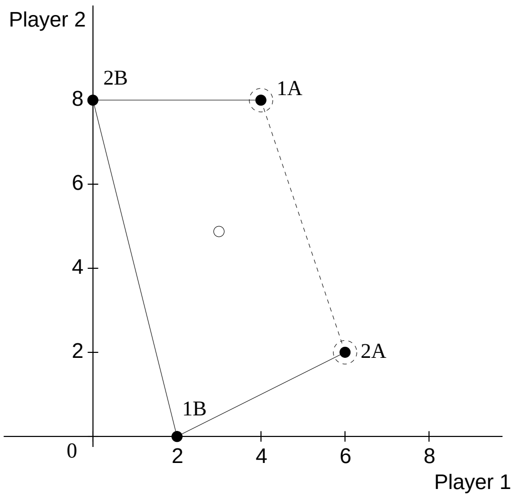
Player 2 should play **A**: $\frac{1}{2}$, **B**: $\frac{1}{2}$ (Player 2's equalising strategy)

Value to Player 1 = 3

If both player's play their equalising strategy none can gain by deviating (mutual maximisation)

Slide 10

Payoff Polygon



Slide 11

Prudential Strategies

(From Straffin 1993)

For nonzero-sum games a player's *prudential strategy* is their optimal strategy in their own "half" of the game

The value of their "half" of the game is called their *security level* (that player can guarantee they will get at least this payoff)

Continuing example above:
Player 1: prudential strategy **1**, security level = 2
Player 2: prudential strategy **A**: $\frac{4}{7}$ **B:** $\frac{3}{7}$, security level = $4\frac{4}{7}$

Prudential strategy not necessarily Pareto optimal

Slide 12

A player's *counter-prudential* strategy is their best response to opponent's prudential strategy

Player 1: **2**, Player 2: **A**

Does this help? Not really!

Definition: (Straffin 1993)
A two-person game is solvable in the strict sense if
– there is at least one Pareto optimal equilibrium outcome
– if more than one Pareto optimal equilibria exist, they are equivalent and interchangeable

Reaction

A *reaction* is a best possible response

It is a correspondence between strategies giving best set of replies

- $R_1(A)$ – Player 1's best replies to Player 2's strategy A
- $R_2(X)$ – Player 2's best replies to Player 1's strategy X

Consider the zero-sum game:

Player 2

Player 1	**A**	**B**	**C**
X	8	4	6
Y	18	0	0
Z	0	2	14

So $R_1(A) = \{Y\}$.

Can indicate this by, say, "circling" Player 1's reaction to each of Player 2's strategies and "squaring" Player 2's reaction to each of Player 1's strategies. This gives a *reaction curve* for pure-strategies in a zero-sum game

Reaction Curves for Mixed-Strategies

Consider the following matrix (Binmore 1992) for a zero-sum game

Player 2

Player 1	**A**	**B**
1	1	4
2	3	2

Player 1's mixed strategy:

- **A:** $1x + 3(1 - x) = 3 - 2x$
- **B:** $4x + 2(1 - x) = 2x + 2$

$x = \frac{1}{4}$

If $x < \frac{1}{4}$ (i.e., Player 1 plays **1** less than $\frac{1}{4}$) then Player 2 would play **A** otherwise $x > \frac{1}{4}$ and Player 2 plays **B** (if $x = \frac{1}{4}$, Player 2 is indifferent)

Slide 15

Player 2's mixed strategy:

1: $1y + 4(1-y) = 4 - 3y$

2: $3y + 2(1-y) = y + 2$

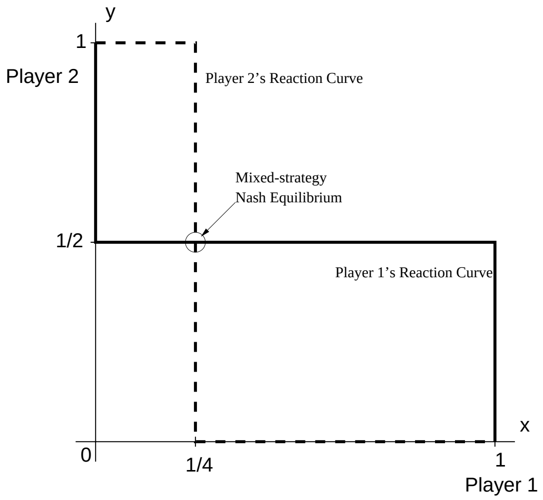
$y = \frac{1}{2}$

If Player 2 plays **A** $< \frac{1}{2}$, Player 1 plays **1**

If Player 2 plays **A** $> \frac{1}{2}$, Player 1 plays **2**

If Player 2 plays **A** $= \frac{1}{2}$, Player 1 indifferent

Slide 16



Slide 17

Reaction Curves for Nonzero-sum Games

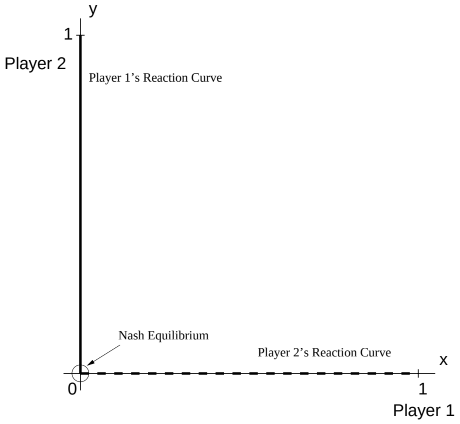


Figure 1: Reaction curve for Prisoner's Dilemma

Slide 18

Reaction Curves for Nonzero-sum Games

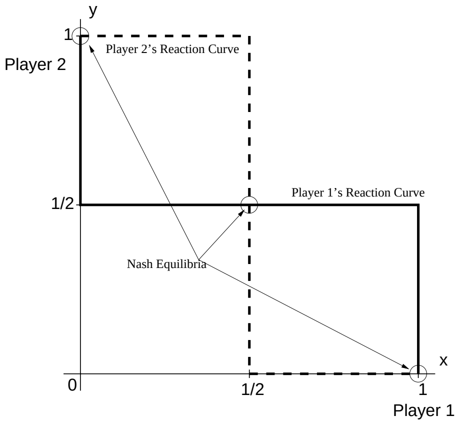


Figure 2: Reaction curve for Chicken

Slide 19

Mixed-Motive Games

Games in which there is no single course of action that is best for each player independently of what the other does are called *mixed-motive*

Players not diametrically opposed nor parallel

Players may want to cooperate rather than compete

There are essentially four non-trivial types of mixed-motive games

Slide 20

Chicken

Two motorists driving towards each other on a collision course. Each can swerve and be a chicken or continue deadly course of action. If one only swerves they get a payoff of 2 units while their opponent gets 4 units. If they both swerve they get 3 units payoff each. If they both continue they only get 1 unit each.

		Motorist 2	
		Swerve	**Continue**
Motorist 1	**Swerve**	(3, 3)*	($\underline{2}$, $\underline{4}$)*
	Continue	($\underline{4}$, $\underline{2}$)*	(1, 1)

Slide 21

Payoffs not constant for all choices

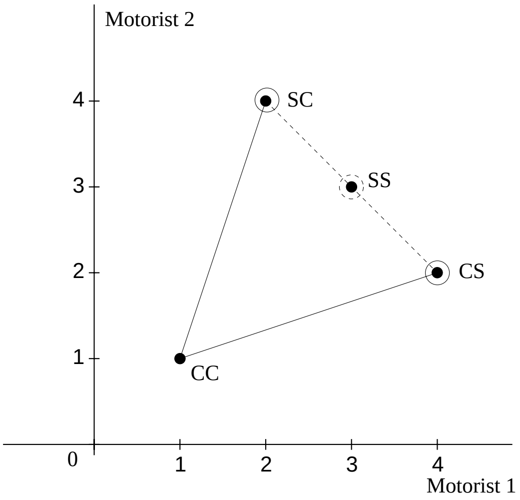
No dominant strategy for either player (i.e., no action for either player giving largest payoff no matter what other player does)

Two equilibria – no incentive to deviate

(Cautious approach – "take best of smallest payoffs" – gives both motorists swerving)

Slide 22

Payoff Polygon for Chicken



Slide 23

Leader

Two drivers entering intersection from opposite sides. When an opening arises they must decide whether to let the other have the right of way or go themselves. If both concede, they will have to wait longer (each receives 2 units payoff). If both go, they will collide (each receives 1 unit payoff). If one goes first there may be time for the other to follow (the leader gets 4 units payoff while follower gets 3 units payoff).

Motorist 2

		Concede	**Go**
Motorist 1	**Concede**	(2, 2)	$(\underline{3}, \underline{4})$*
	Go	$(\underline{4}, \underline{3})$*	(1, 1)

Slide 24

No dominant strategy

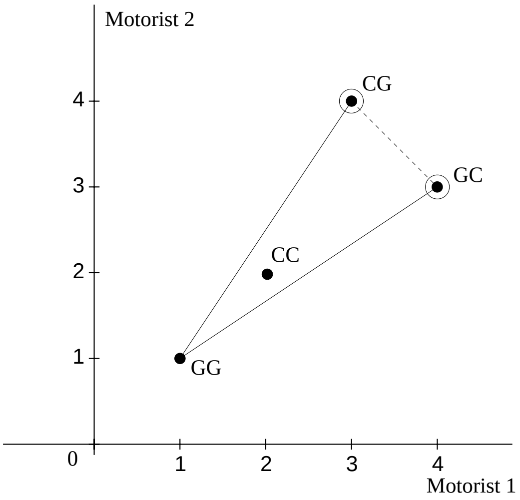
Two pure-strategy Nash Equilibria (each player “prefers” one of these two over the other)

In real-world situations other factors (e.g., cultural — “ladies first” —, psychological, . . .) resolve standoff

(Cautious approach advocates both conceding)

Slide 25

Payoff Polygon for Leader



Slide 26

Battle of the Sexes

Married couple must choose between two entertainment options: opera or prize fight. Wife prefers former but husband prefers latter. However, they would both prefer to be together rather than alone.

		Wife	
		Fight	**Opera**
Husband	**Fight**	($\underline{4}$, $\underline{3}$)*	(2, 2)
	Opera	(1, 1)	($\underline{3}$, $\underline{4}$)*

Slide 27

Neither husband nor wife has a dominant strategy

(Cautious approach gives Husband going to prize fight and wife to Opera)

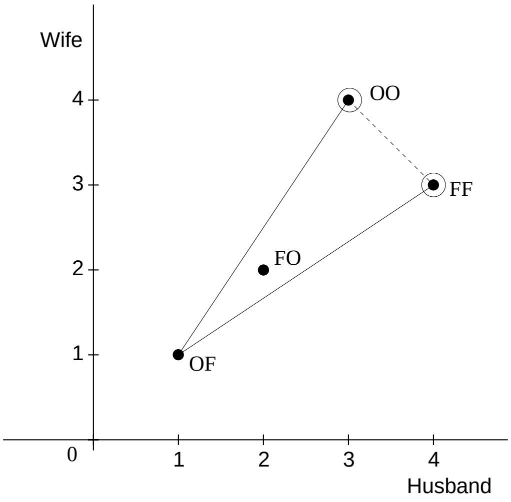
Two pure-strategy Nash equilibria

In contrast to Leader, unilaterally deviating player rewards other player more than themselves

Player can gain by communicating to obtain commitment from other play

Slide 28

Payoff Polygon for Battle of the Sexes



Prisoner's Dilemma

Two prisoners arrested for joint crime and are interogorated in different rooms. If both conceal information (cooperate) they are acquitted with payoff 3 units each. If one defects, they are rewarded with 4 units while the other gains only 1 unit. If both defect they will be convicted of a lesser offence and receive a payoff of 2 units each.

		Prisoner 2	
		Cooperate	**Defect**
Prisoner 1	**Cooperate**	(3, 3)*	(1, $\underline{4}$)*
	Defect	($\underline{4}$, 1)*	($\underline{2}$, $\underline{2}$)

One Nash equilibrium

(Cautious approach is mutual defection "regret free")

Paradox lies in conflict between individual and collective rationality

Individually better for players to defecct

If both cooperate they are each better off

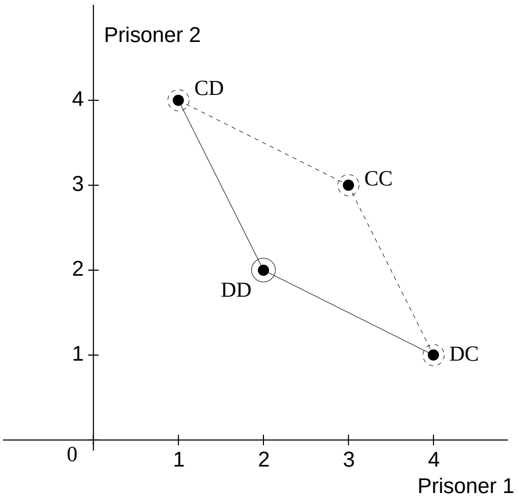
Good principle?: "do unto others as you would have them do unto you"

Doesn't work for Battle of the Sexes

What about a sequence of play?

Slide 31

Payoff Polygon for Prisoner's Dilemma



Slide 32

Iterated Prisoner's Dilemma

Game played more than once

May be beneficial to cooperate early and "defect" later on

If the number of games is finite and known at the outset, the outcome is still for both players to "defect". Why?

Rapoport (1984) suggested TIT FOR TAT: begin by cooperating and then choose whichever strategy your opponent played in the previous iteration of the game

Slide 33

Notes on Nash's Theorem

Generalises the Minimax Theorem by establishing the existence of a solution for both zero-sum and nonzero-sum games

Shows that more than one solution may exist

Extends to the case of finitely many players

Saddle points in a zero-sum game were equivalent and interchangeable

This does not hold for non-zero sum games in which case it may not be clear which to aim for

We have seen in mixed-strategy Nash equilibria that each player plays opponent's "half" of the game. They neglect their own payoffs! Can they do better?

(Note: In zero-sum games all points are Pareto optimal)

Slide 34

Problems with Nash Equilibria?

Equilibrium outcome desirable due to stability

Nash's Theorem shows at least one exists (always)

However, there can be many

Moreover, they can be non-equivalent and non-interchangeable (how do player's coordinate?)

They may not be Pareto optimal (cf., Prisoner's Dilemma)

Conclusion: Solution theory for zero-sum games does not carry over to nonzero-sum games

Nash equilibria essentially describe probabilities that rational player can assign to opponent; not what they should do but what they should believe

Where do we go from here?

(Hamming) “The purpose of Game Theory is insight not solutions”

There are many excellent works on Game Theory

J. Von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944. (The classic but perhaps not the best place to start.)

D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, 1992. (Very mathematical treatment.)

A. Jones *Game Theory*, Ellis Horwood, 1980.

Hit the web?: History:
`http://www.econ.canterbury.ac.nz/hist.htm`
A Game Theory page: `http://www.pitt.edu/alroth/alroth.html`
Prisoner’s Dilemma:
`http://serendip.brynmawr.edu/~ann/pd.html`

Exercise 1

Determine the pure-strategy Nash equilibria in the following nonzero-sum game Which outcomes are Pareto optimal?

Player 2

Player 1		**A**	**B**	**C**
	1	(4, 3)	(5, 1)	(6, 2)
	2	(2, 1)	(8, 4)	(3, 6)
	3	(3, 0)	(9, 6)	(2, 8)

Slide 37

Exercise 2

Determine the mixed-strategy Nash equilibrium in the following nonzero-sum game. Which points are Pareto optimal?

		Player 2	
		A	**B**
Player 1	**1**	(6, 4)	(4, 2)
	2	(8, 6)	(2, 8)

Slide 38

Exercise 3

(Rasmusen 1989) Welfare Game

The government wishes to help paupers only if they search for work. The pauper wishes to search for work only if he can't depend on government aid and may not even find a job if he tries.

The nonzero-sum game matrix may be represented as follows.

		Pauper	
		Work	**Idle**
Government	**Aid**	(3, 2)	(-1, 3)
	No Aid	(-1, 1)	(0, 0)