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Maurice Pagnucco
Knowledge Systems Group
Department of Artificial Intelligence
School of Computer Science and Engineering
The University of New South Wales
NSW 2052, AUSTRALIA
morri@cse.unsw.edu.au

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Mixed Strategies

Mixed strategy: play mixture of strategies according to fixed probabilities (i.e., random factor).

Definition: Expected value of obtaining payoffs $v_1,\ v_2,\ \ldots,\ v_n$ with respective probabilities $p_1,\ p_2,\ \ldots,\ p_n$ is:

$$p_1.v_1\ +\ p_2.v_2\ +\ \ldots\ +\ p_k.v_k$$

That is, a weighted average.

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Mixed Strategies — example

Consider the following game (Williams 1954):

Kershaw proposes to Goldsen:
Goldsen chooses letter: **a** or **i**
Kershaw chooses letter: **f**, **t**, **x**.
If the two letters constitute a word, I pay you \$1 plus \$3 bonus if word is a noun or pronoun. If letters don't form a word, you pay me \$2.

Game matrix:

		Kershaw		
		f	**t**	**x**
Goldsen	**a**			
	i			

What are the payoffs?

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		Kershaw		
		f	**t**	**x**
Goldsen	**a**	-2	1	4
	i	1	4	-2

Is there a saddle point?

Are there any dominated strategies?

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Notice: Kershaw strategy **f** dominated by strategy **t**.

Therefore, consider 2×2 matrix:

		Kershaw	
		f	**x**
Goldsen	**a**	-2	4
	i	1	-2

Absence of a saddle point means that neither player would want to commit to a single strategy with certainty. Other player could take advantage of this!

Solution: randomise choice of strategy.

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Expected Value Principle:
If you know opposing player has adopted a particular mixed strategy and will continue to do so no matter how you play, you should adopt a strategy with the largest expected value.

Consider utility for Goldsen:

a: $-2.x + 4.(1 - x) = 4 - 6.x$
i: $1.x + -2.(1 - x) = -2 + 3.x$

Suppose Kershaw uses a coin to decide his strategy (i.e., fixed probability of $\frac{1}{2}$)

Expected payoff to Goldsen:

a: $4 - 6.\frac{1}{2} = 1$
i: $-2 + 3.\frac{1}{2} = -\frac{1}{2}$

Is there some choice of probabilities that Goldsen cannot take advantage of?

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Equalising Expectations

Goldsen:

a: $-2.x + 4.(1-x) = 4 - 6.x$

i: $1.x + -2.(1-x) = -2 + 3.x$

Goldsen can take no advantage of Kershaw's strategy if these outcomes are equal.

Why?

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$$4 - 6.x = -2 + 3.x$$

$$6 = 9.x$$

$$x = \frac{6}{9} = \frac{2}{3}$$

If Kershaw plays $\frac{2}{3}$**f**, $\frac{1}{3}$**x**.

Goldsen expected values:

a: $-2.\frac{2}{3} + 4.\frac{1}{3} = 0$

i: $1.\frac{2}{3} + -2.\frac{1}{3} = 0$

Therefore payoff to Goldsen *geq* 0

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Do the same for Kershaw's expected values (i.e., Goldsen plays mixed strategy).

Kershaw:

f: $-2.x + 1.(1-x) = 1 - 3.x$

x: $4.x + -2.(1-x) = -2 + 6.x$

$$1 - 3.x = -2 + 6.x$$

$$3 = 9.x$$

$$x = \frac{3}{9} = \frac{1}{3}$$

If Goldsen plays $\frac{1}{3}$**a**, $\frac{2}{3}$**i**

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Kershaw:

f: $-2.\frac{1}{3} + 1.\frac{2}{3} = 0$

x: $4.\frac{1}{3} - 2.\frac{2}{3} = 0$

N.B: Payoff is that to Goldsen (i.e., row player)

Payoff to Goldsen ≤ 0

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Summary of this Game

Value of game = 0

Kershaw's optimal strategy = $\frac{2}{3}\mathbf{f}$, $0\mathbf{t}$, $\frac{1}{3}\mathbf{x}$

Goldsen's optimal strategy = $\frac{1}{3}\mathbf{a}$, $\frac{2}{3}\mathbf{i}$

These values are called the *solution* of the game.

Suppose Kershaw does play **t**?

Goldsen's expected value = $\frac{1}{3}.1 + \frac{2}{3}.4 = 3$!

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An Alternative

(Williams 1954) 2×2 games.

For row player's probabilities:

Step 1. take absolute difference of row entries

Step 2. interchange them to obtain "odds"

Kershaw

		f	**x**	*Diffs*	*Probs*
	a	-2	4	-6	$\frac{3}{9}$
Goldsen	**i**	1	-2	3	$\frac{6}{9}$
	Diffs	-3	6		
	Probs	$\frac{6}{9}$	$\frac{3}{9}$		

For column players do the same but for column entries.

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Beyond 2×2 Games

$2 \times n$ games
(Row player: 2 pure strategies; Column player: $n > 2$ pure strategies)

$m \times 2$ games
(Row player: $m > 2$ strategies; Column player: 2 pure strategies)

Fortunately, if game does not have a saddle point, there is always a solution which is a mixed strategy to a 2×2 *subgame*.

Unfortunately, there can be many 2×2 subgames.

Fortunately, there is an easier way!

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Graphical Technique

Elegant way to find which 2×2 subgame gives solution to game.

Moreover, gives insight into meaning of solution!

Consider the $2 \times n$ matrix (Williams 1954)

Red

		1	**2**	**3**	**4**	**5**	**6**	**7**
Blue	**A**	-6	-1	1	4	7	4	3
	B	7	-2	6	3	-2	-5	7

Does this game have a saddle point?

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1. Draw two vertical axes on the scale of the payoffs
2. Considering each of Red's strategies in turn:
   (a) mark the point on Axis 1 representing the payoff when Blue plays strategy A
   (b) mark the point on Axis 2 representing the payoff when Blue plays strategy B
   (c) draw a line connecting these two points
3. make bold the line segments bounding the figure from below
4. mark the highest point on this boundary

Lines intersecting at this point identify strategies Red should use

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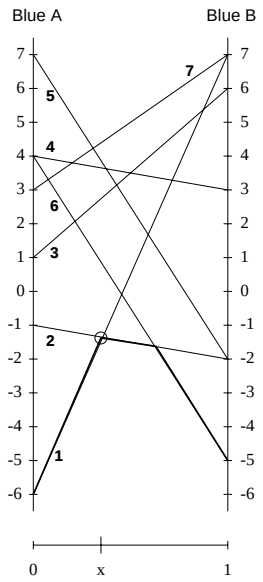

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Rationale

Bold Boundary: Red's best response

Highest point: Blue's way of making Red's payoff as little as possible

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Subgame to Solve

Involves Red 1 and Red 2. That is:

		Red	
		1	**2**
Blue	**A**	-6	-1
	B	7	-2

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Blue's expected value

A: $-6.x - 1.(1-x) = -5x - 1$
B :7.x-2(1-x)=9x-2

$x = \frac{1}{14}$ Expected value=$-1\frac{5}{14}$
Red's mixed strategy: **1:** $\frac{1}{14}$; **2:** $\frac{13}{14}$

Red's expected value

1: $-6.x + 7.(1-x) = 7 - 13x$
2: $-1.x - 2.(1-x) = x - 2$

$x = \frac{9}{14}$ Expected value=$-1\frac{5}{14}$
Blue's mixed strategy; **A:** $\frac{9}{14}$; **B:** $\frac{5}{14}$

Does it make sense? I yes, then ok to proceed.

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Let's go back and look at the graph.

What do you notice about the circled point?

What are its horizontal and vertical coordinates?

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What about $m \times 2$ Games?

Similar ... but different!

Given that the game is zero-sum (what does that mean?) can you suggest what changes in this case?

Shall look at an example in the last part of lecture

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$m \times 2$ Games — Graphical Technique

Draw 2 axes for Column player

Mark lines as previously but for each now player strategy

Now, mark lines bounding graph from *above*

Circle *lowest point* on boundary

What about $n \times m$ games in general?

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Minimax Theorem

von Neumann (1928)

Minimax Theorem:
Every two-person zero-sum game has a solution. That is, every $m \times n$ matrix game has a solution.

That is:

If row player adopts their optimal strategy:

their expected payoff $\geq$ value of game

If column player adopts their optimal strategy:

row player's expected payoff $\leq$ value of game

(no matter what opponent does)

Moreover, the solution is always to be found in some $k \times k$ submatrix of the original game

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Active Strategies — pure strategies involved in solution. These should be played according to determined probabilities. Other (non-active) strategies should not be played at all.

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Rock, Scissors Paper

Player 2

Player 1		**R**	**S**	**P**
	R	0	1	-1
	S	-1	0	1
	P	1	-1	0

Saddle point?

Dominated strategies?

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Player 1 expected values:

1R: $0.x + 1.y - 1.(1 - x - y) = x + 2y - 1$
1S: $-1.x + 0.y + 1.(1 - x - y) = -2x - y + 1$
1R: $1.x - 1.y + 0.(1 - x - y) = x - y$

$x = \frac{1}{3}$; $y = \frac{1}{3}$
Value=0; Player 2: $\frac{1}{3}$R, $\frac{1}{3}$S, $\frac{1}{3}$P

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Player 2 expected values:

2R: $0.x - 1.y + 1.(1 - x - y) = 1 - 2y - x$
2S: $1.x + 0.y - 1.(1 - x - y) = 2x + y - 1$
2R: $-1.x + 1.y + 0.(1 - x - y) = y - x$

$x = \frac{1}{3}$; $y = \frac{1}{3}$
Value=0; Player 1: $\frac{1}{3}$R, $\frac{1}{3}$S, $\frac{1}{3}$P

This will fail if solution involves 1×1 or 2×2 subgame.

Check for saddle point and dominance before equalising expectations

If this fails, look for $k \times k$ subgame (can be tedious!)

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Minimax Theorem — Discussion

There exist probabilities (or odds) by which each player can weight their actions so that they receive exactly the same average return (i.e., value of the game — or its negation for minimising player).

Introducing idea of mixed strategy restores solvability of two-person zero-sum games.

Extended notion of strategy from fixed action (pure strategy) to randomised (probabilistic) choice over space of actions (mixed strategy).

Each player can announce this strategy at the outset without loss.

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Finding Solutions to a Zero-Sum Game

2×2
- Look for a saddle-point
- If none, both players should use *mixed strategies*
  - Probabilities for I should give *equal payoffs* against both pure choices of II
  - The same principle for Player II

$2 \times N$
- There is always a 2×2 subgame with the same equilibrium
- Remove all dominated columns, then try the remaining subgames
- All the solutions can be found by graphical method

Larger games: Linear programming problem

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Solving Two-Person Zero-Sum Games

Solution: "hope for the best, prepare for the worst" (Casti 96)

Check for saddle point(s)

Check for dominated strategies

Equalise expectations (i.e., mixed strategies)

Look for $k \times k$ subgame that can be solved (use graphical technique on $2 \times n$ and $m \times 2$ subgames).

Have patience (or use linear programming techniques)!

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Equilibrium Pair of Strategies

Pair of strategies where player unilaterally deviating from their equilibrium strategy will worsen their expected payoff.

Minimax theorem implies equilibrium pair and minimax pair coincide *for zero-sum game.*

Nash's theorem extends this to non-cooperative games be they zero-sum or nonzero-sum).

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Exercise 1

(Resnick 1987) Commander Smith has landed reinforcements and must get them to the battleground. Two routes are available: over a mountain or around it via a plain. The latter is usually easier. However Commander Jones has been dispatched to intercept Smith. If they meet on the plain Smith will suffer serious losses. If they meet on the mountain his losses will be light. The game matrix is as follows. What should Smith do?

		Jones	
		Mountain	**Plain**
Smith	**Mountain**	-50	100
	Plain	200	-100

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Exercise 2

(Straffin 1993, Ex 3.4)

		Colin		
		A	**B**	**C**
Rose	**A**	-2	0	2
	B	3	1	-1

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Exercise 3

The Hi-Fi (Williams 1954)

A hi-hi company manufactures a highly sought after amplifier. This amplifier depends on a critical component costing $1 (per unit). However, if this component is defective it tends to cost the company $10 on average. Other alternatives are to buy a superior fully guaranteed component at $6 or a component costing $10 which comes with a money-back guarantee. What strategy should the company pursue?

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Exercise 4

(Straffin 1993, Ex 3.8)

		Colin			
		A	**B**	**C**	**D**
Rose	**A**	1	2	2	2
	B	2	1	2	2
	C	2	2	1	2
	D	2	2	2	0

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Exercise 5

(Straffin 1993, Game 3.4)

		Colin					
		A	**B**	**C**	**D**	**E**	**F**
Rose	**A**	4	-4	3	2	-3	3
	B	-1	-1	-2	0	0	4
	C	-1	2	1	-1	2	-3