

# Havannah and TwixT are PSPACE-complete

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**Abstract.** Numerous popular abstract strategy games ranging from HEX and HAVANNAH to LINES OF ACTION belong to the class of connection games. Still, very few complexity results on such games have been obtained since HEX was proved PSPACE-complete in the early eighties.

We study the complexity of two connection games among the most widely played. Namely, we prove that HAVANNAH and TWIXT are PSPACE-complete.

The proof for HAVANNAH involves a reduction from GENERALIZED GEOGRAPHY and is based solely on ring-threats to represent the input graph. On the other hand, the reduction for TWIXT builds up on previous work as it is a straightforward encoding of HEX.

**Keywords:** Complexity, Connection game, Havannah, TwixT, Generalized Geography, Hex, PSPACE

## 1 Introduction

A connection game is a kind of abstract strategy game in which players try to make a specific type of connection with their pieces [5]. In many connection games, the goal is to connect two opposite sides of a board. In these games, players take turns placing or/and moving pieces until they connect the two sides of the board. HEX, TWIXT, and SLITHER are typical examples of this type of game. However, a connection game can also involve completing a loop (HAVANNAH) or connecting all the pieces of a color (LINES OF ACTION).

A typical process in studying an abstract strategy game, and in particular a connection game, is to develop an artificial player for it by adapting standard techniques from the game search literature, in particular the classical Alpha-Beta algorithm [1] or the more recent Monte Carlo Tree Search paradigm [6, 2]. These algorithms explore an exponentially large game tree are meaningful when optimal polynomial time algorithms are impossible or unlikely. For instance, tree search algorithms would not be used for NIM and SHANNON'S EDGE SWITCHING GAME which can be played optimally and solved in polynomial time [7].

The complexity class PSPACE comprizes those problems that can be solved on a Turing machine using an amount of space polynomial in the size of the input. The prototypical example of a PSPACE-complete problem is the Quantified Boolean Formula problem (QBF) which can be seen as a generalization of SAT allowing for

variables to be both existentially and universally quantified. Proving that a game is PSPACE-hard shows that a variety of intricate problems can be encoded via positions of this game. Additionally, it is widely believed in complexity theory that if a problem is PSPACE-hard, then it admits no polynomial time algorithms.

For this reason, studying the computational complexity of games is a popular research topic. The complexity class of CHESS and GO was determined shortly after the very definition of these classes and other popular games have been classified since then [8, 12]. More recently, we studied the complexity of trick taking card games which notably include BRIDGE, SKAT, TAROT, and WHIST [4].

Connection games have received less attention. Besides Even and Tarjan's proof that SHANNON'S VERTEX SWITCHING GAME is PSPACE-complete [9] and Reisch's proof that HEX is PSPACE-complete [20], the only complexity results on connection games that we know of are the PSPACE-completeness of virtual connection detection [14] in HEX, the NP-completeness of dominated cell detection in SHANNON'S VERTEX SWITCHING GAME [3], as well as an unpublished note showing that a problem related to TWIXT is NP-complete [18].<sup>1</sup>

The two games that we study in this paper rank among the most notable connection games. They were the main topic of multiple master's theses and research articles [13, 18, 19, 23, 22, 16, 10], and they both gave rise to competitive play. High-level online competitive play takes place on [www.littlegolem.net](http://www.littlegolem.net). Finally, live competitive play can also be observed between human players at the Mind Sports Olympiads where an international TWIXT championship has been organized every year since 1997, as well as between HAVANNAH computer players at the ICGA Computer Olympiad since 2009.<sup>2</sup>

## 2 Havannah

HAVANNAH is a 2-player connection game played on a hexagonal board paved by hexagons. White and Black place a stone of their color in turn in an unoccupied cell. Stones cannot be taken, moved nor removed. Two cells are neighbors if they share an edge. A group is a connected component of stones of the same color via the neighbor relation. A player wins if they realize one of the three following different structures: a circular group, called *ring*, with at least one cell, possibly empty, inside; a group linking two corners of the board, called *bridge*; or a group linking three edges of the board, called *fork*.

As the length of a game of HAVANNAH is polynomially bounded, exploring the whole game tree can be done with polynomial space, so HAVANNAH is in PSPACE.

In our reduction, the HAVANNAH board is large enough that the gadgets are far from the edges and the corners. Additionally, the gadgets feature ring threats that are short enough that the bridges and forks winning conditions do not have any influence. Before starting the reduction, we define threats and make two observations that will prove useful in the course of the reduction.

<sup>1</sup> For a summary in English of Reisch's reduction, see Maarup's thesis [17].

<sup>2</sup> See [www.boardability.com/game.php?id=twixt](http://www.boardability.com/game.php?id=twixt) and [www.grappa.univ-lille3.fr/icga/game.php?id=37](http://www.grappa.univ-lille3.fr/icga/game.php?id=37) for details.

A *simple threat* is defined as a move which threatens to realize a ring on the next move on a unique cell. There are only two kinds of answers to a simple threat: either win on the spot or defend by placing a stone in the cell creating this very threat. A *double threat* is defined as a move which threatens to realize a ring on the next move on at least two different cells. We will use *threat* as a generic term to encompass both simple and double threats. A *winning sequence of threats* is defined as a sequence of simple threats ended by a double threat for one player such that the opponent's forced move never makes a threat. Thus, when a player is not threatened and can initiate a winning sequence of threats, they do win. To be more concise, we will denote by  $W : a_1, a_2; a_3, a_4; \dots; a_{2n-1}, a_{2n}$  the sequence of moves starting with White's move  $a_1$ , Black's answer  $a_2$ , and so on.  $a_{2n}$  is optional, for the last move of the sequence might be White's or Black's. Similarly,  $B : a_1, a_2; a_3, a_4; \dots; a_{2n-1}, a_{2n}$  denotes the corresponding sequence of moves initiating by Black. We will use the following lemmas multiple times:

**Lemma 1.** *If a player is not threatened, playing a simple threat forces the opponent to answer on the cell of the threat.*

**Lemma 2.** *If a player is not threatened, playing a double threat is winning.*

## 2.1 Generalized Geography.

GENERALIZED GEOGRAPHY (GG) is one of the first two-player games to have been proved PSPACE-complete [21]. It has been used to reduce to multiple games including HEX, OTHELLO, and AMAZONS [20, 11, 8].

In GG, players take turns moving a token from vertex to vertex. If the token is on a vertex  $v$ , then it can be moved to a vertex  $v'$  neighboring  $v$  provided  $v'$  has not been visited yet. A player wins when it is their opponent's turn and the oppoent has no legal moves. An instance of GG is a graph  $G$  and an initial vertex  $v_0$ , and asks whether the first player has a winning strategy in the corresponding game.

We denote by  $P(v)$  the set of predecessors of the vertex  $v$  in  $G$ , and  $S(v)$  the set of successors of  $v$ . A vertex with in-degree  $i$  and out-degree  $o$  is called  $(i, o)$ -vertex. The degree of a vertex is the sum of the in-degree and the out-degree, and the degree of  $G$  is the maximal degree among all vertices of  $G$ . If  $V$  is the set of vertices of  $G$  and  $V'$  is a subset of vertices, then  $G[V \setminus V']$  is the induced subgraph of  $G$  where vertices belonging to  $V'$  have been removed.

Lichtenstein and Sipser have proved that the game remained PSPACE-hard even if  $G$  was assumed to be bipartite and of degree at most 3 [15]. We will reduce from such a restriction of GG to show that HAVANNAH is PSPACE-hard. To limit the number of gadgets we need to create, we will also assume a few simplifications detailed below. An example of a simplified instance of GG can be found in Fig. 1.

Let  $(G, v_0)$  be an instance of GG with  $G$  bipartite and of degree at most 3. We can assume that there is no vertex  $v$  with out-degree 0 in  $G$ . Indeed, if  $v_0 \in P(v)$  then  $(G, v_0)$  is trivially winning for Player 1. Else,  $(G[V \setminus (\{v\} \cup P(v))], v_0)$  is an equivalent instance, since playing in a predecessor of  $v$  is losing.

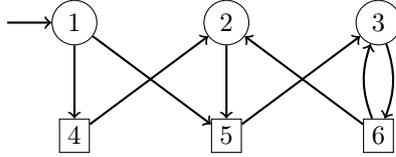


Fig. 1: Example of an instance of GG with vertex 1 as initial vertex.

All edges coming to the initial vertex  $v_0$  can be removed to form an equivalent instance. So,  $v_0$  is a  $(0, 1)$ -, a  $(0, 2)$ -, or a  $(0, 3)$ -vertex. If  $S(v_0) = \{v'\}$ , then  $(G[V \setminus \{v_0\}], v')$  is a strictly smaller instance such that Player 1 is winning in  $(G, v_0)$  if and only if Player 1 is losing in  $(G[V \setminus \{v_0\}], v')$ . If  $S(v_0) = \{v', v'', v'''\}$ , then Player 1 is winning in  $(G, v_0)$  if and only if Player 1 is losing in at least one of the three instances  $(G[V \setminus \{v_0\}], v')$ ,  $(G[V \setminus \{v_0\}], v'')$ , and  $(G[V \setminus \{v_0\}], v''')$ . In those three instances  $v'$ ,  $v''$ , and  $v'''$  are not  $(0, 3)$ -vertices since they had in-degree at least 1 in  $G$ . Therefore, we can also assume that  $v_0$  is  $(0, 2)$ -vertex.

We call an instance with an initial  $(0, 2)$ -vertex and then only  $(1, 1)$ -,  $(1, 2)$ -, and  $(2, 1)$ -vertices a *simplified* instance.

In the following subsections we propose gadgets that encode the different parts of a *simplified* instance of GG. These gadgets have starting points and ending points. The gadgets are assembled so that the ending point of a gadget coincides with the starting point of the next one. The resulting instance of HAVANNAH is such that both players must enter in the gadgets by a starting point and leave it by an ending point otherwise they lose.

## 2.2 Edge gadgets.

Wires, curves, and crossroads will enable us to encode the edges of the input graph. In the representation of the gadgets, White and Black stones form the proper gadget. Dashed stones and gray stones are respectively White and Black stones setting the context.

In the HAVANNAH board we name the 6 directions: North, North-West, South-West, South, South-East, and North-East according to standard designation. While figures and lemmas are mostly presented from White's point of view, all the gadgets and lemmas work exactly the same way with colors reversed.

*The wire gadget.* Basically, a wire teleports moves: one player plays in a cell  $u$  and their opponent has to answer in a possibly remote cell  $v$ .  $u$  is called the starting point of the wire and  $v$  is called its ending point. A wire where White prepares a threat and Black answers is called a WB-wire (Fig. 2a); conversely, we also have BW-wires. We say that WB-wires and BW-wires are *opposite* wires. Note that wires can be of arbitrary length and can be curved with  $120^\circ$  angles (Fig. 2b). On an empty board, a wire can link any pair of cells as starting and ending point provided they are far enough from each other.

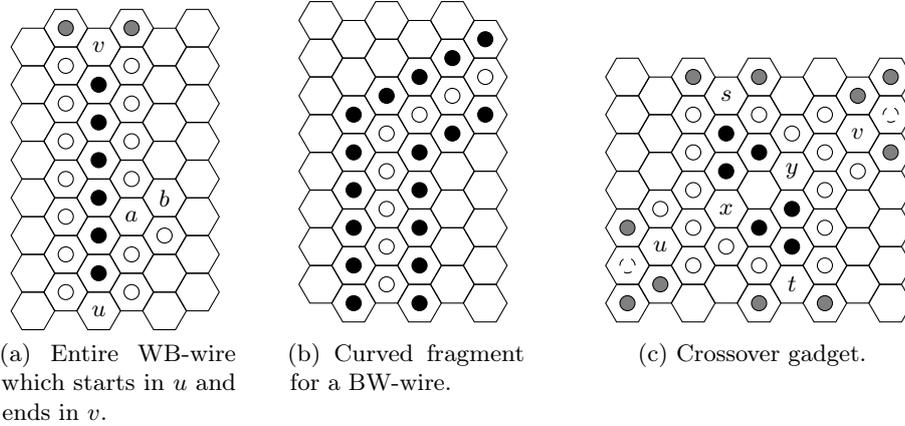


Fig. 2: Edge gadgets.

**Lemma 3.** *If White plays in the starting point  $u$  of a WB-wire (Fig. 2a), and Black does not answer by a threat, Black is forced to play in the ending point  $v$  (possibly with moves at  $a$  and  $b$  interleaved).*

*The crossover gadget.* The input graph of GG might not be planar, so we have to design a crossover gadget to enable two chains of wires to cross. Fig. 2c displays a crossover gadget, we have a South-West BW-wire with starting point  $u$  which is linked to a North-East BW-wire with ending point  $v$ , and a North BW-wire with starting point  $s$  is linked to a South BW-wire with ending point  $t$ .

**Lemma 4.** *In a crossover gadget (Fig. 2c), if White plays in the starting point  $u$ , Black ends up playing in the ending point  $v$  and if White plays in the starting point  $v$  and if White plays in the starting point  $s$ , Black ends up playing in the ending point  $t$ .*

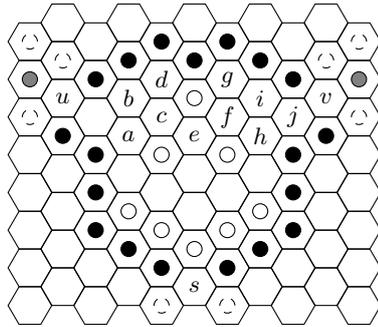
*Proof.* By Lemma 1, if White plays in  $u$ , Black has to play in  $x$ , forcing White to play in  $y$ , forcing finally Black to play in  $v$ . If White plays in  $s$ , again by Lemma 1, Black has to play in  $t$ .  $\square$

Note that the South wire is linked to the North wire irrespective of whether the other pair of wires has been used and conversely. That is, in a crossover gadget two paths are completely independent.

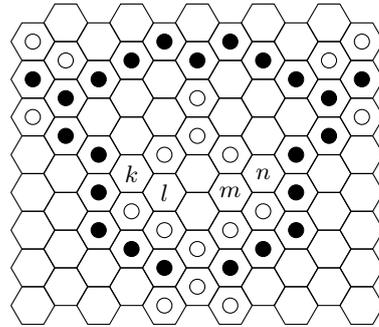
### 2.3 Vertex gadgets.

We now describe the gadgets encoding the vertices. Recall from Section 2.1 that simplified GG instances only feature  $(1, 2)$ -,  $(1, 1)$ -, and  $(2, 1)$ -vertices, and a  $(0, 2)$ -vertex. One can encode a  $(1, 1)$ -vertex with two consecutive opposite wires. Thus, we will only present three vertex gadgets, one for  $(2, 1)$ -vertices, one for  $(1, 2)$ -vertices, and one for the  $(0, 2)$ -vertex.

*The (2,1)-vertex gadget.* A (2,1)-vertex gadget receives two wire ending points. If a stone is played on either of those ending points, it should force an answer in the starting point of a third wire. That simulates a vertex with two edges going in and one edge going out.



(a) Gadget before being used. The wires for the in-edges end in  $u$  and  $v$ , the wire for the out-edge starts in  $s$ .



(b) Gadget after being used and then reentered. White wins with a double threat.

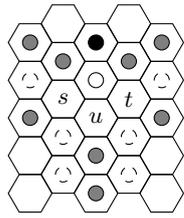
Fig. 3: The (2,1)-vertex gadget links three WB-wires. The North-West and North-East ones end in  $u$  and  $v$ , the South WB-wire starts in  $s$ .

**Lemma 5.** *If Black plays in one of the two possible starting points  $u$  and  $v$  of a (2,1)-vertex gadget (Fig. 3b), and White does not answer by a threat, White is forced to play in the ending point  $s$ .*

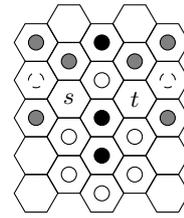
*Proof.* Assume Black plays in  $u$  and White answers by a move which is not in  $s$  nor a threat. This move from White has to be either in  $v$  or in  $j$ , otherwise, Black has a double threat by playing in  $s$  and wins by Lemma 2. Suppose White plays in  $v$ . Now, Black plays in  $s$  with a simple threat in  $j$ , so White has to play in  $j$  by Lemma 1. Then Black has the following winning sequence: B:  $a,b$ ;  $c,d$ ;  $h,i$ ;  $f$ . Black has now a double threat in  $g$  and  $e$  and so wins by Lemma 2. If White plays in  $j$  instead of  $v$ , the argument is similar.

If Black plays in  $v$ , the proof that White has to play in  $s$  is similar.  $\square$

*The (1,2)-vertex and (0,2)-vertex gadgets.* A (1,2)-vertex gadget receives one ending point of a wire (Fig. 4a). If a stone is played on this ending point, it should offer the choice to defend either by playing in the starting point of a second wire, or by playing in the starting point of a third wire. That simulates a vertex with one edge going in and two edges going out. The (0,2)-vertex gadget (or *starting-vertex* gadget) can be seen as a (1,2)-vertex gadget where a stone has already been played on the ending point of the in-edge. The (0,2)-vertex gadget represents the two possible choices of the first player at the beginning of the game.



(a) The (1,2)-vertex gadget. The wire for the in-edge ends in  $u$ .



(b) The (0,2)-vertex gadget representing the starting vertex  $v_0$ .

Fig. 4: In these choice gadgets, White can defend by playing in  $s$  or in  $t$ . A North-West BW-wire starts in  $s$  and a North-East BW-wire starts in  $t$ .

**Lemma 6.** *If Black plays in the starting point  $u$  of a (1,2)-vertex gadget (Fig. 4a), and White does not play a threat, White is forced to play in one of the two ending points  $s$  and  $t$ . Then, if Black does not answer by a threat, they have to play in the other ending point.*

*Proof.* Black plays in  $u$ . Suppose White plays neither in  $s$  nor in  $t$  nor a threatening move. Then Black plays in  $s$ . By Lemma 1, White has to play in  $t$  but Black wins by playing in the ending point of the wire starting at  $s$  by Lemma 3.

Assume White's answer to  $u$  is to play in  $s$ .  $t$  can now be seen as the ending point of the in-wire, so Black needs to play in  $t$  or make a threat by Lemma 3.  $\square$

**Corollary 1.** *If White is forced to play a threat or to open the game in one of the two opening points  $s$  and  $t$  of the (0,2)-vertex gadget (Fig. 4b). Then, if Black does not play a threat, they are forced to play in the other opening point.*

## 2.4 Assembling the gadgets together.

Let  $(G, v_0)$  be a simplified instance of GG, and  $n$  be its number of vertices.  $G$  being bipartite, we denote by  $V_1$  the side of the partition containing  $v_0$ , and  $V_2$  the other side. Player 1 moves the token from vertices of  $V_1$  to vertices of  $V_2$  and player 2 moves the token from  $V_2$  to  $V_1$ . We denote by  $\phi$  the reduction from GG to HAVANNAH. Let us describe the construction of  $\phi((G, v_0))$ . As an example, we provide the reduction from the GG instance from Fig. 1 in Fig. 5.

The initial vertex  $v_0$  is encoded by the gadget displayed in Fig. 4b. Each player 1's (2,1)-vertex is encoded by the (2,1)-vertex gadget of Fig. 3a, and each player 2's (2,1)-vertex is encoded by the same gadget in reverse color. Each player 1's (1,2)-vertex is encoded by the (1,2)-vertex gadget of Fig. 4a, and each player 2's (1,2)-vertex is encoded by the same gadget in reverse color.

All White's vertex gadgets are aligned and all Black's vertex gadgets are aligned on a parallel line. Whenever  $(u, v)$  is an edge in  $G$ , we connect an exit of the vertex gadget representing  $u$  to an entrance of a gadget encoding  $v$  using wires and crossover gadgets. Let  $n$  be the number of vertices in  $G$ , since  $G$  is of

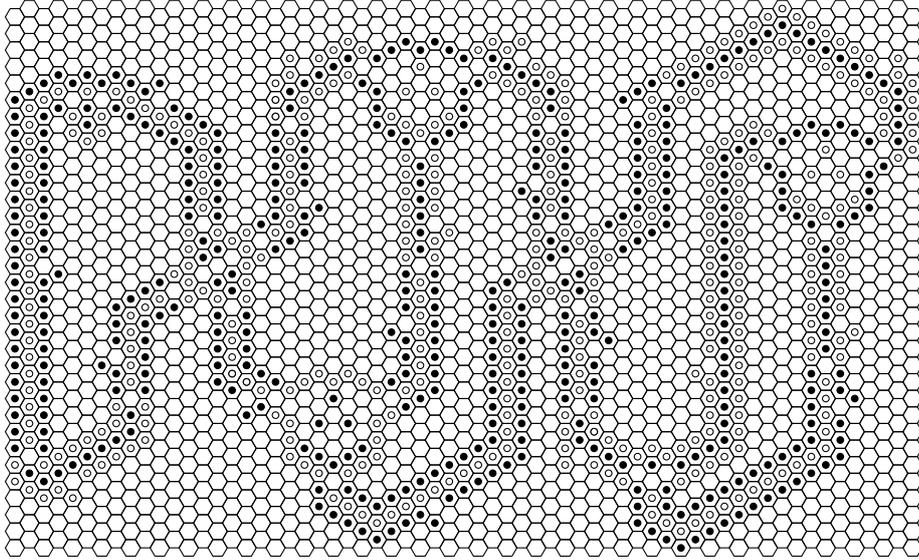


Fig. 5: HAVANNAH gadgets representing the GG instance from Fig. 1.

degree 3, we know that the number of edges is at most  $3n/2$ . The minimal size in terms of HAVANNAH cells for a smallest wire and the size of a crossover are constants. Therefore the distance between Black's line and White's line is linear in  $n$ . Note that, two wires of opposite colors might be needed to connect two vertex gadgets or a vertex gadget and a crossover. Similarly, we can show that the distance between two vertices on Black's line or on White's line is constant.

**Lemma 7.** *If Black reenters a White's (2,1)-vertex gadget (Fig. 3b), and Black has no winning sequence of threats elsewhere, White wins.*

*Proof.* If Black reenters a White's (2,1)-vertex by playing in  $v$ , White plays in  $e$ . As Black cannot initiate a winning sequence, whatever he plays White can defend until Black is not threatening anymore. Then White plays in  $k$  or in  $l$  with a decisive double threat in  $m$  and  $n$ .  $\square$

**Theorem 1.** *HAVANNAH is PSPACE-complete.*

### 3 TwixT

Alex Randolph's TWIXT is one of the most popular connection games. It was invented around 1960 and was marketed as soon as in 1962 [13]. In his book devoted to connection games, Cameron Browne describes TWIXT as one of the most popular and widely marketed of all connection games [5]. We now briefly describe the rules of TWIXT and refer to Moesker's master's thesis for an

introduction and a mathematical approach to the strategy, and the description of a possible implementation [19].

TWIXT is a 2-player connection game played on a GO-like board. At their turn, player White and Black place a pawn of their color in an unoccupied place. Just as in HAVANNAH and HEX, pawns cannot be taken, moved, nor removed. When 2 pawns of the same color are spaced by a knight's move, they are linked by an edge of their color, unless this edge would cross another edge. At each turn, a player can remove some of their edges to allow for new links. The goal for player White (resp. Black) is to link top and bottom (resp. left and right) sides of the board. Note that sometimes, a player could have to choose between two possible edges that intersect each other. The *pencil and paper* version TWIXTPP where the edges of a same color are allowed to cross is also famous and played online.

As the length of a game of TWIXT is polynomially bounded, exploring the whole tree can be done with polynomial space using a minimax algorithm. Therefore TWIXT is in PSPACE.

Mazzoni and Watkins have shown that 3-SAT could be reduced to single-player TWIXT, thus showing NP-completeness of the variant [18]. While it might be possible to try and adapt their work and obtain a reduction from 3-QBF to standard two-player TWIXT, we propose a simpler approach based on HEX. The PSPACE-completeness of HEX has already been used to show the PSPACE-completeness of AMAZONS, a well-known territory game [11].

We now present how we construct from an instance  $G$  of HEX an instance  $\phi(G)$  of TWIXT. We can represent a cell of HEX by the  $9 \times 9$  TWIXT gadgets displayed in Fig. 6. Let  $n$  be the size of a side of  $G$ , Fig. 7 shows how a TWIXT board can be paved by  $n^2$  TWIXT cell gadgets to create a HEX board.

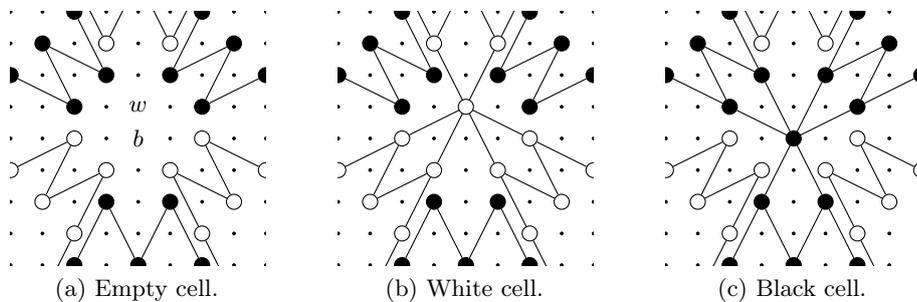


Fig. 6: Basic gadgets needed to represent cells.

It is not hard to see from Fig. 6a that in each gadget of Fig. 7, move  $w$  (resp.  $b$ ) is dominating for White (resp. Black). That is, playing  $w$  is as good for White as any other move of the gadget. We can also see that the moves that are not part of any gadget in Fig. 7 are dominated for both players. As a result, if player Black (resp. White) has a winning strategy in  $G$ , then player Black has a

winning strategy in  $\phi(G)$ . Thus,  $G$  is won by Black if and only if  $\phi(G)$  is won by Black. Therefore determining the winner in TWIXT is at least as hard as in HEX, leading to the desired result.

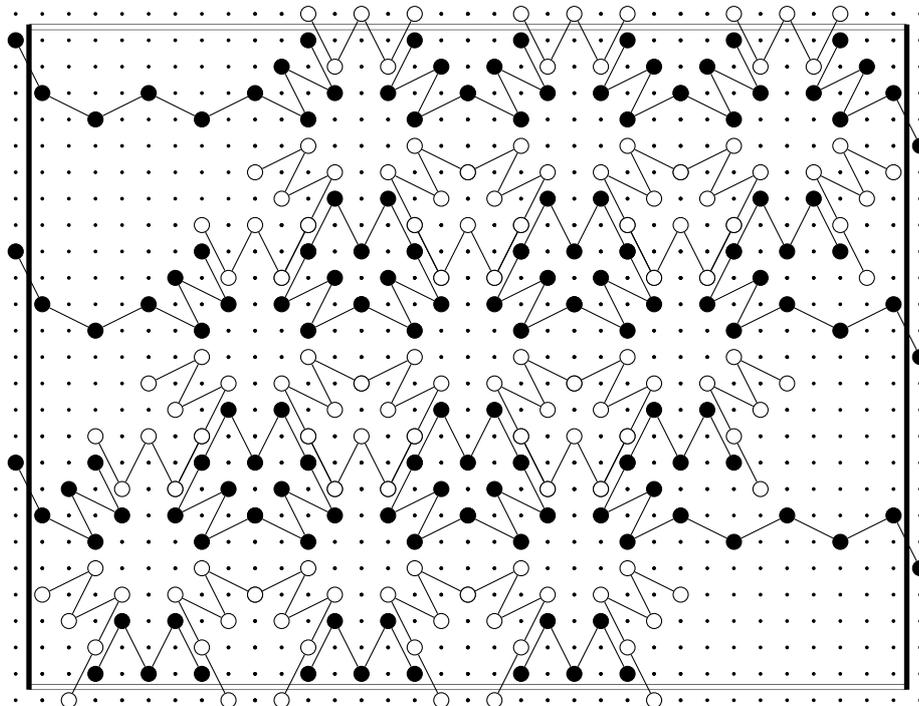


Fig. 7: Empty  $3 \times 3$  HEX board reduced to a TWIXT board.

**Theorem 2.** TWIXT is PSPACE-complete.

Observe that the proposed reduction holds both for the classic version of TWIXT as well as for the *pencil and paper* version TWIXTPP. Indeed, the reduction does not require the losing player to remove any edge, so it also proves that TWIXTPP is PSPACE-hard.

## 4 Conclusion

This paper establishes the PSPACE-completeness of two important connection games, HAVANNAH and TWIXT. The proof for TWIXT is a reduction from HEX and applied to TWIXTPP. The proof for HAVANNAH is more involved and is based on the Generalized Geography problem restricted to bipartite graphs of degree 3.

This HAVANNAH reduction only used the loop winning condition, but it is easy to show that HAVANNAH without the loop winning condition can simulate HEX and is PSPACE-hard as well. For both reductions, the size of the resulting game is only linearly larger than the size of the input instance.

The complexity of other notable connection games remains open. In particular, the following games seem to be good candidates for future work on the complexity of connection games.

In LINES OF ACTION, each player starts with two groups of pieces and tries to connect all their pieces by moving these pieces and possibly capturing opponent pieces [24]. While the goal of LINES OF ACTION clearly makes it a connection game, the mechanics distinguishes it from more classical connection games as no pieces are added to the board and existing pieces can be moved or removed. As a result, it is not even clear that LINES OF ACTION is in PSPACE.

SLITHER is closer to HEX but each move actually consists of putting a new stone on the board and possibly moving another one. Obtaining a PSPACE-hardness result for SLITHER is not so easy since the rules allow a player to influence two different areas of the board in a single turn.

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