A New Equational Foundation for the Fluent Calculus

Hans-Peter Störr and Michael Thielscher

Artificial Intelligence Institute, Department of Computer Science
Dresden University of Technology, 01062 Dresden, Germany
{haps,mit}@inf.tu-dresden.de

Abstract. A new equational foundation is presented for the Fluent Calculus, an established predicate calculus formalism for reasoning about actions. We discuss limitations of the existing axiomatizations of both equality of states and what it means for a fluent to hold in a state. Our new and conceptually even simpler theory is shown to overcome the restrictions of the existing approach. We prove that the correctness of the Fluent Calculus as a solution to the Frame Problem still holds under the new foundation. Furthermore, we extend our theory by an induction axiom needed for reasoning about integer-valued resources.
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1 Introduction

Research in Cognitive Robotics aims at explaining and modeling high-level intelligent agents acting in a complex dynamic world. Among the established predicate calculus formalisms for reasoning about actions, the Fluent Calculus stands out in offering a solution not only to the representational but also the inferential [3] aspect of the fundamental Frame Problem. The basic solution has proven its versatility by allowing extensions regarding a variety of aspects, such as nondeterministic actions, resource-sensitivity, concurrency, ramifications, natural actions in combination with continuous change, sensing actions, and recursive and conditional plans [13, 7, 14, 15]. An implementation of the fluent calculus by means of binary decision diagrams is under way [8].

Central to the Fluent Calculus, which is a many-sorted first-order language, is the representation technique of reification [6]: Terms are used instead of atomic formulas as formal denotations for fluents, i.e., the atomic properties of the world state whose truth-values may change in the course of actions. In the Fluent Calculus these 'atomic' fluent terms are composed to state descriptions by means of a binary function, written as "$\circ$". More precisely, any term of sort *fluent* is also of sort *state*, and if z_1 and z_2 are of sort *state* then so is $z_1 \circ z_2$. For example, if the term $\mathit{Occupied}(x)$ is of sort *fluent*, representing the (temporary) property of a room x to be occupied, and if variable z is of sort *state*, then the term

$$(\mathit{Occupied}(\mathsf{AMT\text{-}101}) \circ \mathit{Occupied}(\mathsf{AMT\text{-}206})) \circ z \tag{1}$$

J. Lloyd et al. (Eds.): CL 2000, LNAI 1861, pp. 733–746, 2000.

describes a world state in which the two rooms AMT-101 and AMT-206 are occupied and in which other fluents z hold.[1]

Based on the concept of state terms, the fundamental Frame Problem is solved in the Fluent Calculus by so-called state update axioms, which specify how the states of the world before and after an action are related [13]: From the Situation Calculus, we adopt the concept of a *situation* as a history of the actions that have been performed [10]. Let the expression $State(s)$ be a denotation of the world state in situation s, let $Do(a,s)$ denote the situation reached after performing action a in situation s, and let the atom $Poss(a,s)$ denote that action a is possible in situation s. Then a state update axiom for an action A with parameters $\boldsymbol{x}$ is of the form,

$$Poss(A(\boldsymbol{x}),s) \supset (\,\Delta(\boldsymbol{x},s) \supset \Gamma_{A,\Delta}[State(Do(A(\boldsymbol{x}),s)), State(s)]\,) \tag{2}$$

where $\Delta(\boldsymbol{x}, s)$ is a first-order formula which describes the conditions on $\boldsymbol{x}$ and situation s under which the two states prior and after the action are related in the way specified by $\Gamma_{A,\Delta}$. In the simple case, $\Gamma_{A,\Delta}$ is a mere equational relation between the states:

$$(\exists \boldsymbol{y})\, State(Do(A(\boldsymbol{x}),s)) \circ \vartheta^-(\boldsymbol{x},\boldsymbol{y}) = State(s) \circ \vartheta^+(\boldsymbol{x},\boldsymbol{y}) \tag{3}$$

where the sub-terms ϑ^- and ϑ^+, which are of sort *state*, contain, respectively, the negative and positive effects of action A under condition Δ.

Consider, for example, the action denoted by $Move(x,y)$ of sending everyone from room x to room y. Suppose that this action has the effect of x no longer being occupied and of room y becoming occupied instead. Suppose further that the action be possible if x is currently occupied and y is not. The following two axioms are a suitable encoding of this specification in the Fluent Calculus:

$$\begin{aligned} &Poss(Move(x,y),s) \supset \\ &\quad State(Do(Move(x,y),s)) \circ Occupied(x) = State(s) \circ Occupied(y) \end{aligned} \tag{4}$$

$$Poss(Move(x,y),s) \equiv Holds(Occupied(x),s) \wedge \neg Holds(Occupied(y),s) \tag{5}$$

where $Holds(f,s)$ means that fluent f holds in situation s.

In order that axioms like these entail reasonable conclusions, an axiomatic account of two properties of states is required:

[1] A word on the notation: Predicate and function symbols, including constants, start with a capital letter whereas variables are in lower case, sometimes with sub- or superscripts. Free variables in formulas are assumed universally quantified. Throughout the paper, action variables are denoted by the letter a, situation variables by the letter s, fluent variables by the letter f, and state variables by the letter z, all possibly with sub- or superscript. Multisets, i.e. collections, that can contain elements more than once, are written as $\dot{\{}f_1,\ldots,f_n\dot{\}}$, and multiset operations are marked by a dot above the operation symbol.

1. What makes two states equal, and what makes them unequal?
2. When does a fluent hold in a state associated to a situation, and when does it not?

An answer to the first question is crucial for solving the representational and inferential Frame Problem by state update axioms whose consequences are equations of the form (3): If a fluent is contained in $State(s)$ and is not among the negative effects $\vartheta^-(\boldsymbol{x}, \boldsymbol{y})$, then the fluent should be contained in $State(Do(A(\boldsymbol{x}), s))$; and if a fluent is not contained in $State(s)$, then it should also not be contained in $State(Do(A(\boldsymbol{x}), s))$ as long as it is not among the positive effects $\vartheta^+(\boldsymbol{x}, \boldsymbol{y})$.[2] An answer to the second question is needed to evaluate both action preconditions and the condition part of state update axioms, and in general to draw any interesting conclusions concerning the values of fluents in situations.

The existing equational foundation of the Fluent Calculus, developed in [9], gives an answer to the two questions based on the equational theory of a commutative monoid along with the notion of unification completeness [12]. In the following section we show the limitations of this approach when it comes to incorporating domain-specific equalities or the definition of functions among domain entities. In Section 3, a new and conceptually simpler equational foundation is developed, which is shown to overcome the restrictions of the existing account. In Section 4, we prove some fundamental properties of the new axiomatization, which in particular ensure that the Fluent Calculus solution to the Frame Problem still is correct under the new foundation. In Section 5, a second-order extension of our theory is presented to enable reasoning about the consumption and production of integer-valued resources. This extension is proved to axiomatically characterize the sort *state* as the finite multisets over the sort *fluent*. We conclude in Section 6.

2 Unification Completeness and Its Limitations

The Fluent Calculus uses classical logic with equality, that is, where the equality relation is assumed to be interpreted as real equality among domain elements. On this basis, the existing equational foundation of the Fluent Calculus consists of the following axioms:

– Equational theory AC1,

$$\begin{aligned}(z_1 \circ z_2) \circ z_3 &= z_1 \circ (z_2 \circ z_3)\\ z_1 \circ z_2 &= z_2 \circ z_1\\ z \circ \emptyset &= z\end{aligned} \qquad \text{(AC1)}$$

(where $\emptyset$ is a constant of sort *state*, denoting the empty collection of fluents);

– An AC1-unification complete theory AC1* (details given below).

[2] We assume throughout the paper that ϑ^+ and ϑ^- are disjunct and do not contain any fluent more than once.

Theory AC1 essentially says that the order in which the fluent terms occur in repeated applications of $\circ$ is irrelevant, so that, say, $\mathit{Occupied}(\mathsf{AMT\text{-}206}) \circ (z \circ \mathit{Occupied}(\mathsf{AMT\text{-}101}))$ and (1) denote the very same state. (Justified by the law of associativity, we will omit parentheses in nested applications of $\circ$ in the rest of the paper.) Based on the equational foundation, the notion of fluents holding in states and situations, resp., is defined via two macros which stand for pure equality sentences:

$$\mathit{Holds}(f, z) \stackrel{\text{def}}{\equiv} (\exists z')\, z = f \circ z' \qquad \text{(Holds)}$$

$$\mathit{Holds}(f, s) \stackrel{\text{def}}{\equiv} \mathit{Holds}(f, \mathit{State}(s))$$

That is, a fluent holds in a state or a situation, resp., if it is contained in the respective state terms.

Negating the left and right hand sides of definition (Holds), a fluent f does not hold in a state z if for all z' we have $z \neq f \circ z'$. Deriving inequalities of this kind requires to axiomatize that states not composed of the same fluents are unequal. The AC1-unification complete theory AC1* serves this purpose [9]. Its definition relies on a complete AC1-unification algorithm, and it comprises an infinite set of axioms which contains the following axiom for any pair of terms t_1, t_2 of sort *state* and without occurrence of function *State*:

$$t_1 = t_2 \supset \bigvee_{\theta \in \Theta_{\mathrm{AC1}}(t_1, t_2)} \theta_= \qquad (6)$$

where $\Theta_{\mathrm{AC1}}(t_1, t_2)$ is a complete [1] set of AC1-unifiers of t_1, t_2 and where $\theta_=$ is the equational formula $x_1 = r_1 \wedge \ldots \wedge x_n = r_n$ if $\theta = \{x_1/r_1, \ldots, x_n/r_n\}$. In particular, if two terms are not AC1-unifiable, then the disjunction evaluates to falsity, hence the implication simplifies to $t_1 \neq t_2$. Inequalities of state terms can thus be derived from their not being AC1-unifiable.

The rigorousness of unification completeness, however, has the important limitation of making it impossible to add simple domain-specific equalities or to define functions among domain entities.

Observation 1. Consider a Fluent Calculus signature with the two constants AMT-101 and *MainLectureHall* of the domain sort *room* and with function *Occupied*: *room* $\mapsto$ *fluent*. Then

$$\mathit{MainLectureHall} = \mathsf{AMT\text{-}101} \qquad (7)$$

and AC1* are inconsistent.

Proof. By the standard interpretation of equality and (7) it follows

$$\mathit{Occupied}(\mathit{MainLectureHall}) = \mathit{Occupied}(\mathsf{AMT\text{-}101})$$

But the terms *Occupied*(*MainLectureHall*) and *Occupied*(AMT-101) of sort *state* are not AC1-unifiable; hence, (6) entails

$$\mathit{Occupied}(\mathit{MainLectureHall}) \neq \mathit{Occupied}(\mathsf{AMT\text{-}101}) \qquad \square$$

Observation 2. Consider a Fluent Calculus signature with constant AMT-206 of domain sort *room*, constant *Peter* of domain sort *person*, and functions $\mathit{RoomOf}: \mathit{person} \mapsto \mathit{room}$ and $\mathit{Occupied}: \mathit{room} \mapsto \mathit{fluent}$. Then

$$\mathit{RoomOf}(\mathit{Peter}) = \mathsf{AMT\text{-}206} \tag{8}$$

and AC1* are inconsistent.

Proof. As above. □

The only way of incorporating domain-dependent equalities or definitions of functions without sacrificing the idea of unification completeness is to use an E-unification complete theory instead of simply AC1* where E consists of the axioms AC1 plus all domain-dependent equations. This approach, however, has severe drawbacks: First, the foundational axioms on equality of state terms are domain-dependent so that they need to be adapted to any additional equation or inequality. Second, if the equational theory E is not finitary [1], then the corresponding unification complete theory includes axioms with an infinite number of disjuncts. Finally and most importantly, the definition of an E-unification complete theory appeals to complete sets of E-unifiers so that the existence of such a set for any two terms needs to be proved for any particular domain axiomatization in order that the definition is not rendered meaningless.

The equational foundation for the Fluent Calculus is accompanied by the following foundational axiom, which stipulates non-multiplicity of fluents in state terms that are associated with a situation:

$$\mathit{State}(s) \neq f \circ f \circ z \tag{NonMult}$$

Assuming non-multiplicity instead of stipulating idempotency of $\circ$ is crucial in order not to annul the solution to the Frame Problem offered by state update axioms. To see why, suppose $\mathit{State}(s) = f \circ z$ for some f, s, z, and consider the equation $\mathit{State}(\mathit{Do}(a, s)) \circ f = \mathit{State}(s)$, where f is specified as negative effect. However, neither idempotency of $\circ$ would allow to conclude that f does not occur in $\mathit{State}(\mathit{Do}(a, s))$, nor would this follow without axiom (NonMult).

3 A New Equational Foundation

The limitations of the existing equational foundation for the Fluents Calculus can be overcome by a paradigm shift away from the inference-oriented viewpoint of unification completeness towards a more semantic-oriented view. Intuitively, two state terms shall be equal only if they contain equal fluents. Indeed, a simple, finite first-order axiomatization of this intuition is possible under which the Fluents Calculus solution to the Frame Problem is still valid.[3]

[3] In a later section, we will show that it is moreover possible to give a finite but second-order extension of these axioms so as to obtain a characterization of equality of *state* terms precisely up to the ordering of the *fluent* sub-terms—in other words, where in every interpretation the sort *state* is isomorphic to the finite multisets over the sort *fluent*.

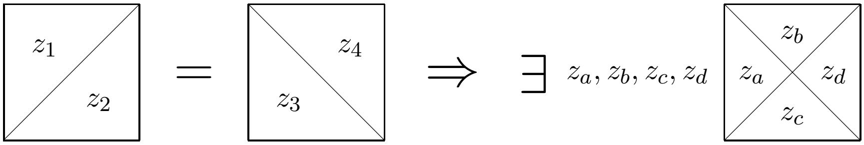

Fig. 1. The Levi axiom: If some state (symbolized by a square) can be partitioned into z_1, z_2 as well as into z_3, z_4, it can be partitioned into z_a, z_b, z_c, z_d such that the same areas denote equal (wrt. (AC1)) parts of the terms.

Definition 1. *The new equational foundation for the Fluent Calculus comprises the following axioms:*

- *Equational theory AC1;*
- *an axiom which specifies that a fluent is an irreducible (wrt. ∘) element of state:*

$$z = f \supset z \neq \emptyset \wedge [z = z' \circ z'' \supset z' = \emptyset \vee z'' = \emptyset] \qquad \text{(Irred)}$$

- *the so-called Levi-axiom:*[4]

$$z_1 \circ z_2 = z_3 \circ z_4 \supset (\exists z_a, z_b, z_c, z_d) \begin{pmatrix} z_1 = z_a \circ z_b \wedge z_3 = z_a \circ z_c \wedge \\ z_2 = z_c \circ z_d \wedge z_4 = z_b \circ z_d \end{pmatrix} \qquad \text{(Levi)}$$

Fig. 1 gives a graphical interpretation of this axiom.[5]

These axioms are domain-independent. By EUNA we denote their union along with a set of domain-dependent unique names-axioms UNA.

To demonstrate the gained expressiveness of the new foundation, recall the observations made in Section 2. Suppose given the initial state

$$State(S_0) = Occupied(\mathsf{AMT\text{-}206}) \qquad (9)$$

[4] The axiom postulated here is proven as a lemma called *Levi's lemma* in trace theory [4]. Since the set of finite multisets with multiset union as an operation is isomorphic to a trace monoid over the same set where all symbols are independent, we turn its role around and postulate this property as an axiom characterizing multisets.

[5] It should be noted that the picture may be a bit misleading: In case z_1, z_2, z_3, z_4 contain multiple occurrences of sub-terms, the states z_a, z_b, z_c, z_d are not necessarily uniquely determined, as the reader may verify with the example $(a \circ a) \circ (a \circ a) = a \circ (a \circ a \circ a)$.

along with the axioms $\mathit{UNA}[\mathit{Occupied}]$, $\mathit{UNA}[\mathsf{AMT\text{-}101}, \mathsf{AMT\text{-}206}]$,[6] and

$$\mathit{MainLectureHall} = \mathsf{AMT\text{-}101} \tag{10}$$

Note that this equation does not contradict the axioms of Def. 1, as opposed to the old foundation of the Fluent Calculus (c.f. Observation 1). Suppose further we want to move the current lecture from room $\mathsf{AMT\text{-}206}$ to the main lecture hall since the former is too small. This action leads from situation S_0 to $\mathit{Do}(\mathit{Move}(\mathsf{AMT\text{-}206}, \mathit{MainLectureHall}), S_0)$. By applying (AC1) and (Irred) to (5) and (9) it follows that $\mathit{Poss}(\mathit{Move}(\mathsf{AMT\text{-}206}, \mathit{MainLectureHall}), S_0)$. Thus, from (4) and (9) we conclude that

$$\begin{aligned} &\mathit{State}(\mathit{Do}(\mathit{Move}(\mathsf{AMT\text{-}206}, \mathit{MainLectureHall}), S_0)) \circ \mathit{Occupied}(\mathsf{AMT\text{-}206}) = \\ &\qquad \mathit{Occupied}(\mathsf{AMT\text{-}206}) \circ \mathit{Occupied}(\mathit{MainLectureHall}) \end{aligned} \tag{11}$$

While axioms (AC1) and (10) suffice to show that (11) is satisfied by

$$\mathit{State}(\mathit{Do}(\mathit{Move}(\mathsf{AMT\text{-}206}, \mathit{MainLectureHall}), S_0)) = \mathit{Occupied}(\mathsf{AMT\text{-}101}) \tag{12}$$

our new axioms are needed to *prove* that (12) holds: Applying (10) to (11) we get

$$\begin{aligned} &\mathit{State}(\mathit{Do}(\mathit{Move}(\mathsf{AMT\text{-}206}, \mathit{MainLectureHall}), S_0)) \circ \mathit{Occupied}(\mathsf{AMT\text{-}206}) = \\ &\qquad \mathit{Occupied}(\mathsf{AMT\text{-}206}) \circ \mathit{Occupied}(\mathsf{AMT\text{-}101}) \end{aligned} \tag{13}$$

According to (Levi) we find z_a, z_b, z_c, z_d such that

$$\mathit{State}(\mathit{Do}(\mathit{Move}(\mathsf{AMT\text{-}206}, \mathit{MainLectureHall}), S_0)) = z_a \circ z_b \tag{14}$$

$$\mathit{Occupied}(\mathsf{AMT\text{-}206}) = z_c \circ z_d \tag{15}$$

$$\mathit{Occupied}(\mathsf{AMT\text{-}206}) = z_a \circ z_c \tag{16}$$

$$\mathit{Occupied}(\mathsf{AMT\text{-}101}) = z_b \circ z_d \tag{17}$$

Employing (Irred), we apply case distinction to (15). The case $z_c = \emptyset$ and $z_d = \mathit{Occupied}(\mathsf{AMT\text{-}206})$ contradicts equation (17) in view of $\mathit{UNA}[\mathit{Occupied}]$, $\mathit{UNA}[\mathsf{AMT\text{-}101}, \mathsf{AMT\text{-}206}]$, and (Irred). In case $z_c = \mathit{Occupied}(\mathsf{AMT\text{-}206})$ and $z_d = \emptyset$, from (17) and (AC1) it follows that $z_b = \mathit{Occupied}(\mathsf{AMT\text{-}101})$; furthermore, from (16) we get $z_a = \emptyset$ according to (Irred). Hence, (15) and (AC1) entail the desired conclusion (12).

The example derivation shows that the new equational foundation successfully handles the domain-dependent equality (10). In a similar fashion we can

[6] For domain dependent assumptions of unique names we adopt from [2] the standard notation $\mathit{UNA}[h_1, \ldots, h_n]$ as an abbreviation for the formula

$$\bigwedge_{i \neq j} h_i(\boldsymbol{x}) \neq h_j(\boldsymbol{y}) \wedge \bigwedge_i (h_i(\boldsymbol{x}) = h_i(\boldsymbol{y}) \supset \boldsymbol{x} = \boldsymbol{y})$$

now introduce functions among domain entities by equations like (8) without producing inconsistency. Admittedly, calculating with pure (Levi) and (Irred) looks rather cumbersome. However, in the next section we will derive two computation rules as logical consequences of our axiomatization, which are of great help when calculating state equations. One of the two rules, for instance, leads directly from (13) to (12).

4 Results

We have seen that our new foundation for the Fluent Calculus allows the incorporation of domain-dependent equations and inequalities. In this section, we prove the crucial result that state update axioms solve the Frame Problem also under the new axioms. More specifically, we prove that the core of a state update axiom, an equation of the form

$$(\exists \boldsymbol{y})\, State(Do(A(\boldsymbol{x}), s)) \circ \vartheta^-(\boldsymbol{x}, \boldsymbol{y}) = State(s) \circ \vartheta^+(\boldsymbol{x}, \boldsymbol{y}) \tag{18}$$

satisfies the following:

1. All fluents in $\vartheta^+(\boldsymbol{x}, \boldsymbol{y})$ (the positive effects of the action) do hold in the successor state $State(Do(A(\boldsymbol{x}), s))$;
2. all fluents in $\vartheta^-(\boldsymbol{x}, \boldsymbol{y})$ (the negative effects of the action) do not hold in the successor state $State(Do(A(\boldsymbol{x}), s))$;
3. all fluents not contained in ϑ^+ or ϑ^- hold in $State(Do(A(\boldsymbol{x}), s))$ if and only if they hold in $State(s)$;
4. the equation is consistent with foundational axiom (NonMult).

The proof is based on two computation rules, the Cancelation Rule and the Distribution Rule. Both are logical consequences of our axiomatization, and they are of great practical value when it comes to calculating with state equations.

Proposition 1. (Cancelation Rule) *In all models of EUNA we have*

$$f \circ z = f \circ z' \supset z = z' \tag{Cancel}$$

Proof. Assume $f \circ z = f \circ z'$. By (Levi) we find z_a, z_b, z_c, z_d such that

$$f = z_a \circ z_b \ \wedge\ z = z_c \circ z_d \ \wedge\ f = z_a \circ z_c \ \wedge z' = z_b \circ z_d$$

By (Irred) we distinguish two cases:

If $z_a = \emptyset$, then (AC1) implies $f = z_b = z_c$, thus $z = f \circ z_d$ and $z' = f \circ z_d$, and hence by symmetry and transitivity of equality, $z = z'$.

If $z_b = \emptyset$, then $f = z_a$ by (AC1). Applying (Irred) we conclude $f \neq \emptyset$; therefore, $f = z_a \circ z_c$ implies $z_c = \emptyset$, again by (Irred). Hence, $z = \emptyset \circ z_d = z'$. □

The Cancelation Rule allows to cancel out equal *fluent* terms on both sides of a state equation. Note, for instance, that by this rule the rather complicated derivation of (12) from (13) of the preceding section follows directly.

Proposition 2. (Distribution Rule) *In all models of EUNA we have*

$$f_1 \neq f_2 \supset f_1 \circ z_1 = f_2 \circ z_2 \supset Holds(f_1, z_2) \qquad \text{(Distrib)}$$

Proof. Assume $f_1 \neq f_2$ and $f_1 \circ z_1 = f_2 \circ z_2$. Following (Irred) we find z_a, z_b, z_c, z_d such that

$$f_1 = z_a \circ z_b \ \wedge\ z_1 = z_c \circ z_d \ \wedge\ f_2 = z_a \circ z_c \ \wedge\ z_2 = z_b \circ z_d$$

By (Irred) we conclude that either $z_a = \emptyset \wedge z_b = f_1$ or $z_a = f_1 \wedge z_b = \emptyset$. The latter case would imply $f_2 = f_1 \circ z_c$, which contradicts (Irred) given that $f_1 \neq f_2$. Thus, $z_a = \emptyset \wedge z_b = f_1$, hence $z_2 = f_1 \circ z_d$, hence $Holds(f_1, z_2)$. □

The Distribution Rule in combination with Cancelation allows to rewrite state equations so as to project onto a particular sub-term. A typical application is to rewrite the equation $State(Do(a, s)) \circ f^- = State(s) \circ f^+$ to $(\exists z)\, (State(s) = f^- \circ z \wedge State(Do(a, s)) \circ f^- = f^- \circ z \circ f^+)$, and then to apply the Cancelation Rule to obtain the projection $(\exists z)\, (State(Do(a, s)) = z \circ f^+ \wedge State(s) = f^- \circ z)$.

We are now in a position to prove the abovementioned main result. As in [13] we make the following assumption of consistency: State update axioms are designed in such a way that if an equation (18) is entailed, then the positive and negative effects, ϑ^+ and ϑ^-, do not share a fluent, contain no fluent more than once and no fluent is specified as positive effect via ϑ^+ if it holds in $State(s)$ itself. From (NonMult) we furthermore know that no fluent occurs twice in $State(s)$.

Theorem 3. *Consider a set UNA of unique names-axioms and let the terms $\vartheta^- = f_1^- \circ \ldots \circ f_m^-$ and $\vartheta^+ = f_1^+ \circ \ldots \circ f_n^+$ be finite, possibly empty sequences of fluent terms joined together with $\circ$ such that $UNA \models f_i^+ \neq f_-^+$ for all i, j, and $UNA \models f_i^- \neq f_j^-$ as well as $UNA \models f_i^+ \neq f_j^+$ for all $i \neq j$. Then in all models for EUNA we have that*

$$z_1 \circ \vartheta^- = z_2 \circ \vartheta^+ \ \wedge \bigwedge_{j=1 \ldots n} \neg Holds(f_j^+, z_2) \ \wedge\ (\forall f)\, \neg Holds(f \circ f, z_2)\,^7$$

implies each of the following.

1. $Holds(f_j^+, z_1)$ *(for all $j = 1, \ldots, n$);*
2. $\neg Holds(f_i^-, z_1)$ *(for all $i = 1, \ldots, m$);*
3. $(\forall f)\, (\neg Holds(f, \vartheta^- \circ \vartheta^+) \supset [Holds(f, z_1) \equiv Holds(f, z_2)])$;
4. $(\forall f)\, \neg Holds(f \circ f, z_1)$.

Proof.

1. Follows from $UNA \models f_i^- \neq f_j^+$ for all f_i^- in ϑ^- by repeated application of the Distribution Rule.

[7] where $Holds(\bar{z}, z) \stackrel{\text{def}}{\equiv} (\exists z')\, z = \bar{z} \circ z'$

2. If $z_1 = f_i^- \circ z'$ for some f_i^- in ϑ^- and some z', then
$$f_i^- \circ z' \circ f_1^- \circ \ldots \circ f_i^- \circ \ldots \circ f_m^- = z_2 \circ \vartheta^+ \tag{19}$$
From $UNA \models f_i^- \neq f_j^+$ for all f_j^+ in ϑ^+ and by n-fold application of the Distribution Rule it follows that $(\exists z'')\, z_2 = f_i^- \circ z''$; hence, (19) implies, using the Cancelation Rule,
$$(\exists z'')\, z' \circ f_1^- \circ \ldots \circ f_i^- \circ \ldots \circ f_m^- = z'' \circ \vartheta^+$$
With a similar argument we conclude that $(\exists z''')\, z'' = f_i^- \circ z'''$; consequently, $(\exists z''')\, z_2 = f_i^- \circ f_i^- \circ z'''$, which contradicts $(\forall f)\, \neg Holds(f \circ f, z_2)$.
3. Follows by repeated application of the Distribution Rule.
4. Follows from $(\forall f)\, \neg Holds(f \circ f, z_2)$ and $\bigwedge_{j=1\ldots n} \neg Holds(f_j^+, z_2)$ with a similar argument as used in the proof of item 2. □

5 An Induction Axiom

While our new equational foundation for the Fluent Calculus does not affect the solution to the basic Frame Problem, the axiomatization presented so far is limited when it comes to modeling resources. Generally, the Fluent Calculus offers a very natural way of reasoning about the production and consumption of integer-valued resources, namely, by simply not letting foundational axiom (NonMult) apply to resources. A state may then contain multiple occurrences of a resource. For example, given that wheels (of a certain diameter) and axles (of a certain length) are different things, i.e., $UNA[Wheel, Axle]$, axioms $EUNA$ entail that $Wheel(6'') \circ Wheel(6'') \circ Axle(3.5') \neq Wheel(6'') \circ Axle(3.5') \circ Axle(3.5')$, read: having available two wheels and one axle is different from holding just one wheel but two axles. An example for a state update axiom talking about resources is the following, which specifies the action $Assemble(l, d)$ of assembling a chassis of length l and with two wheels of diameter d:[8]

$$\begin{aligned} & Holds(Axle(l) \circ Wheel(d) \circ Wheel(d), s) \supset \\ & \quad State(Do(Assemble(l,d), s)) \circ Axle(l) \circ Wheel(d) \circ Wheel(d) \\ & \qquad = State(s) \circ Chassis(l, d) \end{aligned}$$

For an adequate treatment of resources, our axiomatization of Section 3 is insufficient because it admits models in which equations like $z \circ f = z$ are true:

Observation 4. $EUNA \cup \{z \circ f = z\}$ is satisfiable.

Proof. We construct a model as follows. Let the domain for sort *state* be the natural numbers $\mathbb{N}$ (incl. 0) augmented by the element ω. The only domain element of sort *fluent* shall be 1. Let $\emptyset$ be interpreted by 0 and $\circ$ by the function

$$\lambda m, n. \begin{cases} m + n & \text{if } m \neq \omega \text{ and } n \neq \omega \\ \omega & \text{otherwise} \end{cases}$$

[8] Below, $Holds(\bar{z}, s) \stackrel{\text{def}}{=} (\exists z')\, State(s) = \bar{z} \circ z'$.

This function is associative, commutative, and has 0 as unit element; hence, (AC1) holds in the model. Furthermore, $1 \neq 0$, and if $1 = m + n$ then either $m = 0$ or $n = 0$; hence, (Irred) holds. Finally, if $n_1 + n_2 = n_3 + n_4$, then (Levi) is satisfied by

$$\begin{array}{ll} n_a = \min(n_1, n_3) & n_c = n_2 - n_d \\ n_b = n_1 - n_a & n_d = \min(n_2, n_4) \end{array}$$

in case $n_1 + n_2 \neq \omega$. In case $n_1 + n_2 = n_3 + n_4 = \omega$, let us assume without loss of generality that $n_1 = n_3 = \omega$. If none of n_2, n_4 equals ω, then (Levi) is satisfied by

$$\begin{array}{ll} n_a = \omega & n_c = n_2 - n_d \\ n_b = \max(0, n_4 - n_2) & n_d = \min(n_2, n_4) \end{array}$$

else if one and only one of n_2, n_4 equals ω, say n_2, then (Levi) is satisfied by

$$n_a = \omega \ \wedge \ n_b = n_4 \ \wedge \ n_c = \omega \ \wedge \ n_d = 0$$

else if $n_2 = n_4 = \omega$, then (Levi) is satisfied by $n_a = n_b = n_c = n_d = \omega$. Having proved that we have constructed a model for *EUNA*, the claim follows by interpreting z by ω and f by 1, because $\omega + 1 = \omega$. □

The reader may note that this observation does not contradict the Cancelation Rule, which allows only fluents to be canceled out. The observation is to be contrasted to the old foundation of the Fluent Calculus, where the unification complete theory AC1* includes the axiom $z \circ f \neq z$ since the two terms are not AC1-unifiable.

Observation 4 is unproblematic in the non-resource case, because the equation $State(s) \circ f = State(s)$ *is* unsatisfiable in view of foundational axiom (NonMult).[9] But if we remove the non-multiplicity condition in order to deal with resources, then we need means to prevent such unintended models.

In this section, we introduce two additional axioms through which this problem is solved. Speaking algebraically, we extend *EUNA* in such a way that in every model $\mathcal{M}$ of that extension we have that the sort $state^{\mathcal{M}}$ is isomorphic to the set of finite multisets over the sort $fluent^{\mathcal{M}}$, where $\emptyset$ represents the empty multiset and $\circ$ the union of multisets. The additional axioms are, first, an induction axiom, which says that the sort *state* contains exactly the terms which can be constructed by applying $\circ$ to $\emptyset$ and elements of sort *fluent*; and second, an axiom which specifies that $\emptyset$ has no proper divisor:

$$(\forall P)\ [P(\emptyset) \wedge (\forall f, z)\ (P(z) \supset P(f \circ z)) \ \supset\ (\forall z)\, P(z)] \qquad \text{(Ind)}$$

$$z \circ z' = \emptyset \supset z = \emptyset \qquad \text{(ZeroDiv)}$$

EUNA augmented by (Ind) and (ZeroDiv) (we call this theory *EUNA*$^+$) is consistent:

[9] Repeated application of the Distribution Rule to $State(s) \circ f = State(s)$ yields $(\exists z)\, State(s) = f \circ f \circ z$.

Proposition 3. *Axioms $EUNA^+$ are satisfiable.*

Proof. We can construct a model for $EUNA^+$ as follows: Let the domain for the sort *fluent* be a set of singleton multisets $\{\dot{\{}a\dot{\}} : a \in A\}$, and let the domain for the sort *state* be the set of finite multisets over A. It is easy to verify that (AC1), (Irred), and (ZeroDiv) are true. Furthermore, any instance of (Levi) is satisfied by setting

$$z_a = z_1 \dot{\cap} z_3 \qquad z_c = z_2 \dot{\setminus} z_d$$
$$z_b = z_1 \dot{\setminus} z_a \qquad z_d = z_2 \dot{\cap} z_4 \ .$$

The proof for (Ind) is straightforward by well-founded induction over the set of finite multisets over *fluent* with the ordering relation $\dot{\subset}$. □

In the following, we prove stepwise that for each model of $EUNA^+$, the following function ϱ is an isomorphism. The function is a mapping from finite multisets of fluents onto fluent terms:

$$\varrho\Big(\dot{\{}\underbrace{f_1,\ldots,f_1}_{k_1 \text{ times}},\ldots,\underbrace{f_n,\ldots,f_n}_{k_n \text{ times}}\dot{\}}\Big) = \emptyset \circ \underbrace{f_1 \circ \cdots \circ f_1}_{k_1 \text{ times}} \circ \cdots \circ \underbrace{f_n \circ \cdots \circ f_n}_{k_n \text{ times}} \tag{20}$$

Terms which are constructed by applying $\circ$ to $\emptyset$ and terms of sort *fluent*, like the one on the right side of this equation, are called *constructor state terms*.

First, we prove that ϱ is a homomorphism, that is, every equation using $\dot{\emptyset}$ and $\dot{\cup}$ which holds between multisets of fluents, also holds after transforming the operands into the Fluent Calculus via ϱ and where $\emptyset$ and $\circ$ replace $\dot{\emptyset}$ and $\dot{\cup}$.

Proposition 4. *In every model of* (AC1)*, ϱ is a homomorphism from*

$$\langle \mathcal{M}_{fin}(\mathit{fluent});\ \dot{\emptyset};\ \dot{\cup} \rangle$$

into $\langle \mathit{state};\ \emptyset;\ \circ \rangle$, where $\mathcal{M}_{fin}(\mathit{fluent})$ is the set of finite multisets over fluent.

The proof is straightforward using the fact that both $\dot{\cup}$ and $\circ$ are associative and commutative and that $\dot{\emptyset}$ and $\emptyset$ are the respective unit elements.

Splitting a constructor state term with $\circ$ into two parts, the parts are constructor state terms themselves:

Proposition 5. *In every model of $EUNA \cup \{(\text{ZeroDiv})\}$, for every multiset $\dot{z}$ of fluents we have*

$$\varrho(\dot{z}) = z_1 \circ z_2 \supset (\exists \dot{z}_1, \dot{z}_2)\,[\, z_1 = \varrho(\dot{z}_1) \wedge z_2 = \varrho(\dot{z}_2) \wedge \dot{z} = \dot{z}_1 \dot{\cup} \dot{z}_2 \,] \tag{21}$$

Proof. The proof is by induction over the well-founded set of finite multisets with $\dot{\subset}$ as ordering relation. If $\dot{z} = \dot{\emptyset}$ then (21) is trivially satisfied by $\dot{z}_1 = \dot{z}_2 = \dot{\emptyset}$ due to (ZeroDiv). If $\dot{z} \neq \dot{\emptyset}$ then we can find some f and $\dot{z}'$ such that $\dot{z} = \dot{\{}f\dot{\}} \dot{\cup} \dot{z}'$. Assume $\varrho(\dot{z}) = z_1 \circ z_2$, hence $f \circ \varrho(\dot{z}') = z_1 \circ z_2$. We can then apply (Levi) and construct $\dot{z}_1$ and $\dot{z}_2$ using the induction hypothesis for $\dot{z}'$. □

Proposition 5 lays the foundation for proving that ϱ is an injective homomorphism, that is, which maps different multisets onto different elements of *state*:

Proposition 6. *In every model of* $EUNA \cup \{(\text{ZeroDiv})\}$ *we have*

$$\dot{z} \neq \dot{z}' \supset \varrho(\dot{z}) \neq \varrho(\dot{z}') \tag{22}$$

Proof. The proof is by induction over the sum $s = |\dot{z}| + |\dot{z}'|$ of the cardinalities of $\dot{z}$ and $\dot{z}'$. We distinguish three cases. The case of both $\dot{z}$ and $\dot{z}'$ being empty is trivial. If just one of them is empty, say $\dot{z}' = \dot{\emptyset}$, we choose $f \dot{\in} \dot{z}$ and obtain $\varrho(\dot{z}) = f \circ \varrho(\dot{z} \mathbin{\dot{\setminus}} \dot{\{} f \dot{\}})$; and by (ZeroDiv) and (Irred) it follows $\varrho(\dot{z}) \neq \emptyset = \varrho(\dot{z}')$, hence (22). In case $\dot{z} \neq \dot{\emptyset} \wedge \dot{z}' \neq \dot{\emptyset}$ we find f, f' such that $f \dot{\in} \dot{z}$ and $f' \dot{\in} \dot{z}'$. Suppose that $\varrho(\dot{z}) = \varrho(\dot{z}')$. Then we can apply (Levi) to $f \circ \varrho(\dot{z} \mathbin{\dot{\setminus}} \dot{\{} f \dot{\}}) = f' \circ \varrho(\dot{z}' \mathbin{\dot{\setminus}} \dot{\{} f' \dot{\}})$ and prove $\dot{z} = \dot{z}'$ by case distinction and repeated application of (Irred), Proposition 6, and the induction hypothesis. □

We are almost done. What remains to be shown is that *every* element of *state* corresponds to a finite multiset over *fluent* :

Theorem 5. $EUNA^+$ *specifies that the elements of the sort* state *correspond to multisets of elements of sort* fluent*.*

Proof. In addition to Proposition 3 we have to prove that ϱ is an isomorphism. Since we know that it is an injective homomorphism (Propositions 4 and 6), it remains to be shown that ϱ is surjective as well, that is, for every element z of *state* there is some $\dot{z}$ such that $\varrho(\dot{z}) = z$. Let P be a monadic relation over *state* such that $P(z)$ holds iff there is some $\dot{z}$ such that $\varrho(\dot{z}) = z$. Then $P(\emptyset)$, and if $P(z)$ then $P(f \circ z)$ holds as well since $f \circ z = \varrho\left(\dot{z} \mathbin{\dot{\cup}} \dot{\{} f \dot{\}}\right)$. Thus, by (Ind) P holds for all elements of the sort *state*. □

6 Conclusion

We have presented a new, conceptually simpler equational foundation for the Fluent Calculus which allows for incorporating domain-dependent equations, inequalities, and function definitions. In so doing we have overcome an important limitation of the Fluent Calculus in comparison with the Situation Calculus of [10]. The new axiomatization already proved invaluable for a case study where we have successfully applied the Fluent Calculus to the Traffic World, a complex dynamic domain which has recently been posed as a challenge to the scientific community [11] and which involves actions with ramifications in nondeterministic, concurrent, and continuous domains [5, 14].

We have presented two variants of our new equational foundation. The basic axiomatization of Section 3 has been shown sufficient for guaranteeing that the Frame Problem is still solved by state update axioms. In Section 5, we have presented the extended theory $EUNA^+$, which additionally allows for modeling the concept of resources and by which the sort representing states is made isomorphic to the set of finite multisets of fluents. Theory $EUNA^+$ proved useful as the theoretical foundation for the ongoing implementation of planning with resources in the Fluent Calculus by means of BDDs [8].

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