# Iterated Belief Revision, Revised ⋆

Yi Jin [1] and Michael Thielscher

*Department of Computer Science*
*Dresden University of Technology*
{yijin, mit}*@inf.tu-dresden.de*

**Abstract**

The AGM postulates for belief revision, augmented by the DP postulates for iterated belief revision, provide widely accepted criteria for the design of operators by which intelligent agents adapt their beliefs incrementally to new information. These postulates alone, however, are too permissive: They support operators by which all newly acquired information is canceled as soon as an agent learns a fact that contradicts some of its current beliefs. In this paper, we present a formal analysis of the deficiency of the standard postulates alone, and we show how to solve the problem by an additional postulate of *independence*. We give a representation theorem for this postulate and prove that it is compatible with AGM and DP.

*Key words:* Iterated Belief Revision, Implicit Dependence, Conditional Beliefs

## 1 Introduction

The capability of gathering information about the world and revising its beliefs based on the new information is crucial for an intelligent agent. Belief revision therefore is a central topic in Artificial Intelligence. Technically, belief revision is the process of changing the beliefs of an agent to accommodate new, more precise, or more reliable evidence that is possibly inconsistent with the existing beliefs.

The formal study of belief revision took as starting point the work of Alchourrón, Gärdenfors, and Makinson (AGM) during the first half of the 1980s

⋆ This article is a substantial extension of the conference paper [Jin and Thielscher, 2005b].

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[Alchourrón and Makinson, 1982; Alchourrón *et al.*, 1985; Alchourrón and Makinson, 1985]. The AGM framework studies idealized mathematical models of belief revision. Given an underlying logic language $\mathcal{L}$, the beliefs of an agent are represented by a set of sentences in $\mathcal{L}$ (known as *belief set*) which is closed under logical consequence. New evidence is also a sentence in $\mathcal{L}$, and a belief revision operator incorporates the new evidence into the current belief set to obtain a revised belief set. The authors of the original AGM framework have developed their theory under two basic assumptions regarding the new evidence: it is intended to describe facts of the static world; and it is more reliable (hence *prioritized* in the revision process) than the prior beliefs. The latter assumption is often referred to as *primacy of update*. The necessity and ideas of distinguishing belief revision from belief update (suitable for a situation where the new evidence describes a change of the world) was first noticed by Keller and Winslett [1985] and later on formalized in [Katsuno and Mendelzon, 1991a]. Belief revision where the new evidence is not prioritized, is a relatively recent topic studied by many researchers [Fermé and Hansson, 1999; Booth, 2001; Hansson, 1999; Delgrande *et al.*, 2006]. In this paper, we will concentrate on the problem of prioritized belief revision where iterations are necessary.

In situations where the new evidence is consistent with the existing beliefs, the two can just be merged; we call this *mild revision*. More interesting and complicated are situations where the evidence conflicts with the prior beliefs, in which case the agent needs to remove some of its currently held beliefs in order to accommodate the new evidence. This kind of revision is referred to as *severe revision* [Freund and Lehmann, 1994]. To provide general design criteria for belief revision operators, a set of postulates has been developed [Alchourrón *et al.*, 1985]. As first argued by the AGM trio and later frequently repeated by others [Freund and Lehmann, 1994; Darwiche and Pearl, 1997], the guiding principle of the AGM postulates is that of *economy of information*, or *minimal change* of belief sets, which means not to give up currently held beliefs and not to generate new beliefs unless necessary. However, Rott [1999; 2000] has recently pointed out that "it is a pure myth that minimal change principles are the foundation of existing theories of belief revision, at least as far as the AGM tradition is concerned". His argument is mainly based on the fact that so-called *full meet revision* [Alchourrón and Makinson, 1982] discards all prior beliefs in a severe revision and at the same time satisfies all AGM postulates. This implies that the AGM postulates are too weak to capture the principle of minimal change.

For the incremental adaptation of beliefs, the AGM postulates proved to be overly weak, too [Darwiche and Pearl, 1994; Darwiche and Pearl, 1997]. This has led to the development of additional postulates for iterated belief revision by Darwiche and Pearl (DP), among others (e.g., [Freund and Lehmann, 1994; Lehmann, 1995; Boutilier, 1993]).

Still, however, the AGM and DP postulates together are too permissive in that they support belief revision operators which assume arbitrary dependencies among the pieces of information which an agent acquires along its way. These operators have a drastic effect when the agent makes an observation which contradicts its currently held beliefs: The agent is forced to cancel everything it has learned up to this point [Nayak *et al.*, 1996a; Nayak *et al.*, 2003]. In this paper, we first give a formal analysis of this problem of implicit dependence, and then we present, as a solution, an *Independence* postulate for iterated belief revision. We give a representation theorem for our new postulate and prove its consistency by defining a concrete belief revision operator. We also contrast the Independence postulate to the so-called Recalcitrance postulate of [Nayak *et al.*, 1996a; Nayak *et al.*, 2003] and argue that the latter is too strict in that it rejects reasonable belief revision operators.

The rest of the paper is organized as follows. In the next section, we recall the classical AGM approach in a propositional setting as formulated by [Katsuno and Mendelzon, 1991b], followed by the approach of [Darwiche and Pearl, 1994] for iterated belief revision. In Section 3, we formally analyze the problem of the DP postulates to be overly permissive. In Section 4, we present an additional postulate to overcome this deficiency, and we give a representation theorem for the postulate along with a concrete revision operator. We conclude in Section 5 with a detailed comparison to related work. Proofs of the main results can be found in the appendix.

## 2 Background

In this paper, we will deal with a propositional language $\mathcal{L}$ generated from a finite set $\mathcal{P}$ of atomic propositions. The language is that of classical propositional logic, i.e., with the classical consequence relation $\vdash$. We say that two sentences $\alpha$ and $\beta$ are logically equivalent, written as $\alpha \equiv \beta$, iff $\alpha \vdash \beta$ and $\beta \vdash \alpha$. As usual, a propositional interpretation (world) is a mapping from $\mathcal{P}$ to $\{\top, \bot\}$. The set of all interpretations is denoted by $\mathcal{W}$. If an interpretation $w$ truth-functionally maps a sentence $\mu$ to $\top$, then $w$ is called a model of $\mu$ (denoted by $w \models \mu$). Given a sentence $\mu$, we denote by $Mods(\mu)$ the set of all models of $\mu$.

A total pre-order $\leq$ (possibly indexed) is a reflexive, transitive binary relation s.t., either $\alpha \leq \beta$ or $\beta \leq \alpha$ holds for any $\alpha, \beta$. The strict part of $\leq$ is denoted by $<$, that is, $\alpha < \beta$ iff $\alpha \leq \beta$ and $\beta \not\leq \alpha$. As usual, $\alpha = \beta$ abbreviates $\alpha \leq \beta$ and $\beta \leq \alpha$. Given any set $S$ and total pre-order $\leq$, we denote by $\min(S, \leq)$ the set of minimal elements of $S$ wrt. $\leq$.

### 2.1 KM Postulates

Katsuno and Mendelzon (KM) rephrased the AGM postulates for the propositional setting [Katsuno and Mendelzon, 1991b]. The beliefs of an agent are represented by a sentence $\psi$ in $\mathcal{L}$.[2] Any new evidence is a sentence $\mu$ in $\mathcal{L}$, and the result of revising $\psi$ with $\mu$ is also a sentence (denoted by $\psi * \mu$) which belongs to $\mathcal{L}$. This is then the reformulation of the original AGM postulates:

(KM1) $\psi * \mu \vdash \mu$.
(KM2) If $\psi \wedge \mu$ is consistent, then $\psi * \mu \equiv \psi \wedge \mu$.
(KM3) If $\mu$ is consistent, then $\psi * \mu$ is also consistent.
(KM4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 * \mu_1 \equiv \psi_2 * \mu_2$.
(KM5) $(\psi * \mu) \wedge \phi \vdash \psi * (\mu \wedge \phi)$.
(KM6) If $(\psi * \mu) \wedge \phi$ is satisfiable, then $\psi * (\mu \wedge \phi) \vdash (\psi * \mu) \wedge \phi$.

Readers are referred to [Gärdenfors and Makinson, 1988] for the motivation and interpretation of these postulates.

Katsuno and Mendelzon have given a representation theorem for Postulates (KM1)–(KM6) wrt. a revision mechanism based on total pre-orders over possible worlds:

**Definition 1** *A function that maps each belief set $\psi$ to a total pre-order $\leq_\psi$ on $\mathcal{W}$ is called a* faithful assignment over belief sets *iff*

- *If $w_1, w_2 \models \psi$, then $w_1 =_\psi w_2$.*
- *If $w_1 \models \psi$ and $w_2 \not\models \psi$, then $w_1 <_\psi w_2$.*
- *If $\psi \equiv \phi$, then $\leq_\psi = \leq_\phi$.*

The intuitive meaning of $w_1 \leq_\psi w_2$ is that $w_1$ is at least as plausible as $w_2$ from the viewpoint of the agent who possesses the belief set $\psi$. The total pre-order $\leq_\psi$ is also called a *faithful ranking* wrt. $\psi$.

We particularly note that the last condition in Definition 1 says that faithful rankings of logically equivalent belief sets must be identical. This essentially prohibits the possibility that in different situations the agent has the same belief set but with different preferences among the beliefs.

**Theorem 1** *[Katsuno and Mendelzon, 1991b] A revision operator $*$ satisfies Postulates (KM1)–(KM6) iff there exists a faithful assignment that maps a*

[2] As $\mathcal{L}$ is assumed finite, any belief set can be represented as a sentence (modulo logical consequence). In this paper, we therefore do not distinguish belief sets from sentences.

*belief set $\psi$ to a total pre-order $\leq_\psi$ s.t.,*

$$Mods(\psi * \mu) = \min(Mods(\mu), \leq_\psi)$$

Although the KM postulates were meant to be a reformulation of the AGM postulates for propositional logics, there is an important difference: The AGM postulates do not constrain operations wrt. varying belief sets [Alchourrón *et al.*, 1985], whereas Postulate (KM4) stipulates that logically equivalent belief sets revised by logically equivalent sentences must result in logically equivalent (new) belief sets. This essentially implies that a revision operator $*$ is a function on belief sets (modulo logical equivalence). This, in turn, is highly controversial among the belief revisionists; in fact, it is commonly believed that this amounts to too excessive a restriction on the conditions of a faithful assignment over belief sets [Darwiche and Pearl, 1997; Hansson, 1998; Nayak *et al.*, 2003; Freund and Lehmann, 1994].

We follow this consensus and argue that for a faithful reformulation of the AGM postulates, Postulate (KM4) should be weakened as follows:

(KM4′) If $\mu_1 \equiv \mu_2$, then $\psi * \mu_1 \equiv \psi * \mu_2$.

The principle of minimal change is often argued to be the foundation of the AGM postulates. Indeed, Postulate (KM2) says that in the case of a mild revision the agent must retain both the prior beliefs and the new evidence. But how about the case of severe revisions? The following finding unveils the striking fact that the AGM postulates put no constraints at all on the retention of prior beliefs in the case of a severe revision. So-called *full meet revision* [Alchourrón and Makinson, 1982], denoted by $*_a$, is a revision operator which completely "forgets" the prior beliefs when they contradict the new evidence:

$$\psi *_a \mu = \begin{cases} \psi \wedge \mu & \text{if } \psi \nvdash \neg\mu \\ \mu & \text{otherwise} \end{cases} \tag{1}$$

Full meet revision is also called *amnesic revision* by [Rott, 2000]. Despite its radical behavior, amnesic revision perfectly satisfies all of (KM1)–(KM6) [Alchourrón and Makinson, 1982]. Consequently, in order to sufficiently impose the principle of minimal change, the KM postulates must be strengthened.

As proposed by many researchers [Gärdenfors and Makinson, 1988; Spohn, 1988], a general belief revision operator should exploit some kind of extra-logical information concerning the preference over different beliefs to determine the revision strategy. In particular, this preference information should uniquely determine a set of *conditional beliefs*: An agent is said to hold a conditional belief $\alpha \gg \beta$ (with $\alpha, \beta$ sentences in $\mathcal{L}$) precisely when it will believe $\beta$ after a revision with $\alpha$ [Gärdenfors, 1988; Boutilier, 1993]. The *Triviality Theorem* of [Gärdenfors and Makinson, 1988] shows that, when using the AGM postulates, then it is improper to include conditional beliefs into the belief sets. As a consequence, we need to distinguish a belief set (referred to as *propositional beliefs*) from a *belief state* (also called *epistemic state*). The latter contains, in addition to its belief set, the conditional beliefs which determine the revision strategy. In concrete constructions of belief revision operators, this extra-logic preference information could take the form of a relation over all possible worlds (as in Definition 1), or a relation over the set of all sentences [Gärdenfors and Makinson, 1988], or a relation over all subsets of the belief set [Alchourrón *et al.*, 1985].

Darwiche and Pearl [1997] have suggested to make this idea explicit by regarding a belief revision operator as a function on belief states (rather than on belief sets), that is, a function which maps a prior belief state and new evidence to a revised belief state. This has resulted in Postulates (R*1)–(R*6) shown below. From a pragmatic point of view, it is very important that a revision operator delivers a revised belief state instead of a belief set, because only in this way the revision operator can be iterated when another piece of new evidence arrives. This also conforms with the criterion of categorial matching [Hansson, 2003].

As in [Darwiche and Pearl, 1997], for the sake of simplicity we will abuse notation by using interchangeably a belief state $\Psi$ and its belief set $\mathrm{Bel}(\Psi)$. For example, $\Psi$ and $\Psi * \mu$ in Postulate (R*1) refer, respectively, to the current belief state and to the posterior belief state, while $\Psi * \mu \vdash \mu$ is just shorthand for $\mathrm{Bel}(\Psi * \mu) \vdash \mu$. The following are the *modified KM postulates* for revision operators on belief *states*:

(R*1) $\Psi * \mu \vdash \mu$.
(R*2) If $\Psi \wedge \mu$ is consistent, then $\Psi * \mu \equiv \Psi \wedge \mu$.
(R*3) If $\mu$ is consistent, then $\Psi * \mu$ is also consistent.
(R*4) If $\mu_1 \equiv \mu_2$, then $\Psi * \mu_1 \equiv \Psi * \mu_2$. [3]

[3] The original version of (R*4) in [Darwiche and Pearl, 1997] is as follows:
If $\Psi_1 = \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 * \mu_1 \equiv \Psi_2 * \mu_2$
where $\Psi_1 = \Psi_2$ means $\Psi_1$ and $\Psi_2$ are equal. However, Darwiche and Pearl have not

(R*5) $(\Psi * \mu) \wedge \phi \vdash \Psi * (\mu \wedge \phi)$.
(R*6) If $(\Psi * \mu) \wedge \phi$ is satisfiable, then $\Psi * (\mu \wedge \phi) \vdash (\Psi * \mu) \wedge \phi$.

It is easy to observe that these postulates only put constraints on the change of the logical part (propositional beliefs) of the belief state, and those constraints are exactly the same as imposed by the original AGM postulates. Therefore, we consider the modified KM postulates (R*1)–(R*6), which are sometimes referred to as a weakening of the AGM postulates [Freund and Lehmann, 1994; Nayak *et al.*, 2003], to be in fact a proper interpretation of them.

In a way symmetric to Theorem 1, Darwiche and Pearl have given a representation theorem for Postulates (R*1)–(R*6):

**Definition 2** *A function that maps each belief state* $\Psi$ *to a total pre-order* $\leq_\Psi$ *on* $\mathcal{W}$ *is called a* faithful assignment over belief states *iff*

- *If* $w_1, w_2 \models \Psi$*, then* $w_1 =_\Psi w_2$*.*
- *If* $w_1 \models \Psi$ *and* $w_2 \not\models \Psi$*, then* $w_1 <_\Psi w_2$*.*

Note that the conditions for a faithful assignment over belief states are much weaker than those for a faithful assignment over belief sets, since logically equivalent belief sets now are allowed to have distinct faithful rankings.

**Theorem 2** *[Darwiche and Pearl, 1997] A revision operator* $*$ *satisfies Postulates (R*1)–(R*6) iff there exists a faithful assignment that maps a belief state* $\Psi$ *to a total pre-order* $\leq_\Psi$ *such that*

$$Mods(\Psi * \mu) = \min(Mods(\mu), \leq_\Psi)$$

According to the above theorem, for any revision operator $*$ (that satisfies Postulates (R*1)–(R*6)) there exists at least one faithful assignment over belief states for which the specified condition holds. In general, there could be more than one such assignment; however, it is not difficult to see that if $\mathcal{L}$ is finite then this faithful assignment must be unique. In the sequel, we will call this the *faithful assignment corresponding* to $*$.

Theorem 2 only says which models of the new propositional beliefs are obtained after a single revision. In order to allow for successive revisions, in each revision step it must also be fully specified how the conditional beliefs are to be modified. Following the principle of economy of information, some restrictions should be imposed on the change of conditional beliefs, too. By concrete counterexamples, Darwiche and Pearl have shown that the KM postulates

given an explicit definition of the notion of a belief state, let alone a definition of two belief states being equal. This is the reason why we have deliberately refrained from using the equality and reformulated this postulate for the sake of precisison.

alone are too weak to adequately characterize iterated belief revision, because they support unreasonable revision behaviors [Darwiche and Pearl, 1994]. To overcome this deficiency, they have proposed these four additional postulates [Darwiche and Pearl, 1997]:

(C1) If $\beta \vdash \mu$, then $(\Psi * \mu) * \beta \equiv \Psi * \beta$.
(C2) If $\beta \vdash \neg\mu$, then $(\Psi * \mu) * \beta \equiv \Psi * \beta$.
(C3) If $\Psi * \beta \vdash \mu$, then $(\Psi * \mu) * \beta \vdash \mu$.
(C4) If $\Psi * \beta \nvdash \neg\mu$, then $(\Psi * \mu) * \beta \nvdash \neg\mu$.

Motivation and interpretation for these postulates can be found in [Darwiche and Pearl, 1994; Darwiche and Pearl, 1997].

To provide formal justifications, Darwiche and Pearl have given an extension of the above representation theorem for Postulates (C1)–(C4):

**Theorem 3** *[Darwiche and Pearl, 1997] Suppose that a revision operator satisfies Postulates (R*1)–(R*6). The operator satisfies Postulates (C1)–(C4) iff the operator and its corresponding faithful assignment satisfy:*

*(CR1)* *If* $w_1, w_2 \models \mu$, *then* $w_1 \leq_\Psi w_2$ *iff* $w_1 \leq_{\Psi*\mu} w_2$.
*(CR2)* *If* $w_1, w_2 \not\models \mu$, *then* $w_1 \leq_\Psi w_2$ *iff* $w_1 \leq_{\Psi*\mu} w_2$.
*(CR3)* *If* $w_1 \models \mu$ *and* $w_2 \not\models \mu$, *then* $w_1 <_\Psi w_2$ *implies* $w_1 <_{\Psi*\mu} w_2$.
*(CR4)* *If* $w_1 \models \mu$ *and* $w_2 \not\models \mu$, *then* $w_1 \leq_\Psi w_2$ *implies* $w_1 \leq_{\Psi*\mu} w_2$.

This theorem gives an elegant characterization of the seemingly natural constraints that the DP postulates impose on the change of the conditional beliefs: When $\Psi$ is revised by $\mu$, Conditions (CR1) and (CR2) require not to change the relative plausibility ordering of $\mu$-worlds ($\neg\mu$-worlds, respectively); Conditions (CR3) and (CR4) require that if a $\mu$-world $w_1$ is (strictly) more plausible than a $\neg\mu$-world $w_2$, then $w_1$ continues to be (strictly) more plausible than $w_2$.

In addition, [Darwiche and Pearl, 1997] have shown that their four postulates are consistent with the (modified) KM postulates. They did so by defining a concrete revision operator which satisfies both (R*1)–(R*6) and (C1)–(C4).

Theorem 3 implies that the DP postulates together impose constraints on the change of conditional beliefs. The following result shows that it is in fact only Postulate (C2) which puts additional constraints on the retention of propositional beliefs.

**Proposition 1** *Amnesic revision* $*_a$ *satisfies (C1), (C3), and (C4), but violates (C2).*

To our knowledge, this observation has not been formalized elsewhere before.

A different approach to studying iterated belief revision is by defining concrete revision operators. For instance, [Boutilier, 1993] has proposed a specific revision operator (known as *natural revision*) which satisfies the modified KM postulates and also the following one:

(CB) If $\Psi * \mu \vdash \neg\beta$, then $(\Psi * \mu) * \beta \equiv \Psi * \beta$.

It is easy to see that the DP postulates are a weakening of Postulate (CB), in the sense Postulate (CB) implies all of the DP postulates but not vice versa.

As shown by [Darwiche and Pearl, 1997; Boutilier, 1996], Postulate (CB) imposes absolute minimization on the change of conditional beliefs:

**Theorem 4** *Suppose that a revision operator satisfies Postulates (R*1)–(R*6). The operator satisfies Postulate (CB) iff the operator and its corresponding faithful assignment satisfy*

*(CBR) If $w_1, w_2 \models \neg(\Psi * \mu)$, then $w_1 \leq_\Psi w_2$ iff $w_1 \leq_{\Psi*\mu} w_2$*

Note that now $w_2, w_2 \models \Psi * \mu$ is the only case where the relative ordering of $w_1, w_2$ in $\Psi * \mu$ is not determined, since $\leq_{\Psi*\mu}$ must satisfy the conditions of Definition 2. Therefore, Condition (CBR) imposes absolute minimization on the change of conditional beliefs permitted by the modified KM postulates. At first glance, therefore, it seems that Condition (CBR) complies with the principle of economy of information.

However, the following example of Darwiche and Pearl shows that Postulate (CB) is too radical, since a severe revision forces to cancel all previous evidences under any circumstances, which is usually not desirable.

**Example 1** We encounter a strange new animal and it appears to be a bird, so we believe the animal is a bird. As it comes closer to our hiding place, we see clearly that the animal is red, so we believe that it is a red bird. To remove further doubts about the animal birdhood, we call in a bird expert who takes it for examination and concludes that it is not really a bird but some sort of mammal. The question now is whether we should still believe that the animal is red.

As argued in [Darwiche and Pearl, 1997], we have every reason to keep our belief that the animal is red, since birdhood and color are not correlated. However, natural revision requires us to give up the belief of the animal's color: According to Postulate (CB), from $bird{*}red \vdash \neg(\neg bird)$ it follows that $(bird{*}red){*}\neg bird \equiv bird{*}\neg bird$.

The above discussion suggests that the most conservative way of changing conditional beliefs is overly strict and not desirable in general.

While natural revision is the most conservative of all possible DP revision operators, another revision operator, called *lexicographic revision* (with "naked evidence") [Nayak, 1994], sits exactly on the opposite side of the spectrum. Lexicographic revision satisfies, in addition to Postulate (C1) and (C2), another so-called postulate of *Recalcitrance*:

(Rec) If $\beta \nvdash \neg\mu$, then $(\Psi * \mu) * \beta \vdash \mu$.

Semantically, Postulate (Rec) corresponds to the following condition [Nayak *et al.*, 2003]:

(RecR) If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 <_{\Psi*\mu} w_2$.

According to (RecR), all possible worlds satisfying the new evidence become more reliable than those falsifying the new evidence, hence (Rec) is also said to impose the principle of *strong primacy of update* [Konieczny and Pérez, 2000], which is arguably only suitable when the agent has full confidence in the new evidence. Based on its semantic characterization (i.e., Conditions (CR1), (CR2) and (RecR)), it it easy to see that lexicographic revision is the least conservative of all possible DP revision operators, effecting most changes in the relative ordering of worlds permitted by the KM and DP postulates [Booth *et al.*, 2005]. In the next section, we will give a formal analysis of the problems of the DP postulates in general and the problems of the greatest conservatism in particular. The discussion on the problems of the least conservatism is postponed to Section 5.

## 3 The Problem of Implicit Dependence

Although most counter-examples in [Darwiche and Pearl, 1997] against the KM postulates are solved by adding the DP postulates, several open problems remain. Specifically, the DP postulates are consistent with (CB), hence they do not block counter-examples against natural revision.

Recall Example 1, where the DP postulates, in being compatible with (CB), are not strong enough to guarantee that the belief of the animal's color is retained. This can be intuitively explained as follows: After observing the animal's color, we are actually acquiring a new conditional belief as a side-effect, namely, that the animal is red even if it were not a bird, that is, $\neg bird \gg red$. But none of the DP postulates enforces the acquisition of conditional beliefs. In the sequel, we first give a formal analysis of this weakness of the

DP postulates, and then we present an additional postulate by which this problem is overcome.

It is well known (see, e.g., [Gärdenfors and Makinson, 1988]) that if a belief state $\Psi$ suffices to uniquely determine a revision strategy that satisfies the AGM (or the KM) postulates, then the belief state determines a unique, total pre-order $\leq_{\mathrm{Bel}(\Psi)}$ (known as *epistemic entrenchment*) over $\mathcal{L}$ which satisfies the following conditions:

(EE1) If $\alpha \leq_{\mathrm{Bel}(\Psi)} \beta$ and $\beta \leq_{\mathrm{Bel}(\Psi)} \gamma$, then $\alpha \leq_{\mathrm{Bel}(\Psi)} \gamma$.
(EE2) If $\alpha \vdash \beta$, then $\alpha \leq_{\mathrm{Bel}(\Psi)} \beta$.
(EE3) $\alpha \leq_{\mathrm{Bel}(\Psi)} \alpha \wedge \beta$ or $\beta \leq_{\mathrm{Bel}(\Psi)} \alpha \wedge \beta$, for any $\alpha$ and $\beta$.
(EE4) If $\Psi$ is consistent, then $\Psi \nvdash \alpha$ precisely when $\alpha \leq_{\mathrm{Bel}(\Psi)} \beta$ for all $\beta$.
(EE5) If $\beta \leq_{\mathrm{Bel}(\Psi)} \alpha$ for all $\beta$, then $\vdash \alpha$.

If $\alpha <_{\mathrm{Bel}(\Psi)} \beta$, then we say that the *degree* of the belief in $\beta$ is higher than the degree of the belief in $\alpha$ (wrt. $\Psi$).

Given an epistemic entrenchment, the corresponding belief revision operator is defined by the following condition: For any $\beta$,

(C*) $\Psi * \mu \vdash \beta$ if either $\vdash \neg\mu$ or $\neg\mu <_{\mathrm{Bel}(\Psi)} \neg\mu \vee \beta$.

Other forms of total pre-orderings on $\mathcal{L}$ have been proposed, e.g., [Rott, 1991; Williams, 1992]. In fact, a recent result of Rott [2003] shows that such kind of pre-orderings exist even if the revision operator satisfies only (R*1), (R*3), and (R*4). All of these orderings require extra-logical information, that is, they cannot be determined by pure logical relations among the sentences. In the following, we focus on pre-orderings given by epistemic entrenchments; however, our analysis does not depend on this particular choice and can be easily adapted to the other approaches just mentioned.

To begin with, we define the notion of dependence between sentences wrt. a belief state as follows [Fariñas del Cerro and Herzig, 1996]:

**Definition 3** *A sentence $\beta$* depends *on another sentence $\mu$ in belief state $\Psi$ precisely when $\Psi \vdash \beta$ and $\Psi * \neg\mu \nvdash \beta$. Two sentences $\mu, \beta$ are called* dependent in $\Psi$ *if either $\mu$ depends on $\beta$ or $\beta$ depends on $\mu$ in $\Psi$.*

Consider, now, a (non-tautological) new evidence $\mu$. Whenever $\Psi \vdash \beta$, condition (C*) implies that if $\mu \not<_{\mathrm{Bel}(\Psi)} \mu \vee \beta$, then $\beta$ is (implicitly) dependent on $\mu$ in $\Psi$. This kind of dependency could be problematic. In particular, it is possible that two initially independent sentences become, undesirably, dependent after a revision step. In Example 1, for instance, *red* becomes dependent on *bird* after revising by *red* when natural revision is used.

The problem of natural revision is that it assigns the lowest degree of belief to a new evidence without asserting conditional beliefs for independence. Thus the new evidence depends on all other beliefs which survive the revision process. This explains why severe revision necessarily cancels all previous evidences. Of course, this is not merely a problem of natural revision: In the revised belief state $\Psi * \mu$, regardless of the belief degree of the new evidence $\mu$, a belief $\beta$ (logically unrelated to $\mu$) with a lower belief degree will depend on $\mu$ unless the revision operator explicitly asserts the condition $\mu <_{\mathrm{Bel}(\Psi*\mu)} \mu \vee \beta$. In other words, a rational revision operator has to bring about explicitly the conditional belief $\neg\mu \gg \beta$. Symmetrically, a rational revision operator also should take care of the implicit dependence of the new evidence on other beliefs with higher degrees.

## 4 A Postulate of Independence

The analysis in the previous section shows that in order to overcome the problem of implicit dependence, the revision operator must explicitly assert some conditional beliefs. It is easy to see that the DP postulates only require the *preservation* of conditional beliefs when a belief state $\Psi$ is revised with $\mu$: Postulates (C1) and (C2) neither require to add nor to remove certain conditional beliefs; Postulate (C3) requires to retain the conditional belief $\beta \gg \mu$; finally, Postulate (C4) requires not to obtain the new conditional belief $\beta \gg \neg\mu$. Since none of the DP postulates stipulates the addition of independence assumptions, a new postulates is necessary to avoid undesired dependencies.

As already mentioned, the revision process may introduce undesirable dependencies in both directions. That is to say, it could be that the new evidence becomes dependent on existing beliefs, or that it is the other way round. Prior to stating the new postulate, we show that the DP postulates impose some constraints on the retention of the independence information in one direction. In the presence of the KM postulates, Postulate (C2) implies the following (since $(\Psi * \neg\mu) \equiv (\Psi * \mu) * \neg\mu)$:

(WC2) If $\Psi * \neg\mu \vdash \beta$, then $(\Psi * \mu) * \neg\mu \vdash \beta$.

This essentially means that if $\beta$ is not dependent on the new evidence $\mu$ in $\Psi$, then it also does not depend on $\mu$ in $\Psi * \mu$.

In order to ensure the explicit assertion of independence information in the other direction, we propose the following postulate of *Independence* (weak version) dual to (WC2):

(WInd) If $\Psi * \neg\beta \vdash \mu$, then $(\Psi * \mu) * \neg\beta \vdash \mu$.

Suppose $\Psi \vdash \mu$, then Postulate (WInd) guarantees that if some new information $\mu$ does not depend on $\beta$ in $\Psi$, then it also does not depend on $\beta$ in $\Psi * \mu$.

As it is too much to require that the new information $\mu$ is already believed (i.e, $\Psi \vdash \mu$), we propose the following postulate of *Independence* (strong version):

(Ind) If $\Psi{*}\neg\beta \nvdash \neg\mu$ then $(\Psi{*}\mu){*}\neg\beta \vdash \mu$.

It is not difficult to see that (Ind) is a strengthening of (WInd). The new postulate essentially says that if the conditional belief $\neg\beta \gg \neg\mu$ is not held in $\Psi$, then $\mu$ does not depend on $\beta$ in $\Psi * \mu$.

Postulate (Ind) is sufficient to overcome the problem of implicit dependence, as can be shown by reconsidering Example 1. According to (Ind), $(\mathit{bird}{*}\mathit{red}){*}\neg\mathit{bird} \vdash \mathit{red}$, given that $\mathit{bird}{*}\neg\mathit{bird} \nvdash \neg\mathit{red}$. This shows that the new postulate blocks unreasonable behaviors which are admitted by the DP postulates. In Section 5, we will also argue that Postulate (Ind) is not overly strict.

### *4.1 A Representation Theorem*

In order to formally justify our new postulate, we will first provide a representation theorem along the line of Theorem 3. Thereafter, we will design a concrete belief revision operator which satisfies (Ind).

**Theorem 5** *Suppose that a revision operator satisfies Postulates (R*1)–(R*6). The operator satisfies Postulate (Ind) iff the operator and its corresponding faithful assignment satisfy:*

*(IndR) If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_\Psi w_2$ implies $w_1 <_{\Psi * \mu} w_2$.*

The above theorems shows that Postulate (Ind) is quite natural and not overly constrained: Condition (IndR) requires that a world $w_1$ conforming with the new evidence becomes more plausible than a world $w_2$ violating the new evidence only if $w_1$ was at least as plausible as $w_2$.

An immediate consequence of Theorem 3 and 5 is that Postulate (Ind) implies both (C3) and (C4).

**Proposition 2** *Suppose that a revision operator satisfies Postulates (R*1)–(R*6). If the operator satisfies Postulate (Ind), then it also satisfies Postulates (C3) and (C4).*

We suggest to use the modified KM postulates along with Postulates (C1), (C2), and (Ind) to govern iterated belief revision. To show that these postulates together are consistent, we present a concrete revision operator which satisfies all of them. The operator is a modification of Spohn's proposal of revising *ordinal conditional functions* [Spohn, 1988], which can be viewed as a qualitative version of Jeffrey's Rule of probabilistic conditioning [Goldszmidt, 1992].

Originally, an ordinal conditional function (OCF) has been defined as a mapping $k$ from $\mathcal{W}$ to the class of ordinals. As in [Spohn, 1991], for the sake of simplicity we take the signature of an OCF $k$ as $\mathcal{W} \rightarrow \mathbb{N}$, where $k(w)$ is called the *rank* of $w$. Intuitively, the rank of a world represents its degree of implausibility, that is to say, the lower its rank, the more plausible is a world. An OCF encodes both a belief set and the conditional beliefs. The belief set $\mathrm{Bel}(k)$ is the set of sentences which hold in all worlds of rank 0:

$$Mods(\mathrm{Bel}(k)) = \{w | k(w) = 0\} \tag{2}$$

From now on, we use an OCF and its belief set interchangeably; e.g., $\mu \in k$ means $\mu \in \mathrm{Bel}(k)$, and $k \wedge \mu$ denotes $\bigwedge \mathrm{Bel}(k) \wedge \mu$.

Given an OCF $k$, we can induce a ranking of sentences as follows:

$$k(\mu) = \begin{cases} \infty & \text{if } \vdash \mu \\ \min\{k(w) \mid w \models \neg\mu\} & \text{otherwise} \end{cases} \tag{3}$$

Put in words, the rank of a sentence is the lowest rank of a world in which the sentence does not hold.[4] Hence, the higher the rank of a sentence, the firmer the belief in it, and the belief set consists of sentences with rank greater than 0. In fact, it is not hard to see that an OCF $k$ determines an epistemic entrenchment as follows:

$$\alpha \leq_k \beta \text{ iff } k(\alpha) \leq k(\beta) \tag{4}$$

**Proposition 3** *Given an OCF $k$, the binary relation $\leq_k$ defined by (4) satisfies (EE1)–(EE5).*

By a slight modification of Spohn's Conditionalization, we now define a revision operator which we call *reinforcement revision operator*. Like Condition-

[4] In Spohn's original proposal, the rank of a sentence is the lowest rank of a world in which it is true. So the rank of $\beta$ there is equal to $k(\neg\beta)$ here.

alization, reinforcement revision allows to assign different evidence degrees to new evidences; standard KM/DP revision is easily obtained as a special case by using a fixed value in all iterations [Darwiche and Pearl, 1997]. An OCF $k$ is revised according to new evidence $\mu$ with evidence degree $m > 0$ as follows:

$$(k^*_{\mu.m})(w) = \begin{cases} k(w) - k(\neg\mu) & \text{if } w \models \mu \\ k(w) + m & \text{otherwise} \end{cases} \tag{5}$$

Reinforcement revision is distinct from Spohn's Conditionalization [Spohn, 1988] in three aspects. First of all, it is merely a revision operator, whereas Conditionalization defines both a revision and a contraction operator (when the degree of the new information is 0). Secondly, in reinforcement revision the rank of the new evidence in the revised OCF is the sum of its old rank and the evidence degree, whereas in Conditionalization the rank of the new evidence is just its evidence degree. The last, and crucial, difference is that Conditionalization does not satisfy Postulate (Ind).

Assuming the same evidence degree for any new information, satisfiability of the KM postulates along with Postulates (C1), (C2), and (Ind) by reinforcement revision operator is a direct consequence of Theorem 2, 3, and 5.

**Theorem 6** *Assume a fixed evidence degree for any new information. Reinforcement revision satisfies all modified KM postulates, DP postulates, and Postulate (Ind).*

A stronger result shows that all postulates are still satisfied in the general case, where the evidence degrees varies in the course of iterated revision. To begin with, we have the following:

**Theorem 7** *For any $m > 0$, reinforcement revision satisfies all KM postulates (R*1)–(R*6), where* $\mathrm{Bel}(\Psi)$ *and* $\Psi * \mu$ *are, respectively, identified with* $\bigwedge \mathrm{Bel}(k)$ *and* $k^*_{\mu,m}$.

To show the validity of the remaining postulates (in case of varying evidence degrees), we need the following lemma, which fully characterizes the change of belief degrees of non-tautological sentences.

**Lemma 1** *Let $k$ be an arbitrary OCF and $\mu$ a new evidence with degree $m$, then for any non-tautological sentence $\beta$,*

$$k^*_{\mu,m}(\beta) = \begin{cases} k(\beta) + m & \textit{if } \vdash \mu \supset \beta \\ k(\mu \supset \beta) - k(\neg\mu) & \textit{else if } \ k(\mu \supset \beta) = k(\beta) \\ \min(k(\mu \supset \beta) - k(\neg\mu), k(\beta) + m) & \textit{otherwise} \end{cases}$$

As a direct consequence of Lemma 1, it can be seen that our reinforcement revision operator has indeed a reinforcement effect, that is, the evidence degrees of the new information are accumulated.

**Proposition 4** *Let $k$ be an arbitrary OCF and $\mu$ a new non-tautological evidence with degree $m$, then*

$$k^*_{\mu,m}(\mu) = k(\mu) + m$$

From a pragmatic point of view, this is a desirable property in particular for domains where several independent information sources provide new information. In this case, it is appropriate to sum up the evidence degrees of the same information from different sources.

Finally, with the help of Lemma 1, we are able to prove that reinforcement revision satisfies (C1), (C2), and (Ind), regardless of evidence degrees.

**Theorem 8** *For arbitrary $m_1, m_2 > 0$, reinforcement revision satisfies the following conditions:*[5]

*(EC1)* If $\alpha \vdash \mu$, then $(k^*_{\mu,m_1})^*_{\alpha,m_2} \equiv k^*_{\alpha,m_2}$.
*(EC2)* If $\alpha \vdash \neg\mu$, then $(k^*_{\mu,m_1})^*_{\alpha,m_2} \equiv k^*_{\alpha,m_2}$.
*(EInd)* If there exists $m$ such that $k^*_{\neg\beta,m} \nvdash \neg\mu$, then $(k^*_{\mu,m_1})^*_{\neg\beta,m_2} \vdash \mu$.

Theorem 7 and 8 show that Postulate (Ind) is consistent with the KM and DP postulates. On the other hand, (Ind) does not follow from these postulates, as can be seen by the fact that (Ind) is incompatible with (CB), the postulate that characterizes natural revision.

It is worth mentioning that revision operators based on OCFs are particularly suitable for implementations of belief revision. For instance, in [Jin and Thielscher, 2004] we have presented a method (and its implementation) for the revision of belief bases which is equivalent to reinforcement revision. Moreover, we have shown that the complexity of reinforcement revision is lower than that of most well-known operators [Jin and Thielscher, 2005a].

## 5 Related Work and Conclusion

We have suggested to use the modified KM postulates along with Postulates (C1), (C2), and (Ind) to govern iterated belief revision, that is to say,

[5] Note that, as before, we abuse notation by simply writing $k^*_{\mu,m}$ instead of $\bigwedge \mathrm{Bel}(k^*_{\mu,m})$ etc.

any rational iterated revision operator should satisfy all of these postulates. In the belief revision community there is, however, an ongoing controversy on what the proper framework for studying iterated belief revision should be. As in Darwiche and Pearl's original work [1994], revision operators are most commonly viewed as binary functions which map a belief set and the new information to the revised belief set. This is problematic in two aspects. First of all, the revision operators studied in the AGM theory are local in the sense that a fixed belief set is assumed. Such revision operators are more appropriately considered as unary functions, which map the new information $\mu$ to a revised belief set $K * \mu$, with the understanding that $K$ is taken to be the background knowledge [Rott, 1999]. Secondly, the extra-logical preference information should play a role in the revision process. Based on the characterization of revision operators as unary functions, [Nayak *et al.*, 2003] have proposed to view belief revision as dynamic, in the sense that the operator itself (i.e., the revision policy) evolves after each revision by taking the revised belief set as the new background knowledge. While theoretically sound, the idea of dynamic revision is technically quite confusing in the sense that realizing a dynamic revision seems like devising an algorithm which evolves after each run. Most belief revisionists maintain that (iterated) revision operators should be functions on belief states [Lehmann, 1995; Rott, 1999; Darwiche and Pearl, 1994; Williams, 1994; Konieczny and Pérez, 2000], although there is no consensus on what is a belief state.

Furthermore, while Postulate (C1) is almost universally accepted, Postulate (C2) seems to be more problematic. In fact, it is mainly different attitudes towards Postulate (C2) which provoke the dispute over the framework of iterated belief revision. In defense of our framework, we argue that, according to the semantical characterization (Conditions (CR1) and (CR2)), Postulate (C2) seems just as reasonable as Postulate (C1). If being informed about $\mu$ does not change the relative plausibility ordering of $\mu$-worlds, why should the relative ordering of $\neg\mu$-worlds be changed? This idea is also supported by Spohn, who argues that it is only reasonable to change the relative ordering between $\mu$-worlds and $\neg\mu$-worlds [Spohn, 1988].

In the sequel, we will first give a detailed comparison of our framework with the most prominent existing approaches to iterated revision. Thereafter, we will discuss the problems of least conservatism as promised in Section 2.3.

### *5.1 Freund and Lehmann's Proposals*

Freud and Lehmann were the first to point out that Postulate (C2) is inconsistent with the original KM postulates [Freund and Lehmann, 1994]. To avoid the inconsistency, they have suggested to replace the DP postulates by the so-called *minimal influence* postulate:

(MinInf) If $\Psi_1 \vdash \neg\mu$ and $\Psi_2 \vdash \neg\mu$, then $\Psi_1 * \mu \equiv \Psi_2 * \mu$.

According to (MinInf), the revision $\Psi*\mu$ does not depend on $\Psi$ at all in the case of a severe revision. This is of course a very strong restriction, which violates the intuition that the prior beliefs should play a major role. Furthermore, in the presence of the AGM postulates, (MinInf) implies (C1), (C3), (C4) and the following weakening of (C2):

(C2′) If $\Psi \vdash \neg\beta$ and $\beta \vdash \neg\mu$, then $(\Psi * \mu) * \beta \equiv \Psi * \beta$

Strong as it is, Postulate (MinInf) is, on the other hand, too weak to rule out amnesic revision. Moreover, from the fact that the modified KM postulates are consistent with (C2), it follows that the inconsistency of (C2) with regard to the original KM postulates is due to the assumption the latter made on the signature of revision operators (i.e., that they are functions on belief sets). As already discussed, this assumption is not accepted, if not denied, by many researchers. Therefore, the proposal of (MinInf) is in some sense not well-supported.

A conclusion Freund and Lehmann have drawn is that the AGM framework is not the right one in which to study iterated revision. In a later work, Lehmann therefore has proposed an alternative approach to iterated revision, in which a belief state $\Psi$ is a finite sequence of consistent (propositional) sentences $\langle\beta_1 : \ldots : \beta_n\rangle$ (the revision history of the agent) [Lehmann, 1995]. In Lehmann's framework, the iterated revision operator is trivial: $\Psi * \mu$ is simply defined as the concatenation $\langle\Psi : \mu\rangle$ of $\Psi$ and $\mu$. Similarly, we might denote $\langle\Psi_1 : \Psi_2\rangle$ by $\Psi_1 * \Psi_2$. What seems more difficult to define, however, is a mapping "Bel" from a belief state to its belief set. For this purpose, Lehmann has proposed the following set of postulates:

(I1) $\text{Bel}(\Psi)$ is consistent.
(I2) $\mu \in \text{Bel}(\Psi * \mu)$.
(I3) If $\beta \in \text{Bel}(\Psi * \mu)$, then $\mu \supset \beta \in \text{Bel}(\Psi)$.
(I4) If $\mu \in \text{Bel}(\Psi)$, then $\Psi * \Psi_1 \equiv (\Psi * \mu) * \Psi_1$.
(I5) If $\beta \vdash \mu$, then $((\Psi * \mu) * \beta) * \Psi_1 \equiv (\Psi * \beta) * \Psi_1$.
(I6) If $\neg\beta \notin \text{Bel}(\Psi * \mu)$, then $((\Psi * \mu) * \beta) * \Psi_1 \equiv ((\Psi * \mu) * \mu \wedge \beta) * \Psi_1$.
(I7) $\text{Bel}((\Psi * \neg\beta) * \beta) \subseteq \text{Bel}(\Psi) + \beta$.

Readers are referred to [Lehmann, 1995] for the relation between Lehmann's postulates and the AGM postulates. It is worth to mention that Postulate (I5) is in fact just an adaptation of Postulate (C1).

To provide a constructive model, Lehmann has shown that his postulates characterize the so-called *widening ranked revision*. A widening ranked model is a function $\lambda$ which maps an ordinal to a non-empty subset of $\mathcal{W}$ s.t.,

(1) for any $n, m$, if $n \leq m$, then $\lambda(n) \subseteq \lambda(m)$, and
(2) for any $w \in \mathcal{W}$, there exists $n$ with $w \in \lambda(n)$.

Given a widening ranked model $\lambda$, we can inductively define a rank $r(\Psi)$ and a set of worlds $p(\Psi)$ for any belief state $\Psi$:

- $r(\langle\rangle) = 0$ and $p(\langle\rangle) = \lambda(0)$, and
- $r(\langle\Psi : \mu\rangle) = min_o(\Psi, \mu)$ and $p(\langle\Psi : \mu\rangle) = \lambda(r(\langle\Psi : \mu\rangle)) \cap [\mu]$,

where $min_o(\Psi, \mu)$ is the minimal ordinal $n$ s.t., $n \geq r(\Psi)$ and $\lambda(n) \cap [\mu] \neq \emptyset$.

The widening ranked revision (thus, essentially, the mapping Bel) is then defined as follows:

$$Mods(\Psi * \mu) = p(\langle\Psi : \mu\rangle)$$

Lehmann has shown that the widening ranked revision generated from a widening ranked model satisfies Postulates (I1)-(I7). Conversely, any revision operator that satisfies Postulates (I1)-(I7) can be constructed as widening ranked revision. A major problem with widening ranked revision is that it is based on a fixed widening ranked model which is external to the agent's beliefs. Therefore, the agent is supposed to adhere to the same revision policy regardless of its actual beliefs. Moreover, it is not at all clear where the external, extra-logical preference information comes from and how it is to be interpreted. Therefore, this kind of revision has been criticized by Rott as embodying a bad philosophy [Rott, 2003].

### *5.2 Revision Operators with Memory*

Konieczny and Pérez have proposed yet another framework for iterated revision, which also considers, as the agent's belief state, the sequence of consistent sentences the agent has learned [Konieczny and Pérez, 2000]. Like in Lehmann's approach, the revised belief state $\Psi * \mu$ is just the concatenation of $\Psi$ and $\mu$. However, Konieczny and Pérez have suggested a different set of postulates for iterated belief revision, which are essentially a reformulation of the AGM postulates along with the following one: [6]

(H7) $\quad \Psi * \Psi_1 \equiv \Psi * (\wedge \mathrm{Bel}(\Psi_1))$

[6] As $\mathcal{L}$ is assumed finite in [Konieczny and Pérez, 2000], the conjunction $\bigwedge \mathrm{Bel}(\Psi_1)$ is a well-defined sentence.

Postulate (H7) is a kind of associativity law, which expresses the strong confidence in the new information. It is not difficult to see that (H7) implies (Rec) (cf. Section 2.3).

The postulates proposed by Konieczny and Pérez characterize the so-called *revision operators with memory*, which are based on external faithful assignments over belief sets: Given a faithful assignment over belief sets, we can inductively define a ranking $\preceq_\Psi$ of the possible worlds for any belief state $\Psi$:

- $\preceq_{\langle\rangle} = \mathcal{W} \times \mathcal{W}$, and
- for any $w_1, w_2$: $w_1 \preceq_{\langle \Psi : \mu \rangle} w_2$ iff $w_1 \prec_\mu w_2$ or
  $w_1 =_\mu w_2$ and $w_1 \preceq_\Psi w_2$.

The revision operator with memory is then defined as follows:

$$Mods(\Psi * \mu) = \min([\mu], \preceq_\Psi) \tag{6}$$

Just like Lehmann's revision, a revision operator with memory assumes a fixed (external) faithful assignment, which means that the agent never changes its revision policy. Hence, Rott's criticism regarding widening ranked revisions also applies to revision operators with memory.

As a special case, a so-called *basic memory operator* is generated from a *basic faithful assignment* over $\mathcal{L}$ which additionally satisfies the following condition:

- If $w_1, w_2 \models \neg\mu$, then $w_1 =_\mu w_2$.

Put in words, $\preceq_\mu$ partitions $\mathcal{W}$ into two levels, where the lower level contains all $\mu$-worlds while the other level contains all $\neg\mu$-worlds.

In fact, a basic memory operator is equivalent to Nayak's *lexicographic revision* (with "naked evidence") (cf. Section 2.3). Not surprisingly, therefore, Konieczny and Pérez were able to show that basic memory operators also satisfy all DP postulates.

In their later work, Konieczny and Pérez [2002] have suggested to lift the unrealistic restriction by allowing the faithful ranking of the new evidence to be dynamic, meaning that logically equivalent evidences may come with distinct faithful rankings. These new revision operators have therefore been named *dynamic revision operators with memory.*

Konieczny and Pérez have shown that any dynamic revision operator with memory satisfies (C1), (C3), and (C4), but violates (C2). Based on this, they have criticized (C2) as too strong [Konieczny and Pérez, 2000]. In particular, they have proposed the following counterexample:

**Example 2** Consider an electric circuit containing an adder and a multiplier. The atomic propositions *adder_ok* and *multiplier_ok* denote respectively that the adder and the multiplier are working. Initially we have no information about this circuit ($\Psi = \langle\rangle$), and we then learn that the adder and the multiplier are working ($\mu = adder\_ok \wedge multiplier\_ok$). Thereafter, someone tells us that the adder is actually not working ($\beta = \neg adder\_ok$). There is no reason to "forget" that the multiplier is working, whereas imposed by (C2) we have $(\Psi * \mu) * \beta \equiv \Psi * \beta$, since $\beta \vdash \neg\mu$.

In favour of (C2), we give a counterargument to Konieczny and Pérez's criticism. First we observe that a (dynamic) revision operator with memory is not a single revision operator, unlike what the AGM framework attempts to model. Since the new information is coupled with a faithful ranking, a revision operator with memory (except basic memory revision) essentially is a multiple revision operator which revises a belief state with another belief state. After observing that, it is no surprise that (C2) is violated since this postulate is only intended for single revision operators. This argument is supported by the fact that basic memory revision does satisfy (C2). From the perspective of single revision, the behavior imposed by (C2) in Example 2 is perfectly reasonable, since the evidence $\mu$ is supposed to be an atomic piece of information. Note that in case we learned *adder_ok* and *mutiplier_ok* in succession, then thanks to Postulate (Ind) we will retain *multiplier_ok* after the $\neg$*adder_ok*-revision. In fact it is not difficult to see that if we want the revision operator with memory to exhibit the behavior expected by Konieczny and Pérez, then the faithful ranking that comes with $\mu$ should encode the independence of *multiplier_ok* and *adder_ok*. This somehow highlights the subtle distinction between revising by a conjunction of sentences and revising by a set of sentences (with different plausibility degrees) (cf. the discussions in [Nayak *et al.*, 1996b]), which will be further cultivated in the future. Based on the above argument, we consider (C2) a well justified postulate for single revision operators, although it could be too strong for multiple revision operators.

### *5.3 Dynamic Revision Operators*

Independently, [Nayak *et al.*, 1996a] have also noticed the inconsistency between (C2) and the original KM postulates. Their solution to avoid inconsistency has been to view belief revision as dynamic, as mentioned above. By so doing, it becomes possible to safely accept the DP postulates. The framework of dynamic revision operators is not too different from the DP framework, except that the former makes explicit the idea of evolutionary revision policy in its postulates, by distinguishing between an original and a revised policy.

The problem of the DP postulates to be overly permissive has also been studied by Nayak *et al.* [1996a; 2003]. They have suggested to strengthen the DP postulates by the following so-called postulate of Conjunction:[7]

(Conj) If $\mu \nvdash \neg\beta$, then $(\Psi * \mu) * \beta \equiv \Psi * (\beta \wedge \mu)$.

In the presence of the modified KM postulates, (Conj) is strong enough to imply Postulate (Rec).

In the following, we argue that Postulate (Conj), while strengthening the DP postulates, is overly strict. To this end, we show that even Postulate (Rec) is too strong. As shown in Section 2.3, (Rec) corresponds to the least conservatism in the DP framework. Thus, the following argument is also an analysis of the problems of the least conservatism.

Postulate (Rec) says that, as long as $\beta \supset \neg\mu$ is not a tautology, it should be canceled after a successive revision by $\mu$ followed by $\beta$, no matter how strong the initial belief in $\beta \supset \neg\mu$. A simple example shows that this behavior may not be reasonable:

**Example 3** All her childhood, Alice was taught by her parents that a person who has told a lie is not a good person. So Alice believed, initially, that if Bob has told a lie then he is not a good person. After her first date with Bob, she began to believe that he is a good guy. Then a reliable friend of Alice warns her that Bob is in fact a liar, and Alice chooses to believe her. Now, should Alice still believe that Bob is a good guy?

According to Postulate (Rec), Alice should not challenge Bob's morality and still believe he is good, and hence disbelieve what her parents taught her. But in fact it is at least as reasonable to give up the belief that Bob is good. This shows that Postulate (Rec) is too strict a criterion for belief revision operators.

With regard to the postulate we have proposed, it is easy to see that (Ind) is a weakening of Postulate (Rec). This raises the question whether Postulate (Ind) weakens too much. Let us consider an example, taken from [Nayak *et al.*, 1996a], which, at first glance, seems to show that this is indeed the case.

**Example 4** Our agent believes that Tweety is a singing bird. However, since there is no strong correlation between singing and birdhood, the agent is prepared to retain the belief that Tweety sings even after accepting the information that Tweety is not a bird, and conversely, if the agent were to be informed that Tweety does not sing, she would still retain the belief that Tweety is a bird. Imagine that the agent first receives the information that Tweety is in

[7] In [Nayak *et al.*, 2003], (Conj) is written as "if $\mu \nvdash \neg\beta$, then $(\Psi * \mu) *^{\mu} \beta \equiv \Psi * (\beta \wedge \mu)$", where $*^{\mu}$ denotes the evolved operator after a $\mu$-revision. Accordingly, they have reformulated the DP postulates in the same spirit.

fact not a bird, and later learns that Tweety does not sing.

Nayak *et al.* claimed that it is only reasonable to assume that the agent should, in the end, always believe that Tweety is a non-singing non-bird. Indeed, with $\Psi \equiv singing \wedge bird$ it follows from Postulate (Rec) that $(\Psi * \neg bird) * \neg singing \vdash \neg bird$, since $\nvdash \neg singing \supset bird$. Postulate (Ind), on the other hand, does not apply in this case. But the behavior which is claimed to be the only reasonable one is not generally justified. Suppose, for example, the agent initially believes firmly that $\neg singing \supset bird$. It is then possible, after revising by $\neg bird$, that the belief in $\neg singing \supset bird$ is stronger than the belief in $\neg bird$. In this case, after further revising by $\neg singing$, the agent believes that Tweety is a bird after all.

### *5.4 Conclusion*

In this paper, we have formally analyzed the problem of implicit dependence which is intrinsic to belief revision but largely overlooked in the community over the past decade. As (at least a partial) solution to the problem, we have proposed to strengthen the DP theory by a new postulate of independence. The resulting framework for iterated belief revision now consists of the (modified) KM postulates, (C1), (C2), and (Ind). We have informally argued in favor of our new postulate (Ind) by means of examples, and we have given a formal justification by an elegant semantic characterization. Also, a detailed comparison to related work has shown that our new framework is the most satisfactory one thus far in the literature. As a conclusion, we argue that the new framework provides better criteria for the design of rational iterated belief revision operators.

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## Appendix: Proofs

**Proposition 1.** *The amnesic revision $*_a$ satisfies (C1), (C3), and (C4), but violates (C2).*

**Proof** Note that, in the case of the amnesic revision $*_a$, a belief state is identified with its propositional beliefs.

Assume $\vdash \neg\beta$. According to (1), we have $(\Psi *_a \mu) *_a \beta = \beta$ and $\Psi *_a \beta = \beta$. Hence, $*_a$ satisfies (C1), (C2), and (C3). Moreover, (C4) is vacuously satisfied. In the rest of the proof, we consider the case $\nvdash \neg\beta$.

Assume $\beta \vdash \mu$. We consider two cases: **1)** Assume $\Psi \nvdash \neg\mu$. It follows from $\beta \vdash \mu$ that we have $\Psi \wedge \mu \nvdash \neg\beta$. According to (1), if $\Psi \nvdash \neg\beta$ then $(\Psi *_a \mu) *_a \beta = (\Psi \wedge \mu) *_a \beta = \Psi \wedge \mu \wedge \beta$ and $\Psi *_a \beta = \Psi \wedge \beta$; otherwise $(\Psi *_a \mu) *_a \beta = (\Psi \wedge \mu) *_a \beta = \beta$ and $\Psi *_a \beta = \beta$. **2)** Assume $\Psi \vdash \neg\mu$. From $\beta \vdash \mu$, if follows that $\Psi \vdash \neg\beta$. Since $\nvdash \neg\beta$ and $\beta \vdash \mu$, we have $\mu \nvdash \neg\beta$. According to (1), $(\Psi *_a \mu) *_a \beta = \mu *_a \beta = \mu \wedge \beta$ and $\Psi *_a \beta = \beta$. Therefore $*_a$ satisfies (C1).

Assume $\Psi *_a \beta \vdash \mu$. We consider two cases: **1)** Assume $\Psi \nvdash \neg\beta$. According to (1), $\Psi *_a \beta = \Psi \wedge \beta$. It follows from $\Psi \wedge \beta \vdash \mu$ and $\Psi \nvdash \neg\beta$ that we have $\Psi \wedge \mu \nvdash \neg\beta$. According to (1), $(\Psi *_a \mu) *_a \beta$ is either $\Psi \wedge \mu \wedge \beta$ or $\mu \wedge \beta$. **2)** Assume $\Psi \vdash \neg\beta$. According to (1), $\Psi *_a \beta = \beta$. From $\Psi *_a \beta \vdash \mu$ it follows that $\beta \vdash \mu$. Since $*_a$ satisfies (*1), we have $(\Psi *_a \mu) *_a \beta \vdash \mu$. Therefore $*_a$ satisfies (C3).

Assume $\Psi *_a \beta \nvdash \neg\mu$. Obviously, we have $\nvdash \neg\mu$. Consider two cases: **1)** Assume $\Psi \nvdash \neg\beta$. According to (1), $\Psi *_a \beta = \Psi \wedge \beta$. From $\Psi \wedge \beta \nvdash \neg\mu$ and $\Psi \nvdash \neg\beta$ it follows $\Psi \wedge \mu \nvdash \beta$. According to (1), $(\Psi *_a \mu) *_a \beta = \Psi \wedge \mu \wedge \beta$. From $\nvdash \neg\mu$ it follows $\Psi \wedge \mu \wedge \beta \nvdash \neg\mu$. **2)** Assume $\Psi \vdash \neg\beta$. According to (1), $\Psi *_a \beta = \beta$. Since $*_a$ satisfies (*1), we have $(\Psi *_a \mu) *_a \beta \vdash \beta$. From $\nvdash \neg\beta$ and $\beta \nvdash \neg\mu$ it follows $(\Psi *_a \mu) *_a \beta \nvdash \neg\mu$. Therefore $*_a$ satisfies (C4).

The following counterexample shows that $*_a$ violates (C2). Let $\mu$, $\beta$, and $\Psi$ be, respectively, $p$, $\neg p$, and $p \vee q$ ($p, q$ are propositional atoms). Obviously, $\beta \vdash \neg\mu$ holds. According to (1), $(\Psi *_a \mu) *_a \beta = \neg q$ and $\Psi *_a \beta = (p \vee q) \wedge \neg q$. Therefore $*_a$ violates (C2). $\square$

For the proof of the representation theorem, we need the following observation, which is a direct consequence of Theorem 2.

**Lemma 2** *Suppose that a revision operator satisfies Postulates (R*1)–(R*6). If $\nvdash \neg\beta$, then $\Psi * \beta \vdash \mu$ precisely when there exists a world $w$ such that $w \models \mu \wedge \beta$ and $w <_\Psi w'$ for any $w' \models \neg\mu \wedge \beta$, where $\leq_\Psi$ is the corresponding faithful assignment.*

**Theorem 5.** *Suppose that a revision operator satisfies Postulates (R*1)–(R*6). The operator satisfies Postulate (Ind) iff the operator and its corresponding faithful assignment satisfy:*

*(IndR)* *If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_\Psi w_2$ implies $w_1 <_{\Psi * \mu} w_2$.*

**Proof** "$\Leftarrow$": Assume $\Psi * \beta \nvdash \neg\mu$. From Lemma 2, it follows that for any world $w \models \beta \wedge \neg\mu$, there exists another world $w' \models \beta \wedge \mu$ such that $w' \leq_\Psi w$. Hence, since $\leq_\Psi$ is total, there must be a world $w_1$ such that $w_1 \models \mu \wedge \beta$ and $w_1 \leq_\Psi w_2$ for any $w_2 \models \neg\mu \wedge \beta$. Condition (IndR) then implies that $w_1 <_{\Psi*\mu} w_2$ for any $w_2 \models \neg\mu \wedge \beta$. Due to Lemma 2, we have $(\Psi*\mu)*\beta \vdash \mu$.

"$\Rightarrow$": Assume $w_1 \models \mu$, $w_2 \models \neg\mu$, and $w_1 \leq_\Psi w_2$. Let $\beta$ be such that $Mods(\beta) = \{w_1, w_2\}$. From Theorem 2 it follows that $w_1 \in Mods(\Psi*\beta)$. Hence $\Psi*\beta \nvdash \neg\mu$. Postulate (Ind) implies $(\Psi * \mu) * \beta \vdash \mu$. Due to Postulates (R*1) and (R*3), $Mods((\Psi * \mu) * \beta) = \{w_1\}$. From Theorem 2 it follows that $w_1 <_{\Psi*\mu} w_2$. □

**Proposition 3.** *Given an OCF $k$, the binary relation $\leq_k$ defined by (4) satisfies (EE1)–(EE5).*

**Proof** Due to the transitivity of $\leq$ on $\mathbb{N}$, $\leq_k$ satisfies (EE1).

Assume $\alpha \vdash \beta$. By contra-position, we have $\neg\beta \vdash \neg\alpha$. Hence, for any $w \in \mathcal{W}$, if $w \models \neg\beta$ then $w \models \neg\alpha$. According to (3), we have $k(\alpha) \leq k(\beta)$, i.e., $\alpha \leq_k \beta$. Thus $\leq_k$ satisfies (EE2).

Assume $k(\alpha) > k(\alpha \wedge \beta)$ and $k(\beta) > k(\alpha \wedge \beta)$. From (3), it follows that there exists $w$ s.t., $k(w) = k(\alpha \wedge \beta)$ and $w \models \neg\alpha \vee \neg\beta$. Since $k(\alpha) > k(\alpha \wedge \beta)$, according to (3), we have $w \not\models \neg\alpha$, i.e., $w \models \alpha$. From $w \models \neg\alpha \vee \neg\beta$, it follows that $w \models \neg\beta$. It follows from (3) that $k(\beta) \leq k(\alpha \wedge \beta)$, which contradicts $k(\beta) > k(\alpha \wedge \beta)$. Thus $\leq_k$ satisfies (EE3).

Assume $Bel(k)$ is consistent. According to (2), there exists $w_1$ s.t., $k(w_1) = 0$. From (3), it follows that $k(from(w_1)) = 0$. According to (2) and (3), $\mathrm{Bel}(k) \nvdash \alpha$ iff there exists $w$ s.t., $k(w) = 0$ and $w \models \neg\alpha$, i.e., $k(\alpha) = 0$. Since $k(from(w_1)) = 0$, we have $\mathrm{Bel}(k) \nvdash \alpha$ iff $k(\alpha) \leq k(\beta)$, for any $\beta$. Thus $\leq_k$ satisfies (EE4).

Assume $\nvdash \alpha$. According to (3), $k(\top) > k(\alpha)$. Hence, by contra-position, $\leq_k$ satisfies (EE5). □

The following lemma is needed in the proof of Theorem 7.

**Lemma 3** *Let $k$ be an OCF and $\mu$ a new evidence, then for any $m_1, m_2$,*

$$\mathrm{Bel}(k^*_{\mu,m_1}) = \mathrm{Bel}(k^*_{\mu,m_2})$$

***Proof*** According to (5), $k^*_{\mu,m}(w) = 0$ iff $w \models \mu$ and $k(w) = k(\neg\mu)$, which means the value of $m$ does not affect the set of worlds with rank 0 in the revised OCF. From (2) it follows immediately that $\mathrm{Bel}(k^*_{\mu,m_1}) = \mathrm{Bel}(k^*_{\mu,m_2})$. □

**Theorem 7.** *For any $m > 0$, reinforcement revision satisfies all KM postulates (R*1)–(R*6), where $\mathrm{Bel}(\Psi)$ and $\Psi*\mu$ are, respectively, identified with $\bigwedge \mathrm{Bel}(k)$ and $k^*_{\mu,m}$.*

**Proof** Obviously, each OCF $k$ can induce a faithful ranking $\leq_{\mathrm{Bel}(k)}$ of $\mathrm{Bel}(k)$ by letting

$$w_1 \leq_{\mathrm{Bel}(k)} w_2 \text{ iff } k(w_1) \leq k(w_2)$$

According to (5), $k^*_{\mu,m}(w) = 0$ iff $w \models \mu$ and $k(w) = k(\neg\mu)$. From (3), it is easy to see that $k^*_{\mu,m}(w) = 0$ iff $w \in \min(\mathit{Mods}(\mu), \leq_{\mathrm{Bel}(k)})$.

If we fix the value of $m$, then according to Theorem 6, reinforcement revision satisfies all KM postulates (R*1)–(R*6). From Lemma 3, it follows that satisfiability of (R*1)–(R*6) still holds for varying values of $m$. □

**Lemma 1.** *Let $k$ be an arbitrary ordinal conditional function and $\mu$ a new evidence with degree $m$, then for any non-tautological sentence $\beta$,*

$$k^*_{\mu,m}(\beta) = \begin{cases} k(\beta) + m & \textit{if } \vdash \mu \supset \beta \\ k(\mu \supset \beta) - k(\neg\mu) & \textit{else if } \ k(\mu \supset \beta) = k(\beta) \\ \min(k(\mu \supset \beta) - k(\neg\mu), k(\beta) + m) & \textit{otherwise} \end{cases}$$

**Proof** Assume $\vdash \mu \supset \beta$. From $\nvdash \beta$ and (3) it follows that there exists $w_1 \models \neg\beta$ s.t., $k(w_1) = k(\beta)$ and $k(w) \geq k(w_1)$ for any $w \models \neg\beta$. Since $\vdash \mu \supset \beta$, we have $w_1 \models \neg\mu$. According to (5), $k^*_{\mu,m}(w_1) = k(w_1) + m$. Similarly, for any $w \models \neg\beta$ we have $k^*_{\mu,m}(w) = k(w) + m$. Again according to (3) we have $k^*_{\mu,m}(\beta) = k(\beta) + m$.

Assume $\nvdash \mu \supset \beta$ and $k(\mu \supset \beta) = k(\beta)$. From $\nvdash \mu \supset \beta$ and (3) it follows that there exists $w_1 \models \mu \wedge \neg\beta$ s.t., $k(w_1) = k(\mu \supset \beta)$. According to (5), $k^*_{\mu,m}(w_1) = k(w_1) - k(\neg\mu)$. Since $k(\mu \supset \beta) = k(\beta)$, according to (3) we have $k(w) \geq k(w_1)$ for any $w \models \neg\beta$. It follows from (5) that for any $w \models \neg\beta$, $k^*_{\mu,m}(w)$ is either $k(w) - k(\neg\mu)$ or $k(w) + m$. Therefore, according to (3) we have $k^*_{\mu,m}(\beta) = k(\beta) - k(\neg\mu)$.

Assume $\nvdash \mu \supset \beta$ and $k(\mu \supset \beta) \neq k(\beta)$. It is not difficult to see, according to (3), that this is possible only if $k(\mu \supset \beta) > k(\beta)$. From $\nvdash \mu \supset \beta$ and (3), it follows that there exists $w_1 \models \mu \wedge \neg\beta$ s.t., $k(w_1) = k(\mu \supset \beta)$ and $k(w) \geq k(w_1)$ for any $w \models \mu \wedge \neg\beta$. Analogously, there exists $w_2 \models \neg\beta$ s.t., $k(w_2) = k(\beta)$ and $k(w) \geq k(w_2)$ for any $w \models \neg\beta$. Consider two cases: **1)** Assume $k(\mu \supset \beta) - k(\neg\mu) \leq k(\beta) + m$. According to (5), $k^*_{\mu,m}(w_1) = k(w_1) - k(\neg\mu)$. For any $w \models \neg\beta$, according to (5), if $w \models \mu$, then $k^*_{\mu,m}(w) = k(w) - k(\neg\mu) \geq k(w_1) - k(\neg\mu)$; otherwise $k^*_{\mu,m}(w) = k(w) + m \geq k(w_2) + m \geq k(w_1) - k(\neg\mu)$. From (3) it follows that $k^*_{\mu,m}(\beta) = k(\mu \supset \beta) - k(\neg\mu)$. **2)** Assume $k(\mu \supset \beta) - k(\neg\mu) > k(\beta) + m$. Since $k(\mu \supset \beta) > k(\beta)$, according to (3), we have $w_2 \models \neg\mu$. From (5) it follows that $k^*_{\mu,m}(w_2) = k(w_2) + m$. For any $w \models \neg\beta$, according to (5), if $w \models \mu$, then $k^*_{\mu,m}(w) = k(w) - k(\neg\mu) \geq k(w_1) - k(\neg\mu) > k(w_2) + m$; otherwise

$k^*_{\mu,m}(w) = k(w)+m \geq k(w_2)+m$. From (3), it follows that $k^*_{\mu,m}(\beta) = k(\beta)+m$. Therefore, $k^*_{\mu,m}(\beta) = min(k(\mu \supset \beta) - k(\neg\mu), k(\beta) + m)$. □

**Proposition 4.** *Let $k$ be an arbitrary OCF and $\mu$ a new non-tautological evidence with degree $m$, then*

$$k^*_{\mu,m}(\mu) = k(\mu) + m$$

**Proof** This is a direct consequence of Lemma 1. □

**Theorem 8.** *For arbitrary $m_1, m_2 > 0$, reinforcement revision satisfies the following conditions:*[8]

| | |
|---|---|
| *(EC1)* | *If $\alpha \vdash \mu$, then $(k^*_{\mu,m_1})^*_{\alpha,m_2} \equiv k^*_{\alpha,m_2}$.* |
| *(EC2)* | *If $\alpha \vdash \neg\mu$, then $(k^*_{\mu,m_1})^*_{\alpha,m_2} \equiv k^*_{\alpha,m_2}$.* |
| *(EInd)* | *If there exists $m$ such that $k^*_{\neg\beta,m} \nvdash \neg\mu$, then $(k^*_{\mu,m_1})^*_{\neg\beta,m_2} \vdash \mu$.* |

**Proof** If $\vdash \neg\alpha$, Condition (EC1) holds trivially. Assume that $\alpha \vdash \mu$ and $\nvdash \neg\alpha$. By (5),

$$k^*_{\alpha,m_2}(w) = 0 \text{ iff } w \models \alpha \text{ and } k(w) = k(\neg\alpha) \tag{7}$$

Likewise,

$$(k^*_{\mu,m_1})^*_{\alpha,m_2}(w) = 0 \text{ iff } w \models \alpha \text{ and } k^*_{\mu,m_1}(w) = k^*_{\mu,m_1}(\neg\alpha) \tag{8}$$

Since $\alpha \vdash \mu$, for any $w \models \alpha$ we have $k^*_{\mu,m_1}(w) = k(w)-k(\neg\mu)$ by (5). Since $\mu \supset \neg\alpha \equiv \neg\alpha$ and $\nvdash \neg\alpha$, it follows from Lemma 1 that $k^*_{\mu,m_1}(\neg\alpha) = k(\neg\alpha)-k(\neg\mu)$. Hence, (8) is equivalent to

$$(k^*_{\mu,m_1})^*_{\alpha,m_2}(w) = 0 \text{ iff } w \models \alpha \text{ and } k(w) = k(\neg\alpha)$$

This and (7) implies $(k^*_{\mu,m_1})^*_{\alpha,m_2} \equiv k^*_{\alpha,m_2}$. Condition (EC2) can be proved analogously.

We prove Condition (EInd) by contradiction. To begin with, from the assumption that $k^*_{\neg\beta,m} \nvdash \neg\mu$ it follows that $\nvdash \beta$ and $\nvdash \mu \supset \beta$. Furthermore, there exists $w$ such that $k^*_{\neg\beta,m}(w) = 0$, $w \models \neg\beta \wedge \mu$, and $k(w) = k(\beta)$. With the help of (3), this implies $k(\beta) = k(\mu \supset \beta)$.

Now assume that $(k^*_{\mu,m_1})^*_{\neg\beta,m_2} \nvdash \mu$. It follows that there exists $w'$ such that $(k^*_{\mu,m_1})^*_{\neg\beta,m_2}(w') = 0$, $w' \models \neg\beta \wedge \neg\mu$, and $k^*_{\mu,m_1}(w') = k^*_{\mu,m_1}(\beta)$. Since $k(w) =$

[8] Note that, as before, we abuse notation by simply writing $k^*_{\mu,m}$ instead of $\bigwedge \mathrm{Bel}(k^*_{\mu,m})$ etc.

$k(\beta)$ and $w' \models \neg\beta$, we have $k(w') \geq k(w)$. Hence by (5), $k^*_{\mu,m_1}(w') = k(w') + m_1 > k(\beta)$. But from Lemma 1 it follows that $k^*_{\mu,m_1}(\beta) \leq k(\beta)$, since $\nvdash \beta$ and $\nvdash \mu \supset \beta$. This contradicts $k^*_{\mu,m_1}(w') = k^*_{\mu,m_1}(\beta)$. □

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