COMP2521 26T1
Graphs (V)
Digraph Algorithms

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digraph traversal
transitive closure

Directed Graphs (Digraphs)

Reminder: directed graphs are graphs where...

- Each edge (v, w) has a source v and a destination w
- Unlike undirected graphs, $v \rightarrow w \neq w \rightarrow v$

Digraph Applications

application	vertex is...	edge is...
WWW	web page	hyperlink
chess	board state	legal move
scheduling	task	precedence
program	function	function call
journals	article	citation
make	target	dependency

Digraph Traversal

Same as for undirected graphs:

```
bfs(G, src):
    initialise visited array
    mark src as visited
    enqueue src into Q
    while Q is not empty:
        v = dequeue from Q
        for each edge (v,w) in G:
            if w has not been visited:
                mark w as visited
                enqueue w into Q
```

```
dfs(G, src):
    initialise visited array
    dfsRec(G, src, visited)

dfsRec(G, v, visited):
    mark v as visited
    for each edge (v,w) in G:
        if w has not been visited:
            dfsRec(G, w, visited)
```

Digraph Traversal

Application - Web Crawling

Web crawling

Visit a subset of the web...
...to index
...to cache locally

Which traversal method? BFS or DFS?

Note: we can't use a visited array, as we don't know how many webpages there are. Instead, use a visited set.

Digraph Traversal

Application - Web Crawling

Web crawling algorithm:

```
webCrawl(startingUrl, maxPagesToVisit):
    create visited set
    add startingUrl to visited set
    enqueue startingUrl into Q

    numPagesVisited = 0
    while Q is not empty and numPagesVisited < maxPagesToVisit:
        currPage = dequeue from Q

        visit currPage
        numPagesVisited = numPagesVisited + 1

        for each hyperlink on currPage:
            if hyperlink not in visited set:
                add hyperlink to visited set
                enqueue hyperlink into Q
```

Digraph Traversal

Application - Web Crawling

Wiki Game

Given two Wikipedia articles,
navigate from the first article to the second
in as few clicks as possible.

Transitive Closure

Problem: computing reachability

Given a digraph G it is potentially useful to know:

- Is vertex t reachable from vertex s?

Transitive Closure

One way to implement a reachability check:

- Use BFS or DFS starting at s
  - This is $O(V + E)$ in the worst case
  - Only feasible if reachability is an infrequent operation

What about applications that frequently need to check reachability?

Transitive Closure

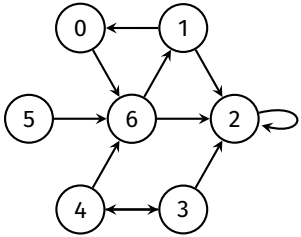
Idea

Construct a $V \times V$ matrix
that tells us whether there is a path (not edge)
from s to t, for $s, t \in V$

This matrix is called the transitive closure (tc) matrix
(or reachability matrix)

`tc[`s`][`t`]` is true if there is a path from s to t, false otherwise

Transitive Closure



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[1]	1	0	1	0	0	0	0
[2]	0	0	1	0	0	0	0
[3]	0	0	1	0	1	0	0
[4]	0	0	0	1	0	0	1
[5]	0	0	0	0	0	0	1
[6]	0	1	1	0	0	0	0

adjacency matrix

	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	1	1	1	0	0	0	1
[1]	1	1	1	0	0	0	1
[2]	0	0	1	0	0	0	0
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[5]	1	1	1	0	0	0	1
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reachability matrix

Transitive Closure

One way to compute reachability matrix:

- Perform BFS/DFS from every vertex

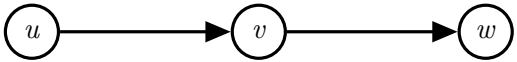
Another way $\Rightarrow$ Warshall's algorithm:

- Simple algorithm that does not require a graph traversal

Warshall's Algorithm

Warshall's algorithm
uses transitivity to compute reachability

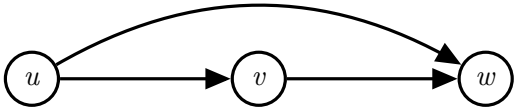
If there is a path from u to v,
and a path from v to w...



Warshall's Algorithm

Warshall's algorithm
uses transitivity to compute reachability

If there is a path from u to v,
and a path from v to w...
then there is a path from u to w



Warshall's Algorithm

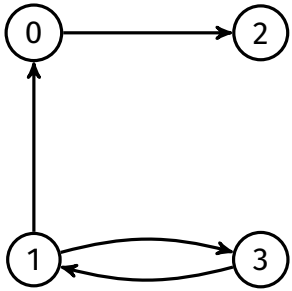
Idea of Warshall's algorithm:

- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0 only, or
  - There is a path from s to t via vertex 0 and/or 1 only, or
  - There is a path from s to t via vertex 0, 1 and/or 2 only, or
  - ...
  - There is a path from s to t via any of the vertices

Warshall's Algorithm

Example:

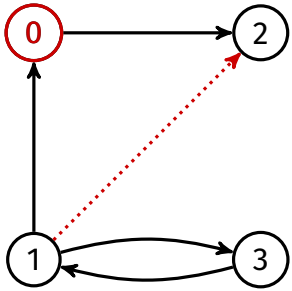
- There is a path from s to t if:
  - There is an edge from s to t, or



Warshall's Algorithm

Example:

- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0 only, or



Warshall's Algorithm

Example:

- There is a path from s to t if:
  - There is an edge from s to t, or
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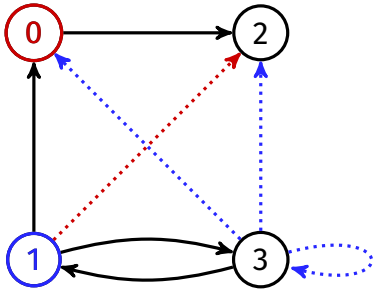
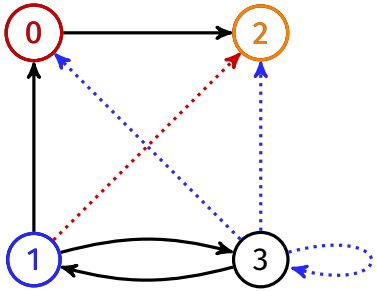


Warshall's Algorithm

Example:

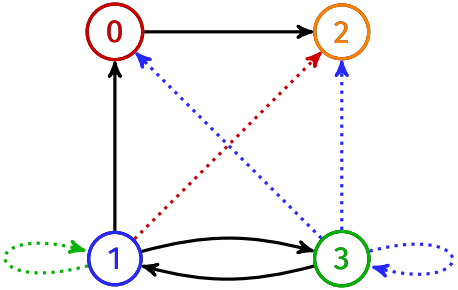
- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0 only, or
  - There is a path from s to t via vertex 0 and/or 1 only, or
  - There is a path from s to t via vertex 0, 1 and/or 2 only, or



Warshall's Algorithm

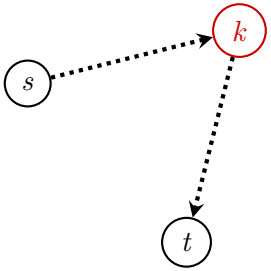
Example:

- There is a path from s to t if:
  - There is an edge from s to t, or
  - There is a path from s to t via vertex 0 only, or
  - There is a path from s to t via vertex 0 and/or 1 only, or
  - There is a path from s to t via vertex 0, 1 and/or 2 only, or
  - There is a path from s to t via vertex 0, 1, 2 and/or 3



Warshall's Algorithm

On the k-th iteration, the algorithm determines if a path exists between two vertices s and t using just 0, ... , k as intermediate vertices

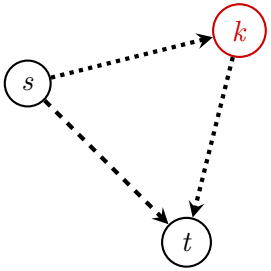


On the k-th iteration

If we have:
(1) a path from s to k
(2) a path from k to t
(using only vertices 0 to $k-1$)

Warshall's Algorithm

On the k-th iteration, the algorithm determines if a path exists between two vertices s and t using just 0, ... , k as intermediate vertices



On the k-th iteration

If we have:
(1) a path from s to k
(2) a path from k to t
(using only vertices 0 to $k-1$)

Then we have a path from s to t
using vertices from 0 to k

```
if tc[s][k] and tc[k][t]:
    tc[s][t] = true
```

Warshall's Algorithm

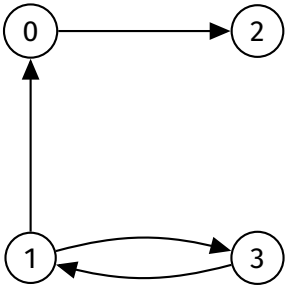
Pseudocode


```
warshall(A):
    Input:  n × n adjacency matrix A
    Output: n × n reachability matrix

    create tc matrix which is a copy of A

    for each vertex k in G: // from 0 to n - 1
        for each vertex s in G:
            for each vertex t in G:
                if tc[s][k] and tc[k][t]:
                    tc[s][t] = true

    return tc
```

Warshall's Algorithm

Example

Find transitive closure of this graph



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[0]	0	0	1	0
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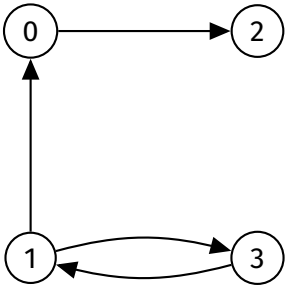
Warshall's Algorithm

Example

Initialise tc with edges of original graph



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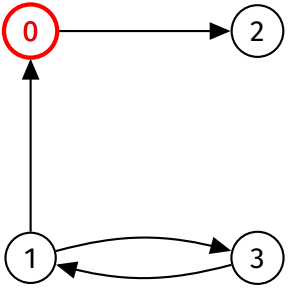
Warshall's Algorithm

Example

First iteration: $k = 0$



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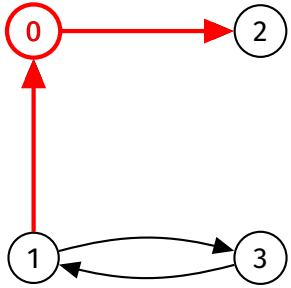
Warshall's Algorithm

Example

First iteration: $k = 0$
There is a path $1 \to 0$ and a path $0 \to 2$



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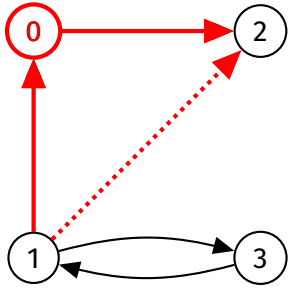
[3]	0	1	0	0

Warshall's Algorithm

Example

First iteration: $k = 0$
There is a path $1 \to 0$ and a path $0 \to 2$
So there is a path $1 \to 2$



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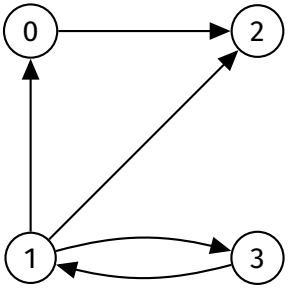
Warshall's Algorithm

Example

First iteration: $k = 0$
Done



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[0]	0	0	1	0
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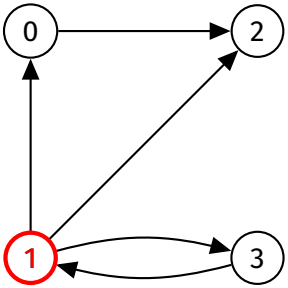
Warshall's Algorithm

Example

Second iteration: $k = 1$



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[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
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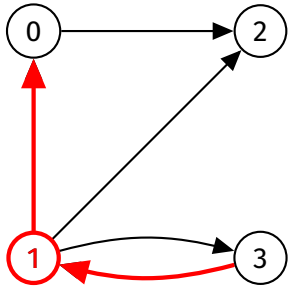
Warshall's Algorithm

Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 0$



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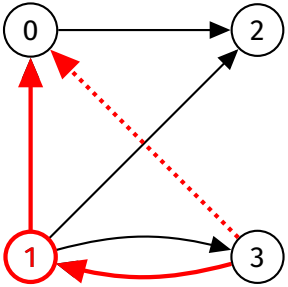
Warshall's Algorithm

Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 0$
So there is a path $3 \to 0$



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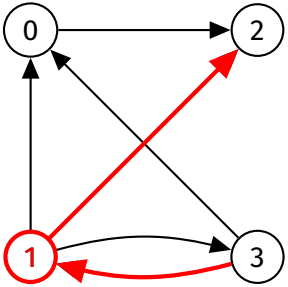
Warshall's Algorithm

Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 2$



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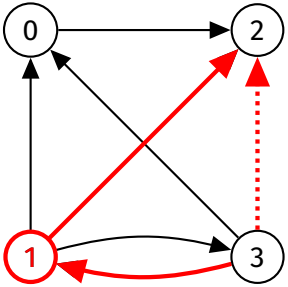
[2]	0	0	0	0
[3]	1	1	0	0

Warshall's Algorithm

Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 2$
So there is a path $3 \to 2$



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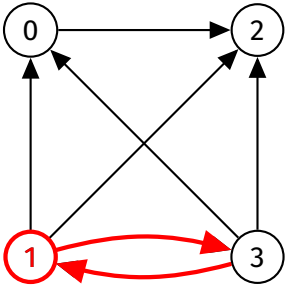
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	0

Warshall's Algorithm

Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 3$



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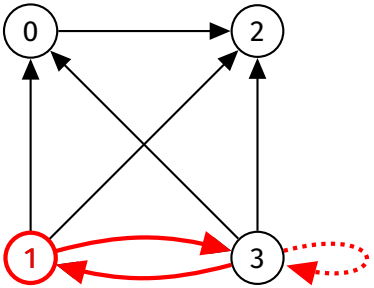
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	0

Warshall's Algorithm

Example

Second iteration: $k = 1$
There is a path $3 \to 1$ and a path $1 \to 3$
So there is a path $3 \to 3$



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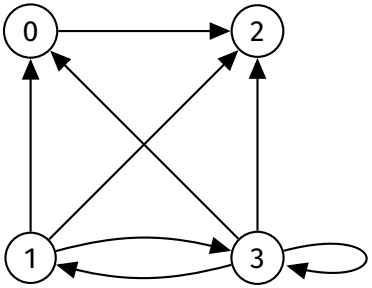
Warshall's Algorithm

Example

Second iteration: $k = 1$
Done



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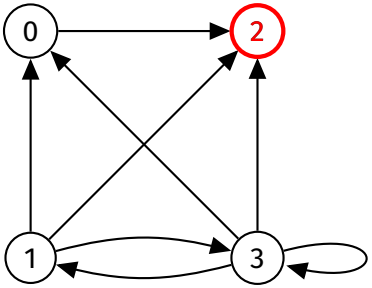
Warshall's Algorithm

Example

Third iteration: $k = 2$

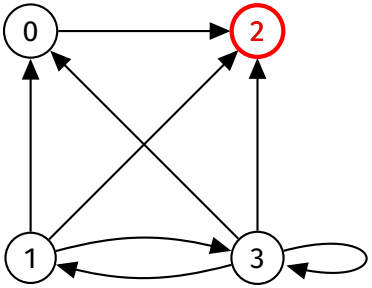


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Example

Third iteration: $k = 2$
No pairs (s, t) such that there are paths $s \to 2$ and $2 \to t$



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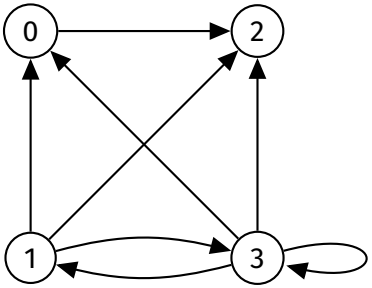
Warshall's Algorithm

Example

Third iteration: $k = 2$
Done



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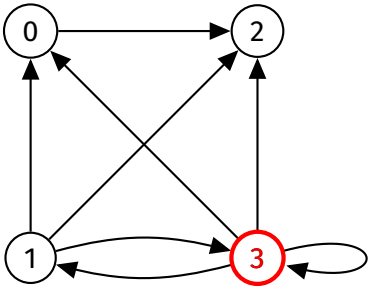
Warshall's Algorithm

Example

Fourth iteration: $k = 3$



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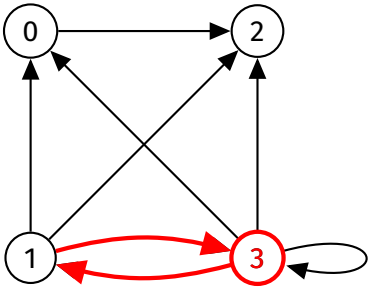
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Example

Fourth iteration: $k = 3$
There is a path $1 \to 3$ and a path $3 \to 1$



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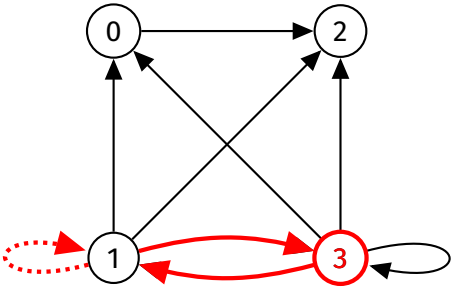
[0]	0	0	1	0
[1]	1	0	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Example

Fourth iteration: $k = 3$
There is a path $1 \to 3$ and a path $3 \to 1$
So there is a path $1 \to 1$

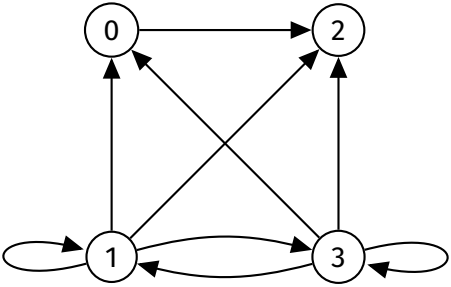


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	1	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Example

Fourth iteration: $k = 3$
Done

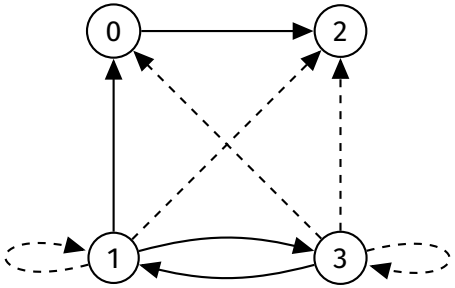


	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	1	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Example

Finished



	[0]	[1]	[2]	[3]
[0]	0	0	1	0
[1]	1	1	1	1
[2]	0	0	0	0
[3]	1	1	1	1

Warshall's Algorithm

Analysis

Analysis:

- Time complexity: $O(V^3)$
  - Three nested loops iterating over all vertices
- Space complexity: $O(V^2)$
  - Can be $O(1)$ if overwriting the input matrix
- Benefit: checking reachability between vertices is now $O(1)$
  - Makes up for slow setup ($O(V^3)$) if reachability is a very frequent operation

Other Algorithms

- Topological sorting
- Strongly connected components:
  - Kosaraju's algorithm
  - Tarjan's algorithm

Feedback

https://forms.office.com/r/ucsJ4r99Ag