

Reinforcement Belief Revision

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Abstract

The capability of revising its beliefs upon new information in a rational and efficient way is crucial for an intelligent agent. The classical work in belief revision focuses on idealized models and is not concerned with computational aspects. In particular, many researchers are interested in the logical properties (e.g., the AGM postulates) that a rational revision operator should possess. For the implementation of belief revision, however, one has to consider that any realistic agent is a finite being and that calculations take time. In this paper, we introduce a new operation for revising beliefs which we call *reinforcement belief revision*. The computational model for this operation allows us to assess it in terms of time and space consumption. Moreover, the operation is proved equivalent to a (semantical) model based on the concept of possible worlds, which facilitates showing that reinforcement belief revision satisfies all desirable rationality postulates.

1 Introduction

Belief revision is the process of adapting the beliefs of an agent to accommodate new, more precise, or more reliable information that is possibly inconsistent with the existing beliefs. The formal study of belief revision took as starting point the work of Alchourrón, Gärdenfors, and Makinson (AGM) during the first half of the 1980s [Alchourrón *et al.*, 1985; Alchourrón and Makinson, 1985]. The AGM framework is an idealized mathematical model of belief revision: It assumes that the beliefs of an agent are represented by a so-called *belief set*, a logically closed set of sentences in some underlying language \mathcal{L} . The new evidence is also represented by a sentence in \mathcal{L} , and a revision operator is then modeled as a function mapping the current belief set and the new evidence to a revised belief set.

To provide general design criteria for belief revision operators, the AGM trio developed a set of so-called *rationality postulates* [Alchourrón *et al.*, 1985]. The guiding principle of the AGM postulates is that of *economy of information* or *minimal change*, which means to not give up currently held beliefs and not generate new beliefs unless necessary. As pointed out by several researchers [Gärdenfors and Makinson, 1988; Spohn, 1988], the AGM postulates do not uniquely determine a revision operator, and

revision operators should, in general, exploit extra-logical information concerning the preference (or plausibility) over different beliefs in order to determine the revision strategy. The *Triviality Theorem* of [Gärdenfors, 1988] shows that, when accepting the AGM postulates, it is improper to include extra-logical preference information into the belief sets. As a consequence, we need to distinguish a belief set from a *belief state* (also called *epistemic state*). The latter contains, in addition to its belief set, the extra-logical preference information (also referred to as *conditional beliefs* [Boutilier, 1993]) which determines the revision strategy. Like in [Darwiche and Pearl, 1997], we will therefore consider revision operators to be functions on belief states in this paper.

For the incremental adaptation of beliefs, the AGM postulates proved to be too weak [Darwiche and Pearl, 1997] due to the excessive freedom they permit on the change of the conditional beliefs. This has led to the development of additional postulates for iterated belief revision by Darwiche and Pearl (DP), among others. Still, however, the AGM and DP postulates together are too permissive in that they support belief revision operators which assume arbitrary dependencies among the pieces of information which an agent acquires along its way. These dependencies have a drastic effect when the agent makes an observation which contradicts its currently held beliefs: The agent is forced to cancel everything it has learnt up to this point [Nayak *et al.*, 1996; Nayak *et al.*, 2003]. To handle dependencies properly, we have recently proposed the postulate of *Independence* as a complement to the AGM+DP postulates [Jin and Thielscher, 2007].

In classical belief revision, the agents are considered to have unlimited memory, time, and deductive ability. For implementing belief revision, however, one has to consider that any realistic agent is a finite being and that calculations take time. Therefore, the beliefs of a realistic agent should always be finitely representable; and a revision operator should not only behave rationally but also consume a reasonable amount of time and space. Arguably, adapting belief revision to realistic settings is far from trivial, as we need an approach which takes these characteristics of finiteness as well as memory and time limitations into account [Wassermann, 1999].

In this paper, we introduce a concrete belief revision operator, called *reinforcement belief revision*. Specifically, we present a computational model for reinforcement revision which operates on finite sets of integer-weighted sentences. This operation is proved equivalent to a possible world-based (semantical) model, which enables us to show that reinforcement belief revision satisfies the desirable rationality postulates. In addition, we will analyze the computational complexity of reinforcement belief revision. Finally, we will place our result in the context of possibility theory, which is a field of research that is intimately related to belief revision [Dubois and Prade, 1992; Dubois *et al.*, 1994]: In [Benferhat *et al.*, 2002], for instance, it is shown that some well-known revision operators can be directly mapped to well-known conditioning operations in possibility theory. In this paper, we will likewise show how reinforcement belief revision can be re-cast in the possibilistic setting. The result is a novel approach to revising possibilistic knowledge states.

The rest of the paper is organized as follows. In the next section, we recall the classical AGM postulates, followed by postulates for iterated revision. In Section 3 we introduce reinforcement belief revision by giving both a semantical and a computational definition. Reinforcement belief revision is assessed in terms of its logical properties and computational complexity. In Section 4, we re-model reinforcement belief revision in the setting of possibility theory. We conclude in Section 5 with a detailed comparison to related work. Proofs of the main results can be found in the appendix.

2 Preliminaries

In this paper, we will deal with a finitary propositional language \mathcal{L} generated from a finite set \mathcal{P} of atomic propositions. It is assumed that \mathcal{L} is equipped with the classical consequence relation \vdash . Given a set S of sentences, $Cn(S)$ consists of all logical consequences of S , that is $Cn(S) = \{\alpha \in \mathcal{L} \mid S \vdash \alpha\}$. Two sentences α and β are logically equivalent, written $\alpha \equiv \beta$, iff $\alpha \in Cn(\{\beta\})$ and $\beta \in Cn(\{\alpha\})$. A propositional *interpretation* (also referred to as a *possible world*) is a mapping from \mathcal{P} to $\{\top, \perp\}$, and the set of all interpretations is denoted by $\Theta_{\mathcal{L}}$. For the sake of simplicity, we may represent an interpretation by the set of atoms to which it assigns \top . An interpretation W is called a *model* of S (denoted by $W \models S$) if it truth functionally maps all sentences in S to \top . The set of all models of S is denoted by $Mods(S)$. Conversely, given a set \mathcal{W} of possible worlds, $Th(\mathcal{W})$ denotes the set of all sentences which are true in each element of \mathcal{W} . For easier readability, in the rest of the paper we will often identify a singleton set with its element. For instance, $Mods(\{\alpha\})$ may also be written as $Mods(\alpha)$.

A belief set K is a logically closed set of sentences, that is, $K = Cn(K)$. The *expansion* of a belief set K by a sentence α is defined as: $K + \alpha \stackrel{\text{def}}{=} Cn(K \cup \{\alpha\})$. For the sake of generality, we deliberately consider a belief state \mathcal{K} as an abstract object from which we can derive a belief set, denoted by $Bel(\mathcal{K})$, and also some extra-logical preference information. In concrete constructions of revision operators, the extra-logic preference information could take the form of a relation over the set of all sentences, or a relation over all possible worlds, etc. Moreover, we say that two belief states $\mathcal{K}_1, \mathcal{K}_2$ are *logically equivalent* (written as $\mathcal{K}_1 \equiv \mathcal{K}_2$) iff $Bel(\mathcal{K}_1) = Bel(\mathcal{K}_2)$.

A total pre-order \leq (possibly indexed) is a reflexive, transitive, binary relation, such that for any α, β : either $\alpha \leq \beta$ or $\beta \leq \alpha$. The strict part of \leq is denoted by $<$, that is, $\alpha < \beta$ iff $\alpha \leq \beta$ and $\beta \not\leq \alpha$. As usual, $\alpha = \beta$ abbreviates $\alpha \leq \beta$ and $\beta \leq \alpha$. Given a total pre-order \leq on S , we denote by $\min(S, \leq)$ the set of all minimal elements of S wrt. \leq . Natural numbers and positive integers (i.e., natural number greater than 0) are denoted by \mathbb{N} and \mathbb{N}^+ , respectively. The *cardinality* of a set S is denoted by $\|S\|$, and the *size* of S (the number of symbols occurring in S) is denoted by $|S|$.

2.1 The AGM Postulates

The *AGM postulates* [Alchourrón *et al.*, 1985] provide a mathematical foundation for belief revision by defining criteria for rational revision operators. The original postulates have been reformulated in [Darwiche and Pearl, 1997] as follows, where $\mathcal{K} * \alpha$ denotes the belief state resulting from the revision of belief state \mathcal{K} by sentence α :

- ($\mathcal{K} * 1$) $Bel(\mathcal{K} * \alpha) = Cn(Bel(\mathcal{K} * \alpha))$
- ($\mathcal{K} * 2$) $\alpha \in Bel(\mathcal{K} * \alpha)$
- ($\mathcal{K} * 3$) $Bel(\mathcal{K} * \alpha) \subseteq Bel(\mathcal{K}) + \alpha$
- ($\mathcal{K} * 4$) If $\neg\alpha \notin Bel(\mathcal{K})$ then $Bel(\mathcal{K}) + \alpha \subseteq Bel(\mathcal{K} * \alpha)$
- ($\mathcal{K} * 5$) $Bel(\mathcal{K} * \alpha)$ is inconsistent only if $\vdash \neg\alpha$
- ($\mathcal{K} * 6$) If $\alpha \equiv \beta$ then $\mathcal{K} * \alpha \equiv \mathcal{K} * \beta$

$$(\mathcal{K} * 7) \quad Bel(\mathcal{K} * (\alpha \wedge \beta)) \subseteq Bel(\mathcal{K} * \alpha) + \beta$$

$$(\mathcal{K} * 8) \quad \text{If } \neg\beta \notin Bel(\mathcal{K} * \alpha) \text{ then } Bel(\mathcal{K} * \alpha) + \beta \subseteq Bel(\mathcal{K} * (\alpha \wedge \beta))$$

Readers are referred to [Gärdenfors and Makinson, 1988] for a detailed account of the motivation and intuition behind these postulates.

Various approaches to constructing AGM revision operators have been proposed in the literature. Here, we will only sketch the basic ideas of two of them which are crucial to the rest of the paper.

2.1.1 Epistemic Entrenchment

[Gärdenfors and Makinson, 1988] have suggested to represent the extra-logical preference information of a belief state by a total pre-order on the underlying language \mathcal{L} . Formally, an *epistemic entrenchment* \leq_K on a belief set K is a binary relation over \mathcal{L} satisfying the following conditions:

$$(EE1) \quad \text{If } \alpha \leq_K \beta \text{ and } \beta \leq_K \gamma, \text{ then } \alpha \leq_K \gamma$$

$$(EE2) \quad \text{If } \alpha \vdash \beta, \text{ then } \alpha \leq_K \beta$$

$$(EE3) \quad \text{For any } \alpha \text{ and } \beta, \alpha \leq_K \alpha \wedge \beta \text{ or } \beta \leq_K \alpha \wedge \beta$$

$$(EE4) \quad \text{When } K \text{ is consistent, } \alpha \notin K \text{ iff } \alpha \leq_K \beta \text{ for all } \beta$$

$$(EE5) \quad \text{If } \beta \leq_K \alpha \text{ for all } \beta, \text{ then } \vdash \alpha$$

Intuitively, $\alpha \leq_K \beta$ means that β is at least as plausible as α .

It has been shown by [Gärdenfors and Makinson, 1988] that the AGM postulates characterize exactly the class of epistemic entrenchment-based revision operators: A revision operator $*$ satisfies $(\mathcal{K} * 1)$ - $(\mathcal{K} * 8)$ iff for any belief state \mathcal{K} there exists an epistemic entrenchment $\leq_{Bel(\mathcal{K})}$ on $Bel(\mathcal{K})$, such that for any sentence α :

$$(C*) \quad Bel(\mathcal{K} * \alpha) = \begin{cases} \mathcal{L} & \text{if } \vdash \neg\alpha \\ \{\beta \in \mathcal{L} \mid \neg\alpha <_{Bel(\mathcal{K})} \neg\alpha \vee \beta\} & \text{otherwise} \end{cases}$$

2.1.2 Faithful Ranking

From quite a different perspective, [Katsuno and Mendelzon, 1991] have proposed a constructive model for belief revision based on possible worlds. Formally, a *faithful ranking* \preceq_K on a belief set K is a total pre-order on the set of all possible worlds $\Theta_{\mathcal{L}}$, such that for any possible worlds W_1, W_2 :

1. If $W_1, W_2 \models K$, then $W_1 =_K W_2$
2. If $W_1 \models K$ and $W_2 \not\models K$, then $W_1 \prec_K W_2$

The intuitive meaning of $W_1 \preceq_K W_2$ is that W_1 is at least as plausible as W_2 .

Similar to epistemic entrenchment-based revision, [Katsuno and Mendelzon, 1991] have shown that the AGM postulates characterize exactly the class of faithful ranking-based revision operators: A revision operator $*$ satisfies $(\mathcal{K} * 1)$ - $(\mathcal{K} * 8)$ iff for any belief state \mathcal{K} there exists a faithful ranking $\preceq_{Bel(\mathcal{K})}$ on $Bel(\mathcal{K})$ such that for any sentence α :

$$Bel(\mathcal{K} * \alpha) = \begin{cases} \mathcal{L} & \text{if } \vdash \neg\alpha \\ Th(\min(Mods(\alpha), \preceq_{Bel(\mathcal{K})})) & \text{otherwise} \end{cases}$$

2.2 Postulates for Iterated Revision

2.2.1 The DP Postulates

It is well-known that the AGM postulates sanction improper response to sequences of new information due to the excessive freedom they allow on the change of the conditional beliefs. Therefore, [Darwiche and Pearl, 1997] have proposed four additional postulates for iterated belief revision:

- (DP1) If $\beta \vdash \alpha$, then $Bel((\mathcal{K} * \alpha) * \beta) = Bel(\mathcal{K} * \beta)$
- (DP2) If $\beta \vdash \neg\alpha$, then $Bel((\mathcal{K} * \alpha) * \beta) = Bel(\mathcal{K} * \beta)$
- (DP3) If $\alpha \in Bel(\mathcal{K} * \beta)$, then $\alpha \in Bel((\mathcal{K} * \alpha) * \beta)$
- (DP4) If $\neg\alpha \notin Bel(\mathcal{K} * \beta)$, then $\neg\alpha \notin Bel((\mathcal{K} * \alpha) * \beta)$

A detailed account of the motivation and interpretation of these postulates can be found in [Darwiche and Pearl, 1997].

To provide formal justifications, [Darwiche and Pearl, 1997] have also given a nice representation result for Postulates (DP1)-(DP4): Let $*$ be a revision operator satisfying Postulates ($\mathcal{K} * 1$)-($\mathcal{K} * 8$), then $*$ satisfies Postulates (DP1)-(DP4) iff the corresponding faithful rankings satisfy the following conditions:¹

- (DPR1) If $W_1, W_2 \models \alpha$, then $W_1 \preceq_{\mathcal{K}} W_2$ iff $W_1 \preceq_{\mathcal{K} * \alpha} W_2$
- (DPR2) If $W_1, W_2 \not\models \alpha$, then $W_1 \preceq_{\mathcal{K}} W_2$ iff $W_1 \preceq_{\mathcal{K} * \alpha} W_2$
- (DPR3) If $W_1 \models \alpha$ and $W_2 \not\models \alpha$, then $W_1 \prec_{\mathcal{K}} W_2$ implies $W_1 \prec_{\mathcal{K} * \alpha} W_2$
- (DPR4) If $W_1 \models \alpha$ and $W_2 \not\models \alpha$, then $W_1 \preceq_{\mathcal{K}} W_2$ implies $W_1 \preceq_{\mathcal{K} * \alpha} W_2$

This result gives an elegant characterization of the seemingly natural constraints that the DP postulates impose on the change of the conditional beliefs: When \mathcal{K} is revised by α , Conditions (DPR1) and (DPR2) require to retain the relative plausible ordering of any two α -worlds ($\neg\alpha$ -worlds, respectively); Conditions (DPR3) and (DPR4) require that if an α -world W_1 is (strictly) more plausible than a $\neg\alpha$ -world W_2 , then W_1 continues to be (strictly) more plausible than W_2 .

2.2.2 Two Radical Revision Operators

The AGM and DP postulates together still do not uniquely determine a revision operator. In the following, we present two interesting operators known from the literature which satisfy all AGM/DP postulates.

[Boutilier, 1993] has proposed a specific revision operator (known as *natural revision*) which satisfies the following:

- (CB) If $\neg\beta \in Bel(\mathcal{K} * \alpha)$, then $Bel((\mathcal{K} * \alpha) * \beta) = Bel(\mathcal{K} * \beta)$

¹For the sake of simplicity, $\preceq_{Bel(\mathcal{K})}$ and $\preceq_{Bel(\mathcal{K} * \alpha)}$ are abbreviated by $\preceq_{\mathcal{K}}$ and $\preceq_{\mathcal{K} * \alpha}$, respectively.

This postulate alone is in fact a strengthening of the set of DP postulates, in the sense that (CB) implies all of them (in the presence of the AGM postulates) but not vice versa.

A semantical characterization for (CB) has been given in [Darwiche and Pearl, 1997]:

$$(CBR) \quad \text{If } W_1, W_2 \not\models Bel(\mathcal{K} * \alpha), \text{ then } W_1 \preceq_{\mathcal{K}} W_2 \text{ iff } W_1 \preceq_{\mathcal{K} * \alpha} W_2$$

Since Condition (CB) forces the agent to cancel the previous observation upon any new observation which contradicts its currently held beliefs, it has been criticized as too radical to serve as a general rationality postulate [Nayak *et al.*, 2003; Jin and Thielscher, 2007].

So-called *lexicographic revision* (with “naked evidence”) satisfies the *postulate of Recalcitrance* [Nayak, 1994; Nayak *et al.*, 2003]:

$$(Rec) \quad \text{If } \beta \not\models \neg\alpha, \text{ then } \alpha \in Bel((\mathcal{K} * \alpha) * \beta)$$

Note that, in the presence of Postulates $(\mathcal{K} * 1)$ - $(\mathcal{K} * 8)$, we can derive (DP3) and (DP4) from (Rec), but neither (DP1) nor (DP2).

The semantical characterization of (Rec) is as follows [Nayak *et al.*, 2003]:

$$(RecR) \quad \text{If } W_1 \models \alpha \text{ and } W_2 \models \neg\alpha, \text{ then } W_1 \prec_{\mathcal{K} * \alpha} W_2$$

As shown in [Konieczny and Pérez, 2000], Postulate (Rec) is only suitable when the agent has full confidence in the new information; hence it, too, cannot serve as a general rationality postulate (cf. [Jin and Thielscher, 2007] for a counterexample).

2.2.3 Postulate of Independence

While both (CB) and (Rec) are too radical, the DP postulates alone are too weak because they do not address the problem of implicit dependence [Jin and Thielscher, 2007]. The additional postulate of *Independence* has been proposed to overcome this weakness:

$$(Ind) \quad \text{If } \neg\alpha \notin Bel(\mathcal{K} * \neg\beta) \text{ then } \alpha \in Bel((\mathcal{K} * \alpha) * \neg\beta)$$

Note that (Ind) is a weakening of (Rec), while it is still strong enough to imply (DP3) and (DP4). Readers are referred to [Jin and Thielscher, 2007] for a detailed discussion on the problem of implicit dependence, and the motivation and interpretation of Postulate (Ind).

The semantical characterization for Postulate (Ind) is as follows:

$$(IndR) \quad \text{If } W_1 \models \alpha \text{ and } W_2 \models \neg\alpha, \text{ then } W_1 \preceq_{\mathcal{K}} W_2 \text{ implies} \\ W_1 \prec_{\mathcal{K} * \alpha} W_2$$

Arguably, Condition (IndR) is quite natural and not overly constrained: When \mathcal{K} is revised by α , Condition (IndR) requires a world W_1 confirming the new information α to become more plausible than a world W_2 violating α , provided that W_1 was at least as plausible as W_2 .

2.3 Cut Base Revision

The constructive models based on epistemic entrenchment and/or faithful ranking seem mathematically elegant. However, in order to construct an implementable revision operator, we need a *feasible representation* of belief states. Obviously, it is infeasible to encode on a computer an explicit belief set, since it is infinite in general. Many researchers [Wobcke, 1992; Nebel, 1994] have therefore suggested to represent the logical contents of a belief state by a *belief base*, which is a finite set of sentences (not logically closed). Moreover, [Nebel, 1998] has argued that the size of the extra-logical preference information of a belief state should be *polynomially bounded* by the size of its belief base.

Cut base revision is a revision operator proposed by [Nebel, 1991] based on a very compact belief representation. Formally, a *prioritized base* $\langle B, \leq_B \rangle$ consists of a belief base B and a total pre-order \leq_B on B . Note that a prioritized base $\langle B, \leq_B \rangle$ can also be represented by a totally ordered family of sets of sentences (B_1, \dots, B_n) such that $\alpha \leq_B \beta$ iff there exist i, j with $\alpha \in B_i$, $\beta \in B_j$, and $i \leq j$.

Given a prioritized base $\langle B, \leq_B \rangle$, the *cut-set* of a sentence α is defined as follows:

$$cut_{<_B}(\alpha) \stackrel{\text{def}}{=} \{\beta \in B \mid \{\gamma \in B \mid \beta \leq_B \gamma\} \not\vdash \alpha\} \quad (1)$$

Put in words, the cut-set of α consists of all sentences in $B_i \cup \dots \cup B_n$ such that if the next lower class B_{i-1} is added, then α becomes entailed.

Let $Cn(B)$ be the belief set induced from a prioritized base $\langle B, \leq_B \rangle$, then according to [Nebel, 1994] the total pre-order \leq_B can be generalized to an epistemic entrenchment $\leq_{Cn(B)}$ on $Cn(B)$ by letting

$$\alpha \leq_{Cn(B)} \beta \text{ iff } cut_{<_B}(\beta) \subseteq cut_{<_B}(\alpha) \quad (2)$$

Formally, cut base revision is defined as follows:

$$\langle B, \leq_B \rangle *_{cut} \alpha \stackrel{\text{def}}{=} cut_{<_B}(\neg\alpha) \cup \{\alpha\} \quad (3)$$

Note that $cut_{<_B}(\neg\alpha)$ is a subset of B ; therefore, the size of $\langle B, \leq_B \rangle *_{cut} \alpha$ is *linearly bounded* in the size of B and α .

In [Nebel, 1994] it has been shown that cut base revision is essentially an epistemic entrenchment-based revision: Suppose that $\langle B, \leq_B \rangle$ is a prioritized base and $\leq_{Cn(B)}$ is the epistemic entrenchment as defined by (2), then for any sentence α :

$$Cn(\langle B, \leq_B \rangle *_{cut} \alpha) = \begin{cases} \mathcal{L} & \text{if } \vdash \neg\alpha \\ \{\beta \in \mathcal{L} \mid \neg\alpha <_{Cn(B)} \neg\alpha \vee \beta\} & \text{otherwise} \end{cases}$$

Nebel's proposal constitutes a nice step from theory to computation. Unfortunately, cut base revision is not an iterated revision operator since it maps a prioritized base and the new information to a flat belief base. It is therefore impossible to do a subsequent revision. This is also referred to by [Hansson, 2003] as the problem of *categorical mis-matching*. It is not difficult to see that categorical mis-matching also occurs in the constructive model based on faithful ranking.

3 Reinforcement Belief Revision

It is an interesting question whether we can construct a satisfactory iterated revision operator by generalizing cut base revision. Unfortunately, the following discussion will

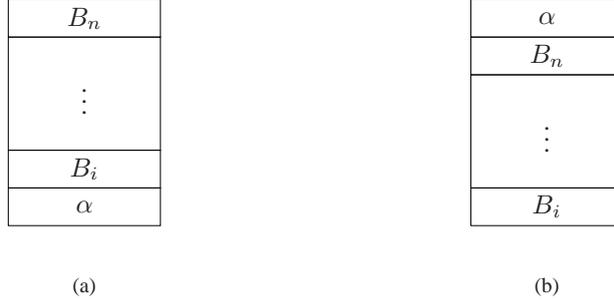


Figure 1: Two naive generalizations of cut base revision

show that at least a naive approach does not work. To make our presentation easier, we assume that a prioritized base is represented by a totally ordered family of classes of sentences (B_1, \dots, B_n) .

Suppose that a prioritized base (B_1, \dots, B_n) is revised by α and $cut_{<_B}(\neg\alpha) = \bigcup\{B_i, \dots, B_n\}$. According to (3), the revised belief base consists of (B_i, \dots, B_n) and α . Arguably, we have every reason to assume that (B_i, \dots, B_n) is ordered as before. The only problem is, where to put the new information α ? Without any further information, we may have two options: to render α less plausible than (B_i, \dots, B_n) as shown in Figure 1(a); or to consider α to be more plausible than (B_i, \dots, B_n) as shown in Figure 1(b).

Formally, we can define the so-called *skeptical cut base revision* as follows:

$$\langle B, \leq_B \rangle *_{cut}^s \alpha \stackrel{\text{def}}{=} \langle B_1 = cut_{<_B}(\neg\alpha) \cup \{\alpha\}, \leq_{B_1} \rangle$$

where $\beta \leq_{B_1} \gamma$ iff $\beta = \alpha$ or $\beta, \gamma \in cut_{<_B}(\neg\alpha)$ and $\beta \leq_B \gamma$. It is not difficult to see that skeptical cut base revision satisfies the AGM postulates as well as Postulate (CB).

Similarly, the so-called *credulous cut base revision* is defined as follows:

$$\langle B, \leq_B \rangle *_{cut}^c \alpha \stackrel{\text{def}}{=} \langle B_1 = cut_{<_B}(\neg\alpha) \cup \{\alpha\}, \leq_{B_1} \rangle$$

where $\beta \leq_{B_1} \gamma$ iff $\gamma = \alpha$ or $\beta, \gamma \in cut_{<_B}(\neg\alpha)$ and $\beta \leq_B \gamma$. Credulous cut base revision satisfies the AGM postulates, (DP1), and (Rec), but violates (DP2).

Note that both skeptical and credulous cut base revision have their own assumptions about the plausibility of the new information. One may argue that these assumptions could be violated in many situations. In the sequel, we will present a generalization of cut base revision which deals with the plausibility of the new information in a more appropriate manner.

In our approach, a belief state is represented by a finite set of integer-weighted sentences. Formally, an *epistemic entrenchment base* (EE base, for short), denoted by $\Xi = \langle B, f \rangle$, consists of a belief base B and a mapping f from B to \mathbb{N}^+ .² The

² EE bases and related notions that will be introduced in this section are not new. Similar belief representations have been used in various computational revision operators [Williams, 1992; Dixon, 1994; Williams, 1994]. Moreover, knowledge bases in possibilistic logic [Dubois *et al.*, 1994] are essentially EE bases modulo a monotonic scale transformation. What makes our work distinct, however, is a novel and better approach to revising EE bases. As we will see in Section 4.2 this also gives us a new approach to revising possibilistic knowledge bases.

belief set of Ξ consists of all logical consequences of B , that is, $Bel(\Xi) = Cn(B)$. For any sentence $\beta \in B$, we call $f(\beta)$ its *evidence degree*. Intuitively, a sentence with a larger evidence degree is considered more plausible.

Given an EE base $\Xi = \langle B, f \rangle$, we denote by $\Xi|_m$ the set of sentences in B whose evidence degree is exactly m :

$$\Xi|_m \stackrel{\text{def}}{=} \{\beta \in B \mid f(\beta) = m\}$$

Moreover, Ξ^m is the set of sentences whose evidence degree is greater than m :

$$\Xi^m \stackrel{\text{def}}{=} \bigcup \{\Xi|_i \mid i \geq m\}$$

The belief degree (also called *rank*) of a sentence β wrt. an EE base $\Xi = \langle B, f \rangle$ is defined as follows:

$$Rank_{\Xi}(\beta) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } B \not\vdash \beta \\ \infty & \text{else if } \vdash \beta \\ \max(\{m \mid \Xi^m \vdash \beta\}) & \text{otherwise} \end{cases} \quad (4)$$

We may consider an EE base as a collection of some uncertain information, in which the rank of a sentence can be interpreted as its certainty. Note that it is possible that $Rank_{\Xi}(\beta) > f(\beta)$ for a sentence $\beta \in B$, in which case we say that β is redundant in Ξ . Therefore, the evidence degree $f(\beta)$ of a sentence $\beta \in B$ is only a lower bound of its belief degree. It is not difficult to see that we can remove redundant sentences from an EE base without affecting the belief degrees of all sentences.

An EE base $\Xi = \langle B, f \rangle$ is a generalized prioritized base in which the cut-set of a sentence α can be obtained as follows:

$$cut_{\Xi}(\alpha) = \{\beta \in B \mid Rank_{\Xi}(\alpha) < f(\beta)\} \quad (5)$$

Note that the notion of cut-set defined by (5) generalizes the one defined by (1).

Not surprisingly, we can also derive an epistemic entrenchment $\leq_{Bel(\Xi)}$ from an EE base $\Xi = \langle B, f \rangle$ by stipulating:

$$\alpha \leq_{Bel(\Xi)} \beta \text{ iff } Rank_{\Xi}(\alpha) \leq Rank_{\Xi}(\beta) \quad (6)$$

In our setting, an iterated revision operator should now be a function which maps an EE base and the new information to a revised EE base. It is worth mentioning that the quantitative nature of EE bases allows to represent more fine-grained beliefs, e.g., the belief that α is “much more plausible” than β . As we will see, the quantitative nature of EE bases also allows for a fine-grained control on belief revision. The discussion in Section 2.3 suggests that one major problem is to find an appropriate evidence degree for the new information in the revised EE base. Obviously, if the new information is a mere sentence α , then the revision operator has to assign to α an evidence degree via a fixed scheme. But it is unlikely that there exists such a fixed scheme suitable for all different kinds of applications. Therefore, based on the same considerations as in [Spohn, 1988], we consider a more general revision scheme in which the new information consists of a sentence α along with an *evidence degree* $m \in \mathbb{N}^+$.

By generalizing cut base revision, we now define so-called *reinforcement base revision* as follows:

$$\langle B, f \rangle *_r \langle \alpha, m \rangle \stackrel{\text{def}}{=} \begin{aligned} & \{ \langle \beta, f(\beta) - Rank_{\Xi}(\neg\alpha) \rangle \mid \beta \in cut_{\Xi}(\neg\alpha) \} \\ & \cup \{ \langle \alpha \vee \beta, f(\beta) + m \rangle \mid \beta \in B \} \\ & \cup \{ \langle \alpha, m \rangle \} \end{aligned} \quad (7)$$

Note that the new sentence α is assigned evidence degree m in the revised EE base. The main difference between cut base revision and reinforcement base revision is that the latter adds a disjunction $\beta \vee \alpha$ for every sentence $\beta \in B$. These disjunctions will obviously not affect the logical contents of the revised EE base, but they are necessary to avoid (undesirable) implicit dependencies.³ It is easy to see that the size of $\langle B, f \rangle *_r \langle \alpha, m \rangle$ is linearly bounded by the size of $\langle B, f \rangle$ and $\langle \alpha, m \rangle$.

To see how reinforcement base revision works, let us consider a classical example introduced by [Darwiche and Pearl, 1997].

Example 1. We are introduced to a lady X who sounds smart and looks rich, so we believe that X is smart and X is rich. Since we profess of no prejudice, we also maintain that X is smart even if found to be poor and conversely, X is rich even if found to be not smart. Now, we obtain some evidence that X is in fact not smart, we remain of course convinced that X is rich. Still, it would be strange for us to say, “if the evidence turns out false, and X turns out smart after all, we would no longer believe that X is rich”. If we currently believe X is smart and rich, then evidence first refuting then supporting that X is smart should not in any way change our opinion about X being rich.

Let s and r represent respectively that X is smart and X is rich. Assume that our initial belief state is encoded by $\Xi = \{\langle r, 1 \rangle, \langle s, 1 \rangle, \langle r \vee s, 2 \rangle\}$ ⁴ and we first learn that X is not smart with evidence degree 2. Since $\text{Rank}_{\Xi}(\neg s) = 1$, according to (7), the revised EE base is $\Xi_1 = \Xi *_r \langle \neg s, 2 \rangle = \{\langle r \vee s, 1 \rangle, \langle \neg s, 2 \rangle, \langle r \vee \neg s, 3 \rangle\}$ ⁵. Suppose we now learn that X is smart with evidence 2. Since $\text{Rank}_{\Xi_1}(\neg s) = 2$, we obtain $\Xi_2 = \Xi_1 *_r \langle s, 2 \rangle = \{\langle r \vee \neg s, 1 \rangle, \langle s, 2 \rangle, \langle r \vee s \vee s, 3 \rangle\}$. It is not difficult to see that $\text{Rank}_{\Xi_2}(r) = 1$, which means that we continue to believe that X is rich (as expected).

As reinforcement base revision is a generalization of cut base revision, one may wonder whether it suffers from a well-known weakness of the latter, namely, the failure of proper belief preservation. Consider the following example:⁶ Suppose that $\Xi = \{\langle p, 1 \rangle, \langle q, 2 \rangle, \langle r, 3 \rangle\}$ is revised by $\langle \neg q, 4 \rangle$. At first glance, it seems indeed undesirable that, in the revised EE base $\{\langle r, 1 \rangle, \langle p \vee \neg q, 5 \rangle, \langle r \vee \neg q, 7 \rangle, \langle \neg q, 4 \rangle\}$, p is not believed anymore and the rank of r is lowered to 1. However, this is indeed the intended behavior, as the following argument shows.

As we have noted in [Jin and Thielscher, 2007], for any revision operator satisfying the AGM postulates a belief state (regardless of its representation) will encode some implicit dependencies among beliefs. More specifically, Condition (C^*) in Section 2.1.1 shows that if $\beta \vee \neg\alpha$ is not more entrenched than $\neg\alpha$, then β will not survive an α -revision (because β depends on $\neg\alpha$, so its removed together with $\neg\alpha$). In the above case, Ξ encodes implicitly that “ p depends on q ” since $p \vee q$ is not considered to be more plausible than q . Therefore, it is perfectly reasonable and inevitable that p is removed after a revision by $\neg q$. An interesting question is how we can obtain a belief state which does not encode undesirable implicit dependencies like “ p depends on q ”? In fact, it is a major advantage of reinforcement base revision that

³We refer readers who are interested in the problem of implicit dependencies to [Jin and Thielscher, 2007] for a detailed discussion.

⁴Readers are invited to check that Ξ is equal to $(\emptyset *_r \langle r, 1 \rangle) *_r \langle s, 1 \rangle$ as well as $(\emptyset *_r \langle s, 1 \rangle) *_r \langle r, 1 \rangle$.

⁵For simplicity, redundant sentences have been removed, and we will always tacitly do so in the rest of the paper.

⁶We are grateful to an anonymous reviewer of this paper for suggesting this example.

it will not introduce undesirable dependencies. For instance, assume that the initial belief state is empty and we subsequently learn p, q, r with individual evidence degrees. In this case, $\Xi' = ((\emptyset *_r \langle p, 1 \rangle) *_r \langle q, 2 \rangle) *_r \langle r, 3 \rangle$ will encode implicitly that p, q, r are independent of each other. Therefore, a revision by the negation of one proposition will not alter the ranks of any of the other propositions. For example, it is not difficult to see that $\text{Rank}_{\Xi''}(p) = 1$ and $\text{Rank}_{\Xi''}(r) = 3$, where $\Xi'' = \Xi' *_r \langle \neg q, 4 \rangle$.

The fact that r is lowered to 1 in the first example, $\{\langle p, 1 \rangle, \langle q, 2 \rangle, \langle r, 3 \rangle\} *_r \langle \neg q, 4 \rangle$, can be explained as follows. Since q depends on r in the initial belief state, r can be considered a strong support of q . Therefore, it is reasonable that we become less certain about r after q is learned to be false. In contrast, the rank of r will remain the same when Ξ' is revised by $\neg q$ because q is not dependent on r in Ξ' . The following result shows that a revision of one sentence in general will not affect the rank of another sentence, unless they are somehow logically related:

Proposition 3.1. *Suppose that Ξ_1 is the result of revising an EE base Ξ by $\langle \neg \alpha, m \rangle$, where α is an arbitrary sentence and m an arbitrary natural number. Then for any sentence β :*

$$\text{Rank}_{\Xi_1}(\beta) \neq \text{Rank}_{\Xi}(\beta) \text{ iff } \text{Rank}_{\Xi_2}(\alpha) \neq \text{Rank}_{\Xi}(\alpha)$$

where $\Xi_2 = \Xi *_r \langle \neg \beta, m' \rangle$ for an arbitrary natural number m' .

We remark that more logical properties of reinforcement base revision will be presented later in this section.

3.1 OCF-Based Semantics

To provide a better intuition about reinforcement base revision, we give an alternative, semantic characterization, which originates in [Jin and Thielscher, 2007] and is based on Spohn's *ordinal conditional functions* (OCF, for short) [Spohn, 1988]. Originally, an OCF k is a mapping from the set of all possible worlds $\Theta_{\mathcal{L}}$ to the class of ordinals. Like in [Spohn, 1991], for mathematical simplicity we take the range of an OCF k to be the natural numbers, \mathbb{N} . For any possible world W , we call $k(W)$ the *rank* of W . Intuitively, the rank of a world represents its degree of implausibility. Hence, the lower its rank, the more plausible is a world.

Any OCF k is a belief state from which we can induce a belief set and a faithful ranking \preceq_k , where $\text{Bel}(k)$ is the set of sentences which hold in all worlds of rank 0:

$$\text{Bel}(k) = \text{Th}(\{W \mid k(W) = 0\}) \quad (8)$$

In contrast to [Spohn, 1988], we do not require the set of possible worlds with rank 0 to be non-empty. Therefore, our approach can also deal with inconsistent belief sets.

An OCF k is extended to a ranking of sentences as follows:

$$k(\beta) = \begin{cases} \infty & \text{if } \vdash \beta \\ \min\{k(W) \mid W \models \neg \beta\} & \text{otherwise} \end{cases} \quad (9)$$

Put in words, the rank of a sentence is the lowest rank of a world in which the sentence does not hold.⁷ Hence, the higher the rank of a sentence, the firmer the belief in it, and the belief set consists of all sentences with rank greater than 0.

⁷In Spohn's original proposal, the rank of a sentence is the lowest rank of a world in which it is true. So the rank of β there is equal to $k(\neg \beta)$ here.

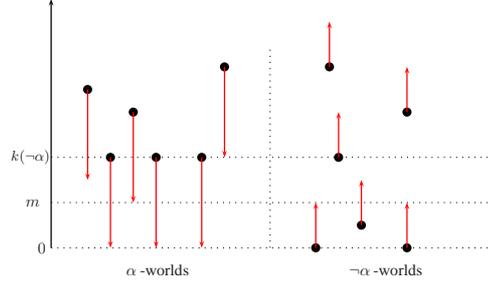


Figure 2: Visualization of reinforcement OCF revision

It is not difficult to see that an OCF k determines an epistemic entrenchment as follows:

$$\alpha \leq_k \beta \text{ iff } k(\alpha) \leq k(\beta) \quad (10)$$

In the sequel, we present a generalized revision operator (called *reinforcement OCF revision*) which also allows to assign different evidence degrees to new information. An OCF k is revised according to new information α with evidence degree $m \in \mathbb{N}^+$ as follows:

$$(k_{\alpha, m}^{r,*})(W) \stackrel{\text{def}}{=} \begin{cases} k(W) - k(\neg\alpha) & \text{if } W \models \alpha \\ k(W) + m & \text{otherwise} \end{cases} \quad (11)$$

Put in words, the rank of an α -world decreases by $k(\neg\alpha)$, whereas the rank of a $\neg\alpha$ -world increases by m .

A graphical illustration of the principle of reinforcement OCF revision is given in Figure 2: Circles to the left (right) of the vertical dotted line represent α -worlds ($\neg\alpha$ -worlds, respectively); the vertical coordinate of a possible world denotes its rank; and the arrow starting from a possible world indicates the change of its rank.

3.1.1 Equivalence Results

To show the equivalence of reinforcement belief revision and reinforcement OCF revision, we need a mapping from EE bases to OCFs. Formally, an OCF k_{Ξ} is derived from an EE base $\Xi = \langle B, f \rangle$ by letting

$$k_{\Xi}(W) = \begin{cases} 0 & \text{if } W \models B \\ \max(\{f(\beta) \mid \beta \in B \text{ and } W \not\models \beta\}) & \text{otherwise} \end{cases} \quad (12)$$

Put in words, the rank of a possible world is the maximal evidence degree of all sentences that it does not satisfy.

The following result shows that an EE base and its induced OCF encode essentially the same belief state.⁸

Proposition 3.2. *Suppose that $\Xi = \langle B, f \rangle$ is an EE base and k_{Ξ} the induced OCF as defined by (12), then for any sentence β :*

$$\text{Rank}_{\Xi}(\beta) = k_{\Xi}(\beta)$$

⁸A similar result can be found in possibilistic logic [Dubois *et al.*, 1994] (cf. Proposition 4.3).

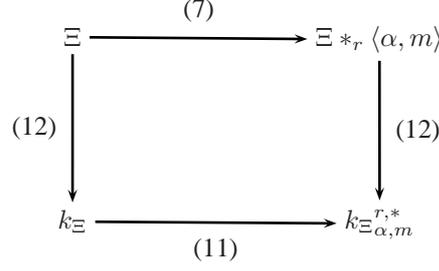


Figure 3: A commuting diagram for reinforcement base revision and reinforcement OCF revision

Recall Example 1. It is easy to see that the EE base $\Xi = \{\langle r, 1 \rangle, \langle s, 1 \rangle, \langle r \vee s, 2 \rangle\}$ induces the OCF k shown in Table 1 (middle column; assuming $\mathcal{P} = \{s, t\}$). Now, with $k(\neg\neg s) = 1$ and following (11) we obtain the OCF $k_{\neg s, 2}^{r,*}$ shown in the right column of Table 1. We remark that this $k_{\neg s, 2}^{r,*}$ is exactly the OCF induced from the EE base $\Xi_1 = \{\langle r \vee s, 1 \rangle, \langle \neg s, 2 \rangle, \langle r \vee \neg s, 3 \rangle\}$, which is the result of revising Ξ by $\neg s$ with evidence degree 2.

possible worlds	k	$k_{\neg s, 2}^{r,*}$
$W_1 = \{s, r\}$	0	2
$W_2 = \{s\}$	1	3
$W_3 = \{r\}$	1	0
$W_4 = \{\}$	2	1

Table 1: An example of reinforcement OCF revision

The following theorem shows that the two revision operators behave symmetrically in general. That is to say, for any EE base Ξ and new information $\langle \alpha, m \rangle$, the revised EE base $\Xi *_r \langle \alpha, m \rangle$ induces exactly the revised OCF $k_{\Xi *_r \langle \alpha, m \rangle}^{r,*}$ (see also Figure 3).

Theorem 3.3. *Suppose that Ξ is an EE base and k_{Ξ} the OCF induced from Ξ . Let $\langle \alpha, m \rangle$ be any new information, then for all possible worlds W :*

$$k_{\Xi_1}(W) = k_{\Xi *_r \langle \alpha, m \rangle}^{r,*}(W)$$

where $\Xi_1 = \Xi *_r \langle \alpha, m \rangle$.

As a direct consequence of Proposition 3.2 and Theorem 3.3, we have the following equivalence result.

Corollary 3.4. *Suppose that Ξ is an EE base and k_{Ξ} the OCF induced from Ξ . Let $\langle \alpha, m \rangle$ be any new information, then for any sentence β :*

$$\text{Rank}_{\Xi_1}(\beta) = k_{\Xi *_r \langle \alpha, m \rangle}^{r,*}(\beta)$$

where $\Xi_1 = \Xi *_r \langle \alpha, m \rangle$.

3.2 Logical Properties

The equivalence result enables us to prove important logical properties of reinforcement base revision by referring to reinforcement OCF revision instead. The reason for doing so is two-fold: On the one hand, it is much easier to deal with the latter due to its simple and intuitive construction. Moreover, reinforcement OCF revision has already been investigated in [Jin and Thielscher, 2007]. In the following, we summarize the most important of the known properties.

Theorem 3.5. [Jin and Thielscher, 2007] *Consider an OCF k .*

1. *Assume an arbitrary but fixed evidence degree for any new information. Then reinforcement OCF revision satisfies the AGM postulates ($\mathcal{K} * 1$)-($\mathcal{K} * 8$) as well as (DP1), (DP2), and (Ind).*

2. *For any $m_1, m_2 \in \mathbb{N}^+$, reinforcement OCF revision satisfies the following:*

- (EDP1) *If $\beta \vdash \alpha$, then $(k_{\alpha, m_1}^{r, *})_{\beta, m_2}^{r, *} \equiv k_{\beta, m_2}^{r, *}$*
- (EDP2) *If $\beta \vdash \neg\alpha$, then $(k_{\alpha, m_1}^{r, *})_{\beta, m_2}^{r, *} \equiv k_{\beta, m_2}^{r, *}$*
- (EInd) *If there exists m such that $k_{-\beta, m_2}^{r, *} \not\vdash \neg\alpha$, then $(k_{\alpha, m_1}^{r, *})_{-\beta, m_2}^{r, *} \vdash \alpha$*

3. *Let $\langle \alpha, m \rangle$ be some new information, then*

$$k_{\alpha, m}^{r, *}(\alpha) = k(\alpha) + m$$

All of these formal properties of reinforcement OCF revision are inherited by reinforcement base revision. Specifically,

1. Assuming an arbitrary but fixed evidence degree for any new information, we obtain a standard iterated revision operator and the desirable rationality postulates are satisfied;
2. This holds also for the more general case of varying evidence degrees;
3. Reinforcement base revision does indeed have a reinforcement effect, which, from a pragmatic point of view, is a desirable effect in particular for domains with several independent information sources.

This is summarized in the following corollary, which is a direct consequence of the above-mentioned properties and the equivalence of reinforcement belief revision and reinforcement base revision.

Corollary 3.6. *Consider an EE-base Ξ .*

1. *Assume an arbitrary but fixed evidence degree for any new information. Then reinforcement base revision satisfies the AGM postulates ($\mathcal{K} * 1$)-($\mathcal{K} * 8$) as well as (DP1), (DP2), and (Ind).*

2. *For any $m_1, m_2 \in \mathbb{N}^+$, reinforcement base revision satisfies the following:*

- (EDP1') *If $\beta \vdash \alpha$, then $(\Xi *_{r, \langle \alpha, m_1 \rangle}) *_{r, \langle \beta, m_2 \rangle} \equiv \Xi *_{r, \langle \beta, m_2 \rangle}$*
- (EDP2') *If $\beta \vdash \neg\alpha$, then $(\Xi *_{r, \langle \alpha, m_1 \rangle}) *_{r, \langle \beta, m_2 \rangle} \equiv \Xi *_{r, \langle \beta, m_2 \rangle}$*

(EInd') If there exists m such that $\Xi *_r \langle \beta, m \rangle \not\vdash \neg\alpha$, then
 $(\Xi *_r \langle \alpha, m_1 \rangle) *_r \langle \beta, m_2 \rangle \vdash \alpha$

3. Let $\langle \alpha, m \rangle$ be some new, non-tautological information. Let $\Xi_1 = \Xi *_r \langle \alpha, m \rangle$, then

$$\text{Rank}_{\Xi_1}(\alpha) = \text{Rank}_{\Xi}(B, \alpha) + m$$

3.2.1 Degree of Syntax Irrelevance

Strictly speaking, reinforcement base revision violates Dalal's principle of *irrelevance of syntax* [Dalal, 1988], because the revised belief state is not purely determined by the logical contents of the original belief state. The principle of irrelevance of syntax is motivated by Newell's influential proposal which states that the behavior of an intelligent system should be specifiable on the *knowledge level* [Newell, 1982]; that is to say, the behavior of the system is determined solely by the contents of its knowledge (in our context, beliefs), independent of the symbolic representation. In the following, we show that reinforcement base revision does indeed not depend on the syntax of an EE base, so that it actually complies with Newell's idea.

To this end, we call two EE bases Ξ_1 and Ξ_2 *epistemically equivalent* iff their induced epistemic entrenchments (as defined by (6)) are equivalent; that is, for any sentences α, β :

$$\alpha \leq_{\text{Bel}(\Xi_1)} \beta \text{ iff } \alpha \leq_{\text{Bel}(\Xi_2)} \beta$$

The following result shows that two epistemically equivalent EE bases yield the same beliefs when revised by the same sentence.

Proposition 3.7. *Let Ξ_1, Ξ_2 be two epistemically equivalent EE bases, then for any sentence α and evidence degrees $m_1, m_2 \in \mathbb{N}^+$:*

$$\text{Bel}(\Xi'_1) = \text{Bel}(\Xi'_2)$$

where $\Xi'_1 = \Xi_1 *_r \langle \alpha, m_1 \rangle$ and $\Xi'_2 = \Xi_2 *_r \langle \alpha, m_2 \rangle$.

Moreover, we call two EE bases Ξ_1 and Ξ_2 *equivalent*, denoted by $\Xi_1 \cong \Xi_2$, iff for any sentence β :

$$\text{Rank}_{\Xi_1}(\beta) = \text{Rank}_{\Xi_2}(\beta)$$

Obviously, two equivalent EE bases are also epistemically equivalent, which in turn means that they are logically equivalent.

It is not difficult to see that two equivalent EE bases lead to equivalent EE bases when revised by the same information.

Proposition 3.8. *Let Ξ_1, Ξ_2 be two equivalent EE bases and $\langle \alpha, m \rangle$ any new information, then*

$$\Xi_1 *_r \langle \alpha, m \rangle \cong \Xi_2 *_r \langle \alpha, m \rangle$$

3.3 Computational Complexity

In practice, computational complexity is a very important criterion for evaluating revision operators [Eiter and Gottlob, 1992; Liberatore, 1997b; Cadoli *et al.*, 1995].⁹ Like in [Eiter and Gottlob, 1992], we will consider the so-called *problem of counterfactual*

⁹We assume the reader to have basic knowledge in complexity theory, as can be found, e.g., in [Papadimitriou, 1994].

(CF, for short), which means to decide whether $\mathcal{K} * \alpha \vdash \beta$ holds for arbitrary belief states \mathcal{K} and sentences α, β .

[Nebel, 1992] has shown that both SAT and VALID can be polynomially (many-to-one) reduced to the problem of CF for a revision operator that satisfies $(\mathcal{K} * 4)$ and $(\mathcal{K} * 5)$. This means that the problem of CF is at least both NP-hard and coNP-hard. It follows immediately that in general the complexity of the problem of CF is higher than NP and coNP, provided that $\text{NP} \neq \text{coNP}$.

For most well-known belief change operators in the literature, the problem of CF has been shown to be at the lower end of so-called *polynomial hierarchy* [Eiter and Gottlob, 1992; Nebel, 1994; Liberatore, 1997a]. In the following, we give a brief introduction to this concept. Let X be a class of decision problems. Let P^X denote the class of decision problems that can be decided in polynomial time by a deterministic Turing machine that is allowed to use a procedure (also referred to as an *oracle*) for deciding a problem $Q \in X$, whereby executing the procedure only costs constant time. Similarly, NP^X denotes the class of decision problems that can be decided in polynomial time by a non-deterministic Turing machine that is allowed to use an oracle for deciding a problem $Q \in X$. Based on these notions, the complexity classes Δ_k^p , Σ_k^p , and Π_k^p are inductively defined for $k = 0, 1, 2, \dots$:

$$\begin{aligned}\Delta_0^p &= \Sigma_0^p = \Pi_0^p = \text{P} \\ \Delta_{k+1}^p &= \text{P}^{\Sigma_k^p} \\ \Sigma_{k+1}^p &= \text{NP}^{\Sigma_k^p} \\ \Pi_k^p &= \text{co}\Sigma_{k+1}^p\end{aligned}$$

Note that $\Sigma_1^p = \text{NP}$ and $\Pi_1^p = \text{coNP}$. The polynomial hierarchy is then defined as $\text{PH} = \bigcup_{k \geq 0} \Delta_k^p = \bigcup_{k \geq 0} \Sigma_k^p = \bigcup_{k \geq 0} \Pi_k^p \subseteq \text{PSPACE}$. It is unknown, but commonly believed, that the inclusion between PH and PSPACE is a proper one.

For problems in Δ_2^p , it is often difficult to determine their exact complexity. But by restricting the number of oracle calls, we obtain an important special class $\Delta_2^p[\text{O}(\log n)]$ of problems that can be decided in polynomial time by using only *logarithmically* many oracle calls. Furthermore, inside of $\Delta_2^p[\text{O}(\log n)]$ are the classes of the so-called *boolean hierarchy*, that is, problems solvable in deterministic polynomial time using a *constant* number of NP-oracle calls.

It has been shown in [Nebel, 1994] that for cut base revision the problem of CF is $\Delta_2^p[\text{O}(\log n)]$ -complete. Since reinforcement base revision generalizes cut base revision, it does not come as a surprise that one can show that the former has the same complexity:

Theorem 3.9. *For reinforcement base revision, the problem of CF is $\Delta_2^p[\text{O}(\log n)]$ -complete.*

This theorem shows that there is not too much room for improvement in terms of computational complexity, since $\Delta_2^p[\text{O}(\log n)]$ inhabits the very low end of the polynomial hierarchy. In general, unless the size of a belief state is restricted, it is hard to imagine that the problem of CF for any realistic operator will only require a constant number of NP-oracle calls. Of course, if the underlying language is constrained to *Horn sentences*, then the problem of CF for reinforcement base revision becomes tractable (i.e., solvable in polynomial time).

It is also an interesting problem to ask how hard it is to compute the revised belief state given an arbitrary belief state and new information. This is the so-called *function problem*, which is not a decision problem like the problem of CF. The complexity

classes for decision problems have natural counterparts for function problems: For example, FP^{NP} (also referred to as NP-easy) represents the set of all function problems which can be solved in polynomial time by a deterministic Turing machine that is allowed to invoke an NP-oracle. It turns out that the problem of computing a revised belief state (EE base) for reinforcement base revision falls in an interesting complexity class, called NP-equivalent. Formally, a function problem is NP-equivalent iff it is both NP-easy and NP-hard. Note that NP-equivalent is the analogue of NP-complete for function problems, in the sense that if one NP-equivalent problem can be solved in polynomial time, then so can be all other NP-equivalent problems.

Theorem 3.10. *For reinforcement base revision, the problem of computing a revised belief state is NP-equivalent.*

4 Possibilistic Reinforcement Revision

Possibility theory (also referred to as possibilistic logic) is another research field which is intimately related to belief revision [Dubois and Prade, 1991; Dubois *et al.*, 1994]. The main difference is that in the former the certainty of information is represented numerically, whereas in the latter we use ordinal measurements [Dubois *et al.*, 1998]. Therefore, many notions used in Section 3 can also be found in possibility theory modulo an innocuous affine transformation. In [Benferhat *et al.*, 2002], for instance, it has been shown that some well-known OCF revision operators can be directly mapped to conditioning operations in possibility theory. In this section, we will likewise show how our reinforcement belief revision can be re-cast in the possibilistic setting. The result is a novel approach to revising possibilistic knowledge states.

In possibility theory, the beliefs of an agent are represented by a possibility distribution π , which maps the set of all possible worlds $\Theta_{\mathcal{L}}$ to the unit interval $[0, 1]$. Contrary to an OCF, a possibility distribution π assigns higher possibility to more plausible worlds, so that $\pi(W) = 0$ indicates an impossible world W while $\pi(W) = 1$ means that nothing prevents W from being the real world.

The belief set $\text{Bel}(\pi)$ encoded by a possibility distribution π consists of all sentences which are true in all most plausible worlds:

$$\text{Bel}(\pi) = \text{Th}(\{W \mid \pi(W) = 1\}) \quad (13)$$

Given a possibility distribution π , we can define two different measures for a sentence β : The *possibility degree* $\Pi_{\pi}(\beta)$ evaluates the extent to which β is consistent with π :

$$\Pi_{\pi}(\beta) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \vdash \neg\beta \\ \max(\{\pi(W) \mid W \models \beta\}) & \text{otherwise} \end{cases} \quad (14)$$

The *necessity degree* $N_{\pi}(\beta)$ evaluates the extent to which β is entailed by π :¹⁰

$$N_{\pi}(\beta) \stackrel{\text{def}}{=} \begin{cases} \infty & \text{if } \vdash \beta \\ 1 - \Pi_{\pi}(\neg\beta) & \text{otherwise} \end{cases} \quad (15)$$

Henceforth, when π is obvious from the context, we denote $N_{\pi}(\beta)$ and $\Pi_{\pi}(\beta)$ simply by $N(\beta)$ and $\Pi(\beta)$, respectively.

¹⁰Originally, the necessity degree is simply defined as $N_{\pi}(\beta) \stackrel{\text{def}}{=} 1 - \Pi_{\pi}(\neg\beta)$. In this paper, we will use this slightly different definition in order to be fully compatible with the notion of belief degree introduced in Section 3.

4.1 Possibilistic Conditionings

In possibility theory, the revision of a possibility distribution π by a totally reliable evidence α is obtained by Bayesian-style conditioning [Benferhat *et al.*, 2002], that is, for any sentence α and any possible world W :

$$\pi_{\alpha}^{*}(W) = \pi(W|\alpha)$$

where $\pi(\cdot|\alpha)$ is the posterior possibility distribution conditioned on α .

Two main methods for conditioning have been proposed in the literature. The so-called *minimum-based conditioning* is more of a qualitative nature:

$$\pi(W|_m\alpha) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \pi(W) = \Pi(\alpha) \text{ and } W \models \alpha \\ \pi(W) & \text{else if } \pi(W) < \Pi(\alpha) \text{ and } W \models \alpha \\ 0 & \text{otherwise} \end{cases}$$

Much like genuine Bayesian conditioning, the so-called *product-based conditioning* re-scales upwards all models of the new evidence α :

$$\pi(W|_{\times}\alpha) \stackrel{\text{def}}{=} \begin{cases} \frac{\pi(W)}{\Pi(\alpha)} & \text{if } W \models \alpha \text{ and } \Pi(\alpha) \neq 0 \\ 1 & \text{else if } W \models \alpha \text{ and } \Pi(\alpha) = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is not difficult to see that both $\pi(\cdot|_m\alpha)$ and $\pi(\cdot|_{\times}\alpha)$ upgrade the necessity degree of α to 1, provided that $\not\models \alpha$. Moreover, they are justified to be called conditioning since both of them satisfy the Bayesian condition $\pi(W) = \pi(W|\alpha) \otimes \Pi(\alpha)$ with $\otimes = \min$ (the minimum operator) for minimum-based conditioning and $\otimes = \times$ (the arithmetic product) for product-based conditioning.

When the new evidence α is not fully certain, it may come with its own degree $w \in [0, 1]$ of possibility. Based on the well-known Jeffrey Rule [Jeffrey, 1965], a possibility distribution π should be conditioned (revised) by an uncertain evidence $\langle \alpha, w \rangle$ as follows:

$$\pi_{\langle \alpha, w \rangle}^{*}(W) \stackrel{\text{def}}{=} \pi(W|\langle \alpha, w \rangle)$$

where

$$\pi(W|\langle \alpha, w \rangle) = \begin{cases} \pi(W|\alpha) & \text{if } W \models \alpha \\ (1-w) \otimes \pi(W|\neg\alpha) & \text{otherwise} \end{cases}$$

Again, $\otimes = \min$ for minimum-based conditioning and $\otimes = \times$ for product-based conditioning. Note that $\pi(\cdot|\langle \alpha, w \rangle)$ coincides with $\pi(\cdot|\alpha)$ in case $w = 1$. Moreover, both minimum-based and product-based conditioning (with uncertain input $\langle \alpha, w \rangle$) have the effect that the posterior necessity degree of α is exactly w .

In [Benferhat *et al.*, 2002] it has been pointed out that there is an intimate relation between OCF-based revision operators and conditioning in possibility theory. In particular, a scale transformation from an OCF k to a possibility distribution π_k can be obtained by letting:

$$\pi_k(W) = e^{-k(W)} \tag{16}$$

Then minimum-based and product-based conditioning with uncertain input can be shown to correspond to two well-known OCF revision operators: adjustment and conditionalization, respectively.¹¹

¹¹A comparison between reinforcement OCF revision and those two operators can be found in Section 5.

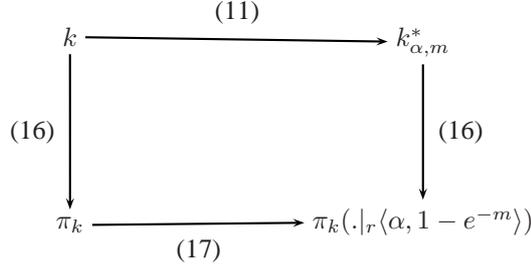


Figure 4: Correspondence between reinforcement OCF revision and conditioning

4.2 Possibilistic Reinforcement Revision

We are now ready to present possibilistic versions for both reinforcement OCF and reinforcement base revision. The equivalence between possibilistic reinforcement revision and genuine reinforcement revision will be shown.

4.2.1 Reinforcement Conditioning

By slightly modifying product-based conditioning, we obtain the following *reinforcement conditioning*:

$$\pi(W |_r \langle \alpha, w \rangle) = \begin{cases} \pi(W |_{\times} \alpha) & \text{if } W \models \alpha \\ (1 - w) \times \pi(W) & \text{otherwise} \end{cases} \quad (17)$$

Note that reinforcement conditioning uses $\pi(W)$ to compute the posterior possibility of $\neg\alpha$ -worlds, instead of $\pi(W | \neg\alpha)$ (as used in product-based conditioning). The main effect of this modification is that the posterior necessity degree of α may not be w anymore. In fact, it can be shown that reinforcement conditioning always strengthens the necessity degree of the new evidence.

Proposition 4.1. *Suppose that a possibility distribution π is revised by a new non-tautological evidence α with possibility degree w . Let $\pi_1 = \pi_{\langle \alpha, w \rangle}^*$, then*

$$N_{\pi_1}(\alpha) = N_{\pi}(\alpha) + w - w \times N_{\pi}(\alpha)$$

Since $w \geq w \times N_{\pi}(\alpha)$, this implies that $N_{\pi_1}(\alpha) \geq N_{\pi}(\alpha)$. Moreover, it is easy to see that $N_{\pi_1}(\alpha) = 1$ iff $w = 1$ or $N_{\pi}(\alpha) = 1$.

We now show that there is actually a close correspondence between reinforcement conditioning and reinforcement OCF revision, given the mapping defined by (16) (cf. also the commuting diagram in Figure 4).

Theorem 4.2. *Suppose that k is an OCF and $\langle \alpha, m \rangle$ (with $m \in \mathbb{N}^+$) some input information. Let $k_{\langle \alpha, m \rangle}^*$ be the revised OCF using reinforcement OCF revision, then for any possible world W :*

$$\pi_{k_{\langle \alpha, m \rangle}^*}(W) = \pi_k(W |_r \langle \alpha, 1 - e^{-m} \rangle)$$

The reader may note that an evidence degree m in OCF revision is monotonically mapped to a possibility degree $1 - e^{-m}$ in possibilistic conditioning.

4.2.2 Reinforcement Possibility Base Revision

Possibilistic bases provide a method for the compact representation of beliefs in possibility theory. Formally, a possibilistic base Σ consists in a belief base B and a mapping g from B to $[0, 1]$, where $g(\beta)$ is called the possibility degree of a sentence $\beta \in B$. Intuitively, sentences with higher certainty have a higher possibility degree.

We remark that a possibilistic base is the same as an EE base modulo a monotonic scale transformation. Therefore, many notions on EE bases can also be applied to possibilistic bases. Specifically, given a possibilistic base $\Sigma = \langle B, g \rangle$, let Σ^w denotes the set of sentences whose possibility degree is at least w :

$$\Sigma^w \stackrel{\text{def}}{=} \{\beta_i \in B \mid g(\beta_i) \geq w\}$$

Moreover, $\Sigma^{>w}$ is the set of sentences with possibility degree greater than w :

$$\Sigma^{>w} \stackrel{\text{def}}{=} \{\beta_i \in B \mid g(\beta_i) > w\}$$

In analogy to the notion of a belief degree, we define the necessity degree of a sentence β as follows:

$$N_{\Sigma}(\beta) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } B \not\vdash \beta \\ \infty & \text{else if } \vdash \beta \\ \max(\{w \mid \Sigma^w \vdash \beta\}) & \text{otherwise} \end{cases} \quad (18)$$

Now we are ready to define a revision operator for possibilistic bases. The basic idea is quite similar to that of reinforcement base revision. Suppose that $\Sigma = \{\langle \beta_1, w_1 \rangle, \dots, \langle \beta_n, w_n \rangle\}$ is a possibilistic base and $\langle \alpha, w \rangle$ the uncertain new information. Let $\bar{w} = N_{\Sigma}(-\alpha)$, then the revised possibilistic base $\Sigma_1 = \Sigma \boxtimes_r \langle \alpha, w \rangle$ is obtained as follows:

$$\begin{aligned} \Sigma_1 = & \{ \langle \beta_i, \frac{w_i - \bar{w}}{1 - \bar{w}} \rangle \mid w_i > \bar{w} \} \\ & \cup \{ \langle \alpha, w \rangle \} \\ & \cup \{ \langle \alpha \vee \beta_i, w_i + w - w \times w_i \rangle \} \end{aligned} \quad (19)$$

In the sequel, we will show that possibilistic reinforcement base revision is equivalent to reinforcement conditioning. Like in Section 3.1.1, we first define a mapping from a possibilistic base $\Sigma = \{\langle \beta_1, w_1 \rangle, \dots, \langle \beta_n, w_n \rangle\}$ to a possibility distribution π_{Σ} :

$$\pi_{\Sigma}(W) = \begin{cases} 1 & \text{if } W \models \{\beta_i, \dots, \beta_n\} \\ 1 - \max(\{w_i \mid W \not\models \beta_i\}) & \text{otherwise} \end{cases} \quad (20)$$

The following result shows that a possibilistic base Σ and its induced possibility distribution π_{Σ} encode essentially the same belief state.

Proposition 4.3. [Dubois *et al.*, 1994] *Suppose that Σ is a possibilistic base and π_{Σ} the induced possibility distribution as defined by (20), then for any sentence β :*

$$N_{\Sigma}(\beta) = N_{\pi_{\Sigma}}(\beta)$$

Finally, we can show the equivalence of reinforcement possibilistic base revision and reinforcement conditioning (cf. Figure 5).

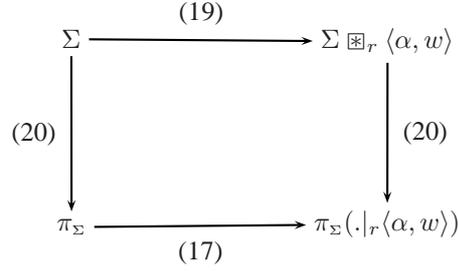


Figure 5: Reinforcement conditioning and possibilistic base revision

Theorem 4.4. *Suppose that Σ is a possibilistic base and π_Σ the possibility distribution induced from Σ . Let $\langle \alpha, w \rangle$ be any new information, then for any possible world W :*

$$\pi_\Sigma(W | \langle \alpha, w \rangle) = \pi_{\Sigma_1}(W)$$

where $\Sigma_1 = \Sigma \boxtimes_r \langle \alpha, w \rangle$.

As a direct consequence of the above equivalence result and Theorem 4.2, reinforcement possibilistic base revision also shares most of the nice properties stated in Section 3.2.

5 Related Work and Conclusion

We have presented a computational model of so-called reinforcement belief revision. This operator has been shown semantically equivalent to an OCF-based model of reinforcement revision. This implies that reinforcement base revision satisfies all desirable rationality postulates but violates those that are too radical. Moreover, we have formally analyzed the time and space complexity of the computational model of reinforcement revision. In this final section, we will give a detailed comparison between our proposal and existing revision operators.

5.1 Syntax Irrelevant Operators

In the literature, there are revision operators which do not exploit any explicit extra-logical preference information. From a representational point of view, this seems to be an advantage; but these revision operators are also criticized as *inflexible* [Nebel, 1998], because they allow little control over which sentences are discarded and which are retained. In the following, we present one well-known syntax irrelevant revision operator proposed by [Dalal, 1988]. We will argue that Dalal’s operator, while sacrificing flexibility, does not have the benefit of lower time and/or space consumption. As shown in [Eiter and Gottlob, 1992; Cadoli *et al.*, 1995], the same holds for most other syntax irrelevant operators.

Dalal’s operator is based on a notion of distance (between possible worlds): When a belief base B is revised by α , the revised belief base should be determined by the models of α that are “closest” to those of B [Dalal, 1988].

Formally, the distance between two possible worlds W_1, W_2 , denoted by $\|\Delta\|(W_1, W_2)$, is the cardinality of their symmetric difference:

$$\|\Delta\|(W_1, W_2) = \|(W_1 \setminus W_2) \cup (W_2 \setminus W_1)\|$$

The distance between a possible world W and a (consistent) belief base B , denoted by $\|\Delta\|^{\min}(B, W)$, is the minimal distance between W and models of B :

$$\|\Delta\|^{\min}(B, W) = \min(\{\|\Delta\|(W_1, W) \mid W_1 \in \text{Mods}(B)\})$$

Given a belief base B , we can induce a faithful ranking $\preceq_{Cn(B)}$ on $Cn(B)$ by stipulating:

$$W_1 \preceq_B W_2 \text{ iff } \|\Delta\|^{\min}(B, W_1) \leq \|\Delta\|^{\min}(B, W_2)$$

Dalal's operator is then defined as follows:

$$Cn(B *_d \alpha) \stackrel{\text{def}}{=} \begin{cases} Th(\min(\text{Mods}(\alpha), \preceq_B)) & \text{if } B \text{ is consistent} \\ Cn(\{\alpha\}) & \text{otherwise} \end{cases}$$

It has been shown by [Katsuno and Satoh, 1991] that Dalal's operator satisfies all AGM postulates.

Note that Dalal's operator does not tell us explicitly how to construct a revised belief base. A negative result in [Cadoli *et al.*, 1995] says that the size of the revised belief base $B *_d \alpha$ can be much larger than that of B and α . More precisely, they have shown that if the size of the revised belief base $B *_d \alpha$ is polynomially bounded by the size of B and α , then $\Sigma_2^p = \Pi_2^p = \text{PH}$. As the equality of these classes is highly unlikely, this essentially shows that Dalal's operator causes super-polynomial space explosion. Moreover, as shown in [Eiter and Gottlob, 1992], the problem of CF for Dalal's operator is $\Delta_2^p[\text{O}(\log n)]$ -complete; however, unlike reinforcement base revision, it remains $\Delta_2^p[\text{O}(\log n)]$ -complete even for Horn theories.

5.2 Theory Base Transmutation

Theory base transmutation is a class of iterated revision operators proposed by [Williams, 1994], among which *conditionalization* and *adjustment* are the most prominent ones. Like reinforcement base revision, both conditionalization and adjustment have intuitive OCF-based semantics. We remark that there are two major differences between reinforcement base revision and theory base transmutation: First of all, Postulate (Ind) is violated by both conditionalization and adjustment, and the latter also violates Postulate (DP2); secondly, theory base transmutation allows the input evidence degree to be 0, and in this case they behave like contraction operators since in theory base transmutation the input evidence degree will be the rank of the input sentence. To facilitate the comparison between reinforcement revision and theory base transmutation, we assume in the following that the input evidence degree is always greater than 0.

5.2.1 Conditionalization

OCF conditionalization was originally introduced in [Spohn, 1988]. It can be viewed as a qualitative version of Jeffrey's Rule of probabilistic conditioning [Goldszmidt, 1992]:

$$(k_{\alpha.m}^{c,*})(W) \stackrel{\text{def}}{=} \begin{cases} k(W) - k(\neg\alpha) & \text{if } W \models \alpha \\ k(W) - k(\alpha) + m & \text{otherwise} \end{cases} \quad (21)$$

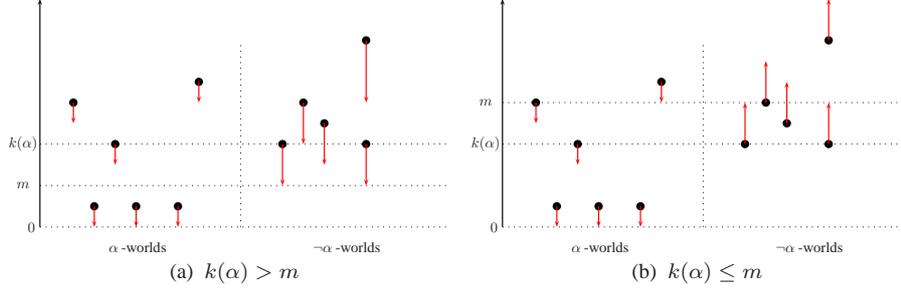


Figure 6: OCF conditionalization

It is easy to see that OCF conditionalization closely resembles reinforcement OCF revision. In particular, they coincide if $k(\alpha) = 0$. Moreover, the following result shows that OCF conditionalization can in fact be decomposed into a sequence of OCF reinforcement revision steps.

Theorem 5.1. *Let k be an OCF and $\langle \alpha, m \rangle$ some new information, then for any possible world W :*

$$k_{\alpha, m}^{c,*}(W) = \begin{cases} k_{\alpha, m-k(\alpha)}^{r,*}(W) & \text{if } k(\alpha) < m \\ ((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) & \text{otherwise} \end{cases}$$

where $m' \in \mathbb{N}^+$ is an arbitrary positive integer.

The above result shows that OCF reinforcement revision behaves much more subtly than conditionalization.

The intuition behind OCF conditionalization is depicted in Figure 6, where we distinguish the two cases $k(\alpha) > m$ and $k(\alpha) \leq m$. OCF conditionalization changes the ranks of α -worlds in exactly the same way as reinforcement OCF revision does. Depending on whether $k(\alpha) > m$ or $k(\alpha) \leq m$, all $\neg\alpha$ -worlds are uniformly moved downwards or upwards, respectively, so that the most plausible $\neg\alpha$ -worlds end up having the rank m , which is necessary to obtain $k_{\alpha, m}^{c,*}(\alpha) = m$. It is easy to prove that OCF conditionalization satisfies (DP1) and (DP2) but violates (Ind) in case $k(\alpha) > m$.

Conditionalization itself is a revision operator on EE-bases defined as follows [Williams, 1992; Benferhat *et al.*, 2002]:

$$\langle B, f \rangle *_c \langle \alpha, m \rangle \stackrel{\text{def}}{=} \begin{cases} \langle B, f \rangle *_r \langle \alpha, m \rangle & \text{if } B \not\vdash \alpha \\ \left\{ \begin{array}{l} \langle \beta, f(\beta) - \text{Rank}_{\Xi}(\alpha) + m \mid f(\beta) > \text{Rank}_{\Xi}(\alpha) \rangle \\ \cup \{ \langle \neg\alpha \vee \beta, f(\beta) \rangle \mid \beta \in B \} \\ \cup \{ \langle \alpha, m \rangle \} \end{array} \right\} & \text{otherwise} \end{cases}$$

Note that conditionalization coincides with reinforcement base revision in case $B \not\vdash \alpha$. It is not difficult to see that $\Xi *_r \alpha \equiv \Xi *_c \alpha$ for any EE base Ξ and new information $\langle \alpha, m \rangle$. Therefore, the complexity of conditionalization is same as that of reinforcement base revision.

Theorem 5.2. *For conditionalization, the problem of CF is $\Delta_2^P[\text{O}(\log n)]$ -complete.*

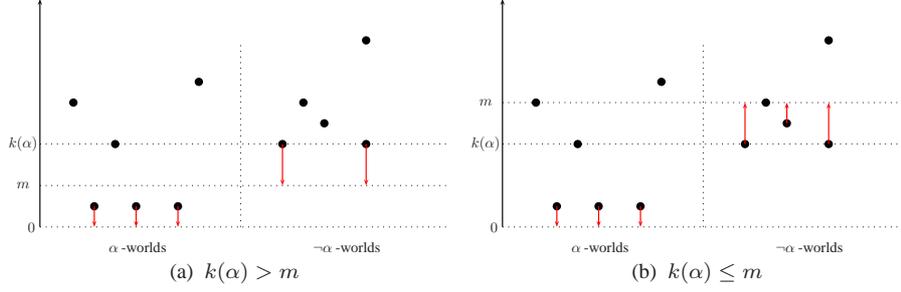


Figure 7: OCF adjustment

However, since conditionalization violates Postulates (Ind), it is easy to find counterexamples. Recall, for instance, Example 1. Suppose that now $\Xi = \{\langle r, 6 \rangle, \langle s, 5 \rangle, \langle r \vee s, 7 \rangle\}$ and we first learn that X is smart with evidence degree 1. This yields $\Xi_1 = \Xi *_c \langle s, 1 \rangle = \{\langle r \vee s, 3 \rangle, \langle r \vee \neg s, 6 \rangle, \langle s, 1 \rangle\}$. Next we learn that X is not rich with evidence degree 2, and obtain $\Xi_2 = \Xi_1 *_c \langle \neg r, 2 \rangle = \{\langle r \vee \neg s, 3 \rangle, \langle \neg r, 2 \rangle, \langle s \vee \neg r, 3 \rangle\}$. Hence, contrary to the intuition, we now believe that X is not smart, since $\text{Rank}_{\Xi_2}(\neg s) = 2$.

5.2.2 Adjustment

OCF adjustment is an operator based on an absolute measure of minimal change [Williams, 1994]:

$$(k_{\alpha, m}^{j, *})(W) \stackrel{\text{def}}{=} \begin{cases} (k_{\alpha}^-)_{\alpha, m}^{\times}(W) & \text{if } m < k(\alpha) \\ k_{\alpha, m}^{\times}(W) & \text{otherwise} \end{cases} \quad (22)$$

where

$$(k_{\alpha}^-)(W) = \begin{cases} 0 & \text{if } W \models \neg\alpha \text{ and } K(W) = k(\alpha) \\ k(W) & \text{otherwise} \end{cases}$$

$$(k_{\alpha, m}^{\times})(W) = \begin{cases} 0 & \text{if } W \models \alpha \text{ and } K(W) = k(\neg\alpha) \\ i & \text{else if } W \models \neg\alpha \text{ and } K(W) < i \\ k(W) & \text{else} \end{cases}$$

Although the definition seems quite complicated, the graphical representation in Figure 7 provides a nice intuition of OCF adjustment. Unlike OCF conditionalization and reinforcement OCF revision, OCF adjustment only changes the ranks of the most plausible α -worlds and $\neg\alpha$ -worlds. Moreover, when $k(a) \leq m$, the relative plausibility ordering of $\neg\alpha$ -worlds is not always preserved (cf. Figure 7(b)). It is easy to verify that OCF adjustment satisfies only Postulate (DP1), but violates (DP2) and (Ind).

Adjustment is an EE-base revision operator defined as follows [Williams, 1992;

Benferhat *et al.*, 2002]:

$$\langle B, f \rangle *_j \langle \alpha, m \rangle = \begin{cases} \left. \begin{array}{l} \{ \langle \beta, f(\beta) \rangle \mid f(\beta) > \text{Rank}_{\Xi}(-\alpha) \} \\ \cup \{ \langle \alpha \vee \beta, f(\beta) \rangle \mid m < f(\beta) \leq \text{Rank}_{\Xi}(-\alpha) \} \\ \cup \{ \langle \alpha, m \rangle \} \end{array} \right\} & \text{if } B \not\sim \alpha \\ \left. \begin{array}{l} \{ \langle \beta, f(\beta) \rangle \mid f(\beta) > \text{Rank}_{\Xi}(\alpha) \} \\ \cup \{ \langle -\alpha \vee \beta, f(\beta) \rangle \mid f(\beta) \leq \text{Rank}_{\Xi}(\alpha) \} \\ \cup \{ \langle \alpha, m \rangle \} \end{array} \right\} & \text{otherwise} \end{cases}$$

Just like conditionalization, adjustment has the same complexity as reinforcement base revision:

Theorem 5.3. *For adjustment, the problem of CF is $\Delta_2^P[\text{O}(\log n)]$ -complete.*

Similar to conditionalization, it is easy to find examples in which adjustment leads to counter-intuitive conclusions. Consider, for instance, Example 1 again. Suppose that $\Xi = (\emptyset *_r \langle r, 1 \rangle) *_r \langle s, 1 \rangle = \{ \langle r, 1 \rangle, \langle s, 1 \rangle, \langle r \vee s, 2 \rangle \}$ and we first learn that X is not smart with evidence degree 2, followed by learning the opposite with the same evidence degree. With adjustment we obtain the revised EE-bases $\Xi_1 = \Xi *_j \langle \neg s, 2 \rangle = \{ \langle r \vee s, 2 \rangle, \langle \neg s, 2 \rangle \}$ and $\Xi_2 = \Xi_1 *_j \langle s, 2 \rangle = \{ \langle s, 2 \rangle \}$. Hence, contrary to the intuition, we do not believe that X is rich anymore.

Acknowledgments

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A Proofs

Proposition 3.2. *Suppose that $\Xi = \langle B, f \rangle$ is an EE base and k_Ξ the induced OCF as defined by (12), then for any sentence β :*

$$\text{Rank}_\Xi(\beta) = k_\Xi(\beta)$$

Proof. Assume $\vdash \beta$. Then $\text{Rank}_\Xi(\beta) = k_\Xi(\beta) = \infty$.

Assume $\not\vdash \beta$ and $\text{Rank}_\Xi(\beta) = i$. It follows from (4) that $\Xi^{i+1} \not\vdash \beta$ and $\Xi^i \vdash \beta$. Let W be a possible world such that $W \models \Xi^{i+1} \cup \{\neg\beta\}$. From $\Xi^i \vdash \beta$, it follows that $W \not\models \Xi^i$. Therefore, there must be a sentence $\beta_i \in \Xi|_i$ such that $W \not\models \beta_i$. It follows from (12) that $k_\Xi(W) = i$. Let W_1 be any possible world such that $k_\Xi(W_1) < k_\Xi(W)$. It follows from (12) that $W_1 \models \Xi^i$; hence, $W_1 \models \beta$. According to (9), we have $k_\Xi(\beta) = i$. \square

Theorem 3.3. *Suppose that Ξ is an EE base and k_Ξ the OCF induced from Ξ . Let $\langle \alpha, m \rangle$ be any new information, then for all possible worlds W :*

$$k_{\Xi_1}(W) = k_{\Xi_{\alpha, m}^{r, *}}(W)$$

where $\Xi_1 = \Xi *_r \langle \alpha, m \rangle$.

Proof. Let $k_\Xi(W) = i$. By (12), we have $W \models \Xi^{i+1}$ and there exists a sentence $\beta_i \in \Xi|_i$ such that $W \models \neg\beta_i$.

If $W \models \neg\alpha$, then (11) implies that $k_{\Xi_{\alpha, m}^{r, *}}(W) = i + m$. It follows from (7) that $\Xi_1^{i+m+1} = \Xi^{i+m+1} \cup \{\beta \vee \alpha \mid \beta \in \Xi^{i+1}\}$ and $\alpha \vee \beta_i \in \Xi_1|_{i+m}$. Obviously, we have $W \models \Xi_1^{i+m+1}$ and $W \not\models \alpha \vee \beta_i$. Hence, according to (12), $k_{\Xi_1}(W) = i + m$.

If $W \models \alpha$, then (11) implies that $k_{\Xi_{\alpha, m}^{r, *}}(W) = i - k_\Xi(\neg\alpha)$. Let $\text{Rank}_\Xi(\neg\alpha) = r$. It follows from (7) that $\Xi_1^{i-r+1} \subseteq \Xi^{i+1} \cup \{\beta \vee \alpha \mid \beta \in B\} \cup \{\alpha\}$ and $\beta_i \in \Xi_1|_{i-r}$. It is obvious that $W \models \Xi_1^{i-r+1}$. It follows from (12) that $k_{\Xi_1}(W) = i - r = i - \text{Rank}_\Xi(\neg\alpha)$. According to Proposition 3.2, we obtain that $k_{\Xi_1}(W) = i - k_\Xi(\neg\alpha)$. \square

Some of the proofs below rely on the following lemma.

Lemma A.1. *Suppose that Ξ is an EE base and $\langle \alpha, m \rangle$ is any new information. Let $\Xi_1 = \Xi *_r \langle \alpha, m \rangle$, then for any non-tautologous sentence β :*

$$\text{Rank}_{\Xi_1}(\beta) = \begin{cases} t + m & \text{if } \vdash \alpha \rightarrow \beta \\ t' - \bar{r} & \text{else if } t' = t \\ \min(t' - \bar{r}, t + m) & \text{otherwise} \end{cases}$$

where $\bar{r} = \text{Rank}_\Xi(\neg\alpha)$, $t = \text{Rank}_\Xi(\beta)$, and $t' = \text{Rank}_\Xi(\alpha \rightarrow \beta)$.

Proof. This is a direct consequence of the following property, whose proof can be found in [Jin and Thielscher, 2007], and Theorem 3.3: Let k be an OCF and $\langle \alpha, m \rangle$ any new information, then for any non-tautological sentence β ,

$$k_{\alpha, m}^{r, *}(\beta) = \begin{cases} k(\beta) + m & \text{if } \vdash \alpha \rightarrow \beta \\ k(\alpha \rightarrow \beta) - k(\neg\alpha) & \text{else if } k(\alpha \rightarrow \beta) = k(\beta) \\ \min(k(\alpha \rightarrow \beta) - k(\neg\alpha), k(\beta) + m) & \text{otherwise} \end{cases}$$

□

Proposition 3.1. *Suppose that Ξ_1 is the result of revising an EE base Ξ by $\langle \neg\alpha, m \rangle$, where α is an arbitrary sentence and m an arbitrary natural number. Then for any sentence β :*

$$\text{Rank}_{\Xi_1}(\beta) \neq \text{Rank}_{\Xi}(\beta) \text{ iff } \text{Rank}_{\Xi_2}(\alpha) \neq \text{Rank}_{\Xi}(\alpha)$$

where $\Xi_2 = \Xi *_r \langle \neg\beta, m' \rangle$ for an arbitrary natural number m' .

Proof. Assume, without loss of generality, that $\text{Rank}_{\Xi_1}(\beta) = \text{Rank}_{\Xi}(\beta)$ and $\text{Rank}_{\Xi_2}(\alpha) \neq \text{Rank}_{\Xi}(\alpha)$. From Lemma A.1, it follows that $\not\vdash \neg\alpha \rightarrow \beta$ and $\text{Rank}_{\Xi}(\beta) = \text{Rank}_{\Xi_1}(\beta) = \text{Rank}_{\Xi}(\neg\alpha \rightarrow \beta) - \text{Rank}_{\Xi}(\alpha)$. Therefore, $\text{Rank}_{\Xi}(\alpha) = \text{Rank}_{\Xi}(\neg\alpha \rightarrow \beta) - \text{Rank}_{\Xi}(\beta)$. Note that $\text{Rank}_{\Xi}(\neg\alpha \rightarrow \beta) = \text{Rank}_{\Xi}(\neg\beta \rightarrow \alpha)$. Therefore, Lemma A.1 implies that $\text{Rank}_{\Xi_2}(\alpha) = \text{Rank}_{\Xi}(\neg\alpha \rightarrow \beta) - \text{Rank}_{\Xi}(\beta)$, which contradicts $\text{Rank}_{\Xi_2}(\alpha) \neq \text{Rank}_{\Xi}(\alpha)$.

□

Proposition 3.7. *Let Ξ_1, Ξ_2 be two epistemically equivalent EE bases, then for any sentence α and evidence degrees $m_1, m_2 \in \mathbb{N}^+$:*

$$\text{Bel}(\Xi'_1) = \text{Bel}(\Xi'_2)$$

where $\Xi'_1 = \Xi_1 *_r \langle \alpha, m_1 \rangle$ and $\Xi'_2 = \Xi_2 *_r \langle \alpha, m_2 \rangle$.

Proof. Let $\beta \in \text{Bel}(\Xi'_1)$. It follows that $\text{Rank}_{\Xi'_1}(\beta) > 0$. According to Lemma A.1, it must then be the case that either $\vdash \alpha \rightarrow \beta$ or $\text{Rank}_{\Xi_1}(\alpha \rightarrow \beta) > \text{Rank}_{\Xi_1}(\neg\alpha)$.

If $\vdash \alpha \rightarrow \beta$, then according to Lemma A.1 we have that $\text{Rank}_{\Xi'_2}(\beta) = \text{Rank}_{\Xi_2}(\beta) + m_2 > 0$.

If $\text{Rank}_{\Xi_1}(\alpha \rightarrow \beta) > \text{Rank}_{\Xi_1}(\neg\alpha)$, then also $\text{Rank}_{\Xi_2}(\alpha \rightarrow \beta) > \text{Rank}_{\Xi_2}(\neg\alpha)$ since Ξ_2 is epistemically equivalent to Ξ_1 . Again, it follows from Lemma A.1 that $\text{Rank}_{\Xi'_2}(\beta) > \text{Rank}_{\Xi_1}(\alpha \rightarrow \beta) - \text{Rank}_{\Xi_1}(\neg\alpha) > 0$.

The above discussion shows that $\text{Bel}(\Xi'_1) \subseteq \text{Bel}(\Xi'_2)$. The converse, $\text{Bel}(\Xi'_2) \subseteq \text{Bel}(\Xi'_1)$, can be shown symmetrically. □

Proposition 3.8. *Let Ξ_1, Ξ_2 be two equivalent EE bases and $\langle \alpha, m \rangle$ some new information, then*

$$\Xi_1 *_r \langle \alpha, m \rangle \cong \Xi_2 *_r \langle \alpha, m \rangle$$

Proof. A direct consequence of Lemma A.1. □

Theorem 3.9. *For reinforcement base revision, the problem of CF is $\Delta_2^P[\text{O}(\log n)]$ -complete.*

Proof. We first show that the problem is in $\Delta_2^P[\text{O}(\log n)]$. It is easy to see that in order to compute a revised EE base $\Xi_1 = \Xi *_r \langle \alpha, m \rangle$, we mainly need to calculate $\text{Rank}_{\Xi}(\neg\alpha)$ (cf. Algorithm 2 in Appendix B). Algorithm 1 in Appendix B shows that $\text{Rank}_{\Xi}(\neg\alpha)$ can be computed with at most logarithmic many NP-oracle calls. Once the revised EE base Ξ_1 has been computed, we just need one additional NP-oracle to decide whether Ξ_1 entails β . Therefore, the problem is in $\Delta_2^P[\text{O}(\log n)]$.

To show that the problem is $\Delta_2^P[\text{O}(\log n)]$ -hard, we give a polynomial (many-to-one) reduction from the problem of CF for cut base revision (which is $\Delta_2^P[\text{O}(\log n)]$ -hard [Nebel, 1994]). Given any prioritized base $\langle B, \leq_B \rangle$, we can construct a EE base $\Xi = \langle B, f \rangle$ by assigning evidence degree 1 to all sentences in the lowest class and evidence degree 2 to all sentences in the next higher class, and so on. It is easy to see that for any sentences α and β : $\langle B, \leq_B \rangle *_\text{cut} \alpha \vdash \beta$ iff Ξ_1 entails β where $\Xi_1 = \Xi *_r \langle \alpha, 1 \rangle$. Thus, it follows that the problem of CF for reinforcement base revision is $\Delta_2^P[\text{O}(\log n)]$ -hard. \square

Theorem 3.10. *For reinforcement base revision, the problem of computing a revised belief state is NP-equivalent.*

Proof. The proof of Theorem 3.9 shows that the problem is NP-easy. To show that it is also NP-hard, it suffices to observe that the problem of satisfiability of an arbitrary sentence β can be reduced to the problem of computing the revision $\{\langle \beta, 1 \rangle\} *_r \langle \top, 1 \rangle$: According to (7), the revised belief base includes $\langle \beta, 1 \rangle$ iff $\text{cut}_{\{\langle \beta, 1 \rangle\}}(\perp) = \{\beta\}$ and $\text{Rank}_{\{\langle \beta, 1 \rangle\}}(\perp) = 0$. The latter is equivalent to β being satisfiable according to (5) and (4). \square

Proposition 4.1. *Consider a possibility distribution π revised by a new non-tautological evidence α with possibility degree w . Let $\pi_1 = \pi_{\langle \alpha, w \rangle}^*$, then*

$$N_{\pi_1}(\alpha) = N_{\pi}(\alpha) + w - w \times N_{\pi}(\alpha)$$

Proof. Assume $N_{\pi}(\alpha) = 1 - \Pi_{\pi}(\neg\alpha) = w'$. According to (14), we have $\max(\{\pi(W) \mid W \models \neg\alpha\}) = 1 - w'$. From (17), it follows that $\max(\{\pi_1(W) \mid W \models \neg\alpha\}) = (1 - w') \times (1 - w)$. Then (14) implies $\Pi_{\pi_1}(\neg\alpha) = (1 - w') \times (1 - w)$. Thus, according to (15), we have $N_{\pi_1}(\alpha) = 1 - (1 - w') \times (1 - w) = w + w' - w \times w'$. \square

The following observation is needed for the proof of Theorem 4.2.

Lemma A.2. *Let k be an OCF, and $P(k)$ the possibility distribution as defined by (16), then for any sentence α :*

$$\Pi_{\pi_k}(\alpha) = e^{-k(\neg\alpha)}$$

Proof. Recall that $\Pi_{\pi}(\alpha)$ is defined as $\max(\{\pi(W) \mid W \models \alpha\})$. Let W_1 be a possible world such that $\pi_k(W_1) = \Pi_{\pi_k}(\alpha) = \max(\{\pi_k(W) \mid W \models \alpha\})$. Since $f(x) = e^{-x}$ is monotonically decreasing, we have $k(W_1) = \min(\{k(W) \mid W \models \alpha\})$. It follows from (9) that $k(W_1) = k(\neg\alpha)$. Thus $\Pi_{\pi_k}(\alpha) = e^{-k(\neg\alpha)}$. \square

Theorem 4.2. Suppose that k is an OCF and $\langle \alpha, m \rangle$ (with $m \in \mathbb{N}^+$) some input information. Let $k_{\langle \alpha, m \rangle}^*$ be the revised OCF using reinforcement OCF revision, then for any possible world W :

$$\pi_{k_{\langle \alpha, m \rangle}^*}(W) = \pi_k(W|_r \langle \alpha, 1 - e^{-m} \rangle)$$

Proof. Assume $W \models \alpha$. According to (16), $\pi_{k_{\langle \alpha, m \rangle}^*}(W) = e^{-k_{\alpha, m}^*(W)}$. It follows from (11) that $e^{-k_{\alpha, m}^*(W)} = e^{-(k(w) - k(\neg\alpha))}$. On the other hand, (17) implies $\pi_k(W|_r \langle \alpha, 1 - e^{-m} \rangle) = \frac{\pi_k(W)}{\Pi_{\pi_k}(\alpha)}$. It follows from (16) and Lemma A.2 that $\pi_k(W|_r \langle \alpha, 1 - e^{-m} \rangle) = \frac{e^{-k(w)}}{e^{-k(\neg\alpha)}}$. Hence $\pi_{k_{\langle \alpha, m \rangle}^*}(W) = \pi_k(W|_r \langle \alpha, 1 - e^{-m} \rangle)$.

Assume $W \not\models \alpha$. Similar to the above, according to (16) and (11), $\pi_{k_{\langle \alpha, m \rangle}^*}(W) = e^{-(k(W) + m)}$. It follows from (17) that $\pi_k(W|_r \langle \alpha, 1 - e^{-m} \rangle) = \pi_k(W) \times e^{-m}$. (16) implies $\pi_k(W|_r \langle \alpha, 1 - e^{-m} \rangle) = e^{-k(W)} \times e^{-m}$. Hence $\pi_{k_{\langle \alpha, m \rangle}^*}(W) = \pi_k(W|_r \langle \alpha, 1 - e^{-m} \rangle)$. \square

Theorem 4.4. Suppose that Σ is a possibilistic base and π_Σ the possibility distribution induced from Σ . Let $\langle \alpha, w \rangle$ be any new information, then for any possible world W :

$$\pi_\Sigma(W|_r \langle \alpha, w \rangle) = \pi_{\Sigma_1}(W)$$

where $\Sigma_1 = \Sigma \boxtimes_r \langle \alpha, w \rangle$.

Proof. Analogous to the proof of Theorem 3.3. \square

Theorem 5.1. Let k be an OCF and $\langle \alpha, m \rangle$ some new information, then for any possible world W :

$$k_{\alpha, m}^{c,*}(W) = \begin{cases} k_{\alpha, m - k(\alpha)}^{r,*}(W) & \text{if } k(\alpha) < m \\ ((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) & \text{otherwise} \end{cases}$$

where $m' \in \mathbb{N}^+$ is an arbitrary positive integer.

Proof.

Assume $k(\alpha) < m$, then it follows directly from (21) and (11) that $k_{\alpha, m}^{c,*}(W) = k_{\alpha, m - k(\alpha)}^{r,*}(W)$.

Assume $k(\alpha) \geq m$. Suppose $W \models \alpha$. According to (21), $k_{\alpha, m}^{c,*}(W) = k(W) - k(\neg\alpha)$, whereas (11) implies that $((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) = k_{-\alpha, m'}^{r,*}(W) - k_{-\alpha, m'}^{r,*}(\neg\alpha)$. Again, (11) implies that $k_{-\alpha, m'}^{r,*}(W) = k(W) + m'$. From item 3 of Theorem 3.5, it follows that $k_{-\alpha, m'}^{r,*}(\neg\alpha) = k(\neg\alpha) + m'$. Therefore, $((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) = k(W) - k(\neg\alpha)$. Thus, $((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) = k_{\alpha, m}^{c,*}(W)$.

Suppose $W \not\models \alpha$. It follows from (21) that $k_{\alpha, m}^{c,*}(W) = k(W) - k(\alpha) + m$, whereas (11) implies that $((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) = k_{-\alpha, m'}^{r,*}(W) + m$. According to (11), $k_{-\alpha, m'}^{r,*}(W) = k(W) - k(\alpha)$. Therefore, $((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) = k(W) - k(\alpha) + m$. Thus, $((k_{-\alpha, m'}^{r,*})_{\alpha, m}^{r,*})(W) = k_{\alpha, m}^{c,*}(W)$. \square

Theorem 5.2. *For conditionalization, the problem of CF is $\Delta_2^P[\text{O}(\log n)]$ -complete.*

Proof. It suffices to show that $\Xi *_{r} \alpha \equiv \Xi *_{c} \alpha$ for any EE base Ξ and new information $\langle \alpha, m \rangle$. If $B \not\vdash \alpha$, this holds trivially. Assume $B \vdash \alpha$. Then $\Xi *_{r} \alpha \equiv \Xi \equiv \Xi *_{c} \alpha$, since both conditionalization and reinforcement base revision satisfy the AGM postulates. \square

Theorem 5.3. *For adjustment, the problem of CF is $\Delta_2^P[\text{O}(\log n)]$ -complete.*

Proof. It suffices to show that $\Xi *_{r} \alpha \equiv \Xi *_{j} \alpha$ for any EE base Ξ and new information $\langle \alpha, m \rangle$. Assume $B \vdash \alpha$. Then $\Xi *_{r} \alpha \equiv \Xi \equiv \Xi *_{j} \alpha$, since both adjustment and reinforcement base revision satisfy the AGM postulates. Assume $B \not\vdash \alpha$. The only difference between $\Xi *_{r} \alpha$ and $\Xi *_{j} \alpha$ is that the former contains additional sentences of the form $\alpha \vee \beta_i$. Since both revised bases contain α , it follows that $\Xi *_{r} \alpha \equiv \Xi \equiv \Xi *_{j} \alpha$. \square

B Algorithms

Input : $\Xi = \{\langle \beta_1, e_1 \rangle, \dots, \langle \beta_n, e_n \rangle\}$ such that $e_i \leq e_{i+1}$, β
Output : $\text{Rank}_{\Xi}(\beta)$
begin
 if $\{\beta_1, \dots, \beta_n\} \not\vdash \beta$ **then return** 0 ;
 else if $\vdash \beta$ **then return** ∞ ;
 else
 $i = 1$; $j = n$;
 while $i \leq j$ **do**
 $k = i + \lceil \frac{j-i}{2} \rceil$;
 if $\Xi^{e_k} \vdash \beta$ **then**
 if $\Xi^{e_{k+1}} \not\vdash \beta$ **then**
 return e_k ;
 else $i = k + 1$;
 else $j = k$;
 end
 end
end

Algorithm 1: Computing $\text{Rank}_{\Xi}(\beta)$

Input : $\Xi = \{\langle \beta_1, e_1 \rangle, \dots, \langle \beta_n, e_n \rangle\}, \alpha, m$
Output : Ξ_1 such that $\Xi_1 = \Xi *_r \langle \alpha, m \rangle$
begin
 $\Xi_1 = \{ \}$;
 $\bar{r} = \text{Rank}_{\Xi}(\neg \alpha)$;
 for $i = 1 \dots n$ **do**
 if $e_i > \bar{r}$ **then**
 $\Xi_1 = \Xi_1 \cup \{ \langle \beta_i, e_i - \bar{r} \rangle \}$;
 end
 $\Xi_1 = \Xi_1 \cup \{ \langle \beta_i \vee \alpha, e_i + m \rangle \}$;
 end
 $\Xi_1 = \Xi_1 \cup \{ \langle \alpha, m \rangle \}$;
 return Ξ_1 ;
end

Algorithm 2: Reinforcement base revision

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